SYMMETRY-PRESERVING REGULARIZATION MODELS OF TURBULENT CHANNEL FLOW

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ABSTRACT

We consider regularizations of the convective term in the Navier-Stokes equations that preserve the conservation and symmetry properties. These regularizations restrain the production of small scales of motion by vortex stretching in an unconditionally stable manner, meaning that the velocity cannot blow up in the energy-norm (in 2D also: enstrophy-norm). The regularization model is successfully tested for numerical simulations of fully-developed turbulent channel flow (Re$_\tau$=180 and Re$_\tau$=395).

INTRODUCTION

Most turbulent flows cannot be computed directly from the (incompressible) Navier-Stokes equations,

$$\partial_t u + C(u,u) + D(u) + \nabla p = 0, \qquad (1)$$

because they possess far too many scales of motion. The computationally almost numberless small scales result from the convective term $C(u,v) = (u \cdot \nabla)v$, which allows for the transfer of energy from scales as large as the flow domain to the smallest scales that can survive viscous dissipation. In the quest for a dynamically less complex mathematical formulation, we consider smooth approximations (regularizations) of the nonlinearity:

$$\partial_t u_\epsilon + \widetilde{C}(u_\epsilon, u_\epsilon) + D(u_\epsilon) + \nabla p_\epsilon = 0. \qquad (2)$$

The regularized system (2) should be more amenable to solve numerically, while the leading modes of u_ϵ have to approximate the corresponding modes of the Navier-Stokes solution u.

The first outstanding approach in this direction goes back to Leray [1], who took $\widetilde{C}(u,u) = C(\bar{u}, u)$ and proved that a moderate filtering of the transport velocity is sufficient to regularize a turbulent flow. Here, the filtering operation is denoted by a bar; the residual will be indicated by a prime. The Navier-Stokes-α-model forms another example of regularization modeling. In this model, the convective term becomes $\widetilde{C}_r(u,u) = C_r(u, \bar{u})$, where C_r denotes the convective operator in rotational form: $C_r(u,v) = (\nabla \times u) \times v$.

In large-eddy simulation, the Navier-Stokes equations are filtered spatially, and the resulting non-closed term is modelled:

$$\partial_t \overline{u}_\epsilon + C(\overline{u}_\epsilon, \overline{u}_\epsilon) + D(\overline{u}_\epsilon) + \nabla \overline{p}_\epsilon = f(\overline{u}_\epsilon), \qquad (3)$$

where $f(\overline{u}_\epsilon)$ represents the model. The regularization (2) falls in with this concept if $\widetilde{C}$ is taken such that

$$\overline{\widetilde{C}(u_\epsilon, u_\epsilon)} = C(\overline{u}_\epsilon, \overline{u}_\epsilon) - f(\overline{u}_\epsilon). \qquad (4)$$

Indeed under this condition, Eq. (2) is equivalent to (3): we can filter (2) first and thereafter compare the filtered version of (2) term-by-term with (3) to identify the closure model $f(\overline{u}_\epsilon)$. Eq. (4) relates the regularization $\widetilde{C}(u_\epsilon, u_\epsilon)$ one-to-one to the closure model for any invertible filter (the Gaussian filter, for instance). Hence, Eq. (2) is formally equivalent to a LES for any invertible filter.

The regularization method basically alters the nonlinearity to restrain the production of small scales of motion, see e.g. [2]-[3]. In doing so, one can preserve certain fundamental properties of the convective operator in the Navier-Stokes equations exactly. We propose to preserve the symmetry properties that are intimately tied up with the conservation of energy, enstrophy (in 2D) and helicity.

SYMMETRY AND CONSERVATION

The evolution of the energy follows from differentiating (u, u) with respect to time and rewriting $\partial_t u$ with the help of (1). In this way, we get a convective contribution given by $(C(u, u), u)$. This term cancels, because the trilinear form $(C(u, v), w)$ is skew-symmetric with respect to v and w:

$$(C(u, v), w) = -(v, C(u, w)) \tag{5}$$

The evolution of the enstrophy is obtained by taking the inner product of the Navier-Stokes equations with the vector field $-\Delta u$. The resulting convective contribution vanishes in two spatial dimension, since in 2D:

$$(C(u, v), \Delta v) = (u, C(\Delta v, v)), \tag{6}$$

for any u and v. Note that the right-hand side vanishes for $u = v$ because of (5). The evolution of the helicity follows from the inner product of Eq. (1) with the vorticity ω and the inner product of the curl of Eq. (1) with the velocity u. Taking these inner products results into the convective contribution $(C(u, u), \omega) + (C(u, \omega), u) - (C(\omega, u), u)$, which vanishes as an immediate consequence of the skew symmetry (5). Therefore, the helicity is conserved in the absence of viscous dissipation ($D = 0$).

