A MECHANISM FOR THE FORMATION OF JETS AND VORTICES IN ROTATING FLOWS

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ABSTRACT

Numerical simulations are used to isolate a mechanism for the formation of (i) vortical columns in 3D homogeneous rotating flow and (ii) zonal jets on the β-plane. The dynamics of 'reduced models' are computed by restricting the nonlinear term to include a subset of triad interactions in Fourier space. Reduced models of both near-resonant and non-resonant triads are studied. At moderately small Rossby and Rhines numbers, near resonances are responsible for the generation of large-scale jets and vortices from small-scale fluctuations. In the absence of large-scale drag, near resonances reproduce the asymmetry properties of both full systems: predominance of cyclones in 3D, and stronger westward jets on the β-plane. Including large-scale drag on the β-plane, the full system tends to a constant-energy state characterized by thinner and stronger eastward jets, and zonally averaged profiles that are linearly stable; near resonances exhibit weaker asymmetry, and the averaged profiles are marginally stable.

INTRODUCTION

Three-dimensional (3D), homogeneous rotating flow and β-plane flow are two of the simplest models used to understand wave dynamics in geophysical and planetary flows (see , e.g., [1]). In both flows, the dynamics involve the complex interaction of turbulence and dispersive waves. The wave frequencies $\sigma(\mathbf{k})$ are given by the dispersion relations, where $\sigma(\mathbf{k}) \propto k_z/|\mathbf{k}|$ in 3D homogeneous rotation, and $\sigma(\mathbf{k}) \propto k_x/|\mathbf{k}|^2$ on the β-plane. Here (x, y, z) are the zonal, meridional and vertical directions, respectively. Both systems have so-called slow modes with zero frequency, and those are z-independent modes (e.g. vortical columns) in 3D and x-independent modes (jets) in 2D. Laboratory and numerical experiments of β-plane and 3D rotating flows have shown that large-scale slow modes are spontaneously generated from isotropic, small-scale forcing (see, e.g., recent work by [2], [3] and references therein). Furthermore, at moderately small values of the Rossby and Rhines numbers, asymmetries develop between cyclones and anticyclones in 3D, and between eastward and westward jets on the β-plane. We explore the extent to which near-resonant triad interactions capture the formation of large-scale, coherent structures and asymmetries exhibited by the full dynamics [3], [4].

GOVERNING EQUATIONS

The equations for 3D, homogeneous, incompressible flow in a frame rotating about the $\hat{\mathbf{z}}$-axis at constant rate Ω are given by

$$\partial_t \mathbf{u} + Ro^{-1}\hat{\mathbf{z}} \times \mathbf{u} + (\nabla \times \mathbf{u}) \times \mathbf{u}$$

$$= -\nabla P + Re^{-1}\nabla^2 \mathbf{u} + \mathbf{F}_u \qquad (1)$$

with incompressibility $\nabla \cdot \mathbf{u}(x, y, z) = 0$, and where P is a pressure and $\mathbf{F}_u$ is an external force. The Rossby and Reynolds numbers are, respectively, $Ro = U/(2\Omega L)$ and $Re = UL/\nu$, where L is a characteristic length and U is a characteristic velocity.

The β-plane model is a local, planar approximation to 2D motion on the surface of a sphere, which accounts for mid-latitude variation of the normal component of the Coriolis parameter. The equation for the vertical vorticity $\zeta(x, y) = v_x(x, y) - u_y(x, y)$ is given by

$$\partial_t \zeta + J(\nabla^{-2}\zeta, \zeta) + (Rh)^{-1}\partial_x \nabla^{-2}\zeta$$

$$= Re^{-1}\nabla^2\zeta - \Lambda\zeta + F_\beta, \qquad (2)$$

where F_β is an external force, $J(g, h) = g_x h_y - g_y h_x$ is the Jacobian, $Rh = U/(\beta L^2)$ is the Rhines number, $\Lambda = rU/L$ is a non-dimensional drag coefficient, r is the (dimensional) drag coefficient, and the parameter β is the linear variation of the normal component of the Coriolis parameter with latitude [5].

In the inviscid, linear limit and in the absence of external forcing, both (1) and (2) admit wave solutions

$$\mathbf{u} \text{ or } \zeta \propto \exp\left[i\left(\mathbf{k} \cdot \mathbf{x} - \sigma(\mathbf{k})\frac{t}{R_*}\right)\right] \; + \; \text{c.c.}, \quad (3)$$

where c.c. denotes the complex conjugate, $R_* = Ro$ or Rh, and $\mathbf{x}$, $\mathbf{k}$ are 3D vectors for 3D rotation and 2D vectors for the β-plane system. The wave frequencies $\sigma(\mathbf{k})$ are given by the dispersion relations,

$$\sigma(\mathbf{k}) = \pm\frac{k_z}{k} \;\; (3\text{D}), \quad \sigma(\mathbf{k}) = -\frac{k_x}{k^2} \;\; (2\text{D}) \qquad (4)$$

where $k = |\mathbf{k}|$. Both dispersion relations allow resonant triad interactions with $\sigma(\mathbf{k}) + \sigma(\mathbf{p}) + \sigma(\mathbf{q}) = 0$, but these interactions cannot transfer energy directly from fast waves to slow modes [6], [7]. For small R_* and on long time scales $T = O(1/R_*)$, near-resonant interactions become important [8], with $\sigma(\mathbf{k}) + \sigma(\mathbf{p}) + \sigma(\mathbf{q}) = O(R_*)$.

NUMERICAL SIMULATIONS

Our goal is to isolate the dominant mechanism for the spontaneous self-organization of small-scale fluctuations into large-scale vortices in 3D, and large-scale jets on the β-plane. To this end we numerically investigate reduced models including subsets of triad interactions in Fourier space. We here focus on reduced models of near resonances with

$$|\sigma(\mathbf{k}) + \sigma(\mathbf{p}) + \sigma(\mathbf{q})| \leq R_* \qquad (5)$$

and non-resonances with

$$|\sigma(\mathbf{k}) + \sigma(\mathbf{p}) + \sigma(\mathbf{q})| > R_*. \qquad (6)$$

Since the physical-space form of the nonlinear term for a reduced model is not explicitly known, fully spectral rather than pseudo-spectral methods must be used for reduced-model calculations. Thus, our resolutions are restricted to 64^3 Fourier modes in 3D, and 384^2 (256^2) modes on the β-plane without (with) linear drag. The forcing is white in time with spatial, two-point correlation given by a Gaussian peaked at intermediate wavenumber k_f and energy input rate ε_f. We choose moderately small parameter values $Ro = (\varepsilon_f k_f^2)^{1/3}/(2\Omega) = 0.085$ and $Rh = (\varepsilon_f k_f^5)^{1/3}/\beta = 0.5$ that are relevant to synoptic-scale winds and atlantic ocean currents at mid-latitudes [5].

Results for 3D Rotating Flow

Figure 1 compares same-time energy spectra for a full simulation of (1) in a periodic box, and the reduced model of near resonances defined by (5). Both runs have $Ro = 0.085$, $k_f = 10$ and the same energy input rate ε_f. The run of near resonances reproduces the energy spectrum of the full simulation surprisingly well, considering that the near-resonant triads are only about 12% of the total number of triad interactions. The relatively small number of near-resonant triads is less efficient than the total number of triads for extracting energy from the 3D isotropic force, and hence figure 1 shows more energy in the forced wavenumbers for the run of near resonances than for the full simulation. In both calculations, energy injected into 3D modes at intermediate wavenumbers $k \approx 10$ is partially transferred to larger 2D slow modes with $k_z = 0$, corresponding to vortical columns in physical space. Hence $E(k) \approx E(k_h; k_z = 0)$ at large scales (figure 1 shows only $E(k_h; k_z = 0)$ for the near-resonances run to avoid clutter). Near resonances are *more* efficient for transferring energy to large-scale vortices than the full set of triads, as evidenced by an increased rate of transfer to large scales (not shown), and by the smaller overall peak wavenumber in the run of near resonances (at $k = 1$) compared to the full simulation (with overall peak wavenumber at $k = 2$). The latter implies that non-resonances given by (6) slow the transfer of energy from isotropic 3D small scales to 2D large scales with $k_z = 0$. Consistent with higher resolution 128^3 simulations [9] at the same $Ro = 0.085$ and with peak forcing wavenumber $k_f = 24$, the large-scale spectra scale approximately as k^{-3} indicating that the transfer of energy from small to large scales is not an inverse cascade among 2D modes only.

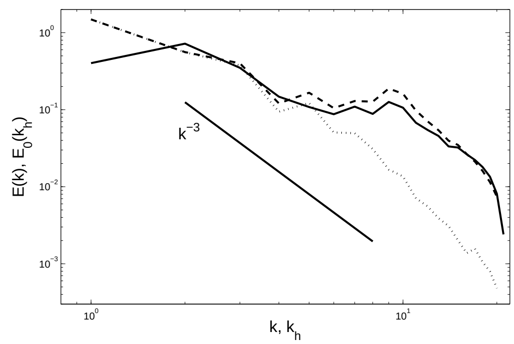

Fig. 1. Energy spectrum $E(k)$ (solid) for a full simulation of 3D rotation, compared to spectra for the reduced model of near resonances: $E(k)$ (dashed) and $E_0(k_h) = E_0(k_h; k_z = 0)$ (dotted). Both runs have $Ro = 0.085$ and instantaneous spectra are measured at the same time. A k^{-3} line is also shown for reference.

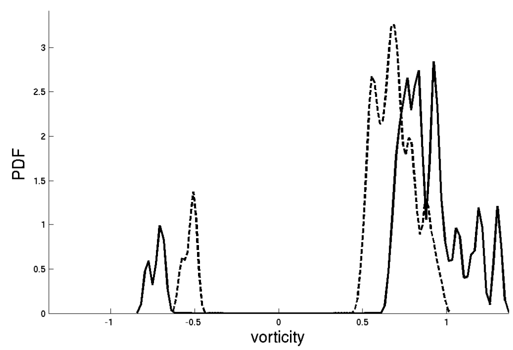

Fig. 2. PDF of vertical vorticity in the $\hat{\mathbf{z}}$-averaged velocity fields for 3D rotation: near resonances (solid) and the full simulation (dashed) at the same time.

Figure 2 shows the strong asymmetry between cyclones (with positive vorticity) and anticyclones (with negative vorticity), and the asymmetry in favor of cyclones is at least as strong in the run of near resonances as for the full simulation. Here PDFs are sampled over points $Q = u_x v_y - u_y v_x > 0.25 \max Q$. In contrast, the reduced model of non-resonances does not show significant energy transfer from intermediate forced scales to larger scales.

Results for β-Plane Flow

Figures 3 and 4 compare same-time snapshots of spectra for a full simulation of β-plane flow and the reduced model of near resonances. For resolution 384^2 and $Rh = 0.5$, near-resonant triads interactions are about 33% of the total number of triad interactions. Other parameter values are $\Lambda = 0$ (no linear drag), and $k_f = 80$. In both cases, $E(k) \approx E_0(k_y)$ at large scales, showing that the large-scale flow is predominantly zonal. Here $E(k)$ is the 2D energy spectrum; $E_0(k_y)$ ($E_0(k_x)$) is the zonal (meridional) spectrum, with energy in a small sector $\pi/12$ about $k_x = 0$ ($k_y = 0$). As can be seen in figure 5, the time developing β-plane flow without linear drag has a clear asymmetry in favor of stronger westward jets. That asymmetry is enhanced in the simulation of near resonances only. The reduced model of non-resonant triads (6) does not produce strongly zonal flows.

When linear drag is included with $\Lambda = 2.2\times10^{-3}$ and the flow approaches a state of constant energy, then the asymmetry of the full β-plane flow reverses to favor stronger and thinner eastward jets. The zonally averaged velocity $u_{avg}(y)$ is linearly stable with $d\zeta_{avg}(y)/dy > -\beta$ everywhere [10], where $\zeta_{avg}(y)$ is the zonally averaged vorticity. In the reduced model of near resonances, the zonally averaged velocity and vorticity profiles develop a secondary structure which is not present in the full β-plane flow. A weaker form of the jet asymmetry emerges, and the zonally averaged velocity $u_{avg}(y)$ is marginally stable with $d\zeta_{avg}(y)/dy = -\beta$ at several values of y.

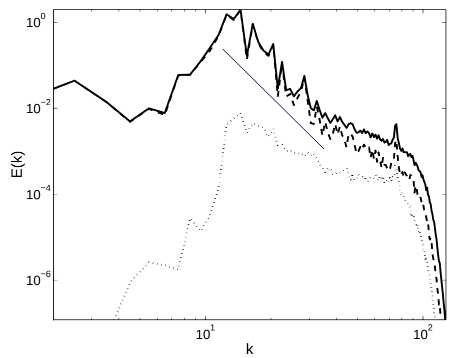

Fig. 3. Energy spectra $E(k)$ (solid), $E_0(k_y)$ (dashed) and $E_0(k_x)$ (dotted) for a full β-plane simulation with $Rh = 0.5$ and $\Lambda = 0$. The line is k^{-5}.

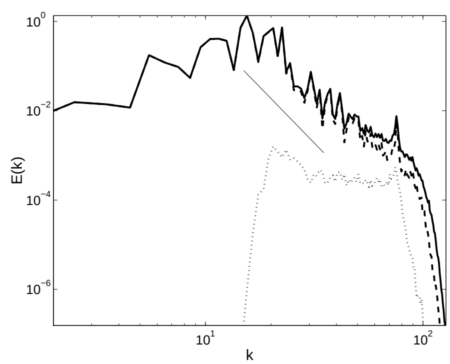

Fig. 4. Energy spectra $E(k)$ (solid), $E_0(k_y)$ (dashed) and $E_0(k_x)$ (dotted) for near resonances on the β-plane with $Rh = 0.5$ and $\Lambda = 0$. The line is k^{-5}.

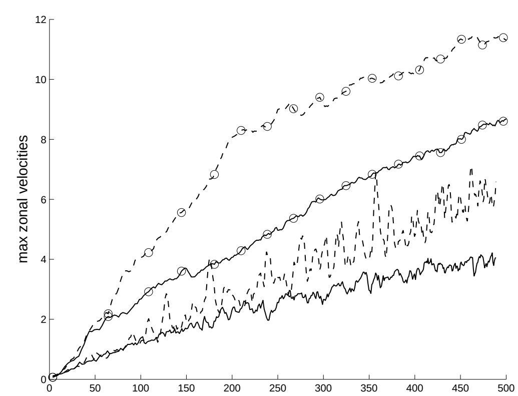

Fig. 5. Maximum eastward (solid) and westward (dashed) velocities for $Rh = 0.5$ and $\Lambda = 0$: full simulation (no symbols); near resonances (circles).

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