Regularizations of particular interest are the ones that conserve the energy, the enstrophy (in 2D) and the helicity (in 3D) in the absence of viscous dissipation. Therefore, we aim to regularize C in such manner that the underlying symmetries (given by Eq. (5) and Eq. (6)) are preserved. This criterion yields the following class of regularizations

$$\begin{aligned}
\widetilde{C}_2(u, v) &= \overline{C(\overline{u}, \overline{v})} \\
\widetilde{C}_4(u, v) &= C(\overline{u}, \overline{v}) + \overline{C(\overline{u}, v')} + \overline{C(u', \overline{v})} \\
\widetilde{C}_6(u, v) &= C(\overline{u}, \overline{v}) + C(\overline{u}, v') + C(u', \overline{v}) \\
&+ \overline{C(u', v')}
\end{aligned}$$

The difference between $\widetilde{C}_n(u, u)$ and $C(u, u)$ is of the order ϵ^n (where n=2,4,6) for symmetric filters with filter length ϵ. The approximations $\widetilde{C}_n(u, u)$ are stable by construction, meaning that the velocity cannot blow up in the energy-norm (in 2D: enstrophy-norm).

NONLINEAR TRANSPORT MECHANISM

To see how the above regularizations restrain the production of small scales of motion, we take the curl of Eq. (2), with $\widetilde{C} = \widetilde{C}_n$,

$$\partial_t \omega_\epsilon + \widetilde{C}_n(u_\epsilon, \omega_\epsilon) + D(\omega_\epsilon) = \widetilde{C}_n(\omega_\epsilon, u_\epsilon).$$

This equation resembles the vorticity equation that follows from the Navier-Stokes equations: the only difference is that C is replaced by its regularization $\widetilde{C}_n$. The Navier-Stokes equations yield the vortex-stretching term

$$C(\omega, u) = \overline{S}\overline{\omega} + \overline{S}\omega' + S'\overline{\omega} + S'\omega,$$

where $S = \frac{1}{2}(\nabla u + \nabla u^{\mathrm{T}})$ is the deformation tensor. The regularized vortex stretching terms become

$$\begin{aligned} C_2(\omega, u) &= \overline{\overline{S}\overline{\omega}} \\ C_4(\omega, u) &= \overline{S}\overline{\omega} + \overline{\overline{S}\omega'} + \overline{S'\overline{\omega}} \\ C_6(\omega, u) &= \overline{S}\overline{\omega} + \overline{S}\omega' + S'\overline{\omega} + \overline{S'\omega'}, \end{aligned}$$

respectively. Qualitatively, vortex stretching leads to the production of smaller and smaller scales, *i.e.*, to a continuous, local increase of both S' and ω'. Consequently, at the positions where vortex stretching occurs, the terms with S' and ω' will eventually amount considerably to $C(\omega, u)$. Since the regularizations $C_n(\omega, u)$ diminish these terms, they counteract the production of smaller and smaller scales by means of vortex stretching and may eventually stop the continuation of the vortex stretching process. In this way, the symmetry-preserving regularization method restrains the convective production of smaller and smaller scales of motion by means of vortex stretching.

A detailed study of the triadic interactions shows that $\widetilde{C}_n(u, u)$ approximates the local interactions between large scales of motion ($\epsilon|k| < 1$) up to n-th order. Hence, the triadic interactions between large scales of motion are only slightly altered. All interactions involving longer wavevectors (smaller scales of motion) are reduced. The amount by which the interactions between the wavevector-triple (k, p, q) are lessened depends on the length of the legs of the triangle $k = p+q$. In case $n = 4$, for example, all triadic interactions for which at least two legs are (much) longer than $1/\epsilon$ are (strongly) attenuated, whereas interactions for which at least two legs are (much) shorter than $1/\epsilon$ are reduced to a small degree only.

RESULTS FOR TURBULENT CHANNEL FLOW

As a first step in the application of symmetry-preserving regularization, the approximation $\widetilde{C}_4$ is tested for a turbulent channel flow by means of a comparison with direct numerical simulations at Re_τ = 180 and Re_τ = 395. The numerical discretization of the convective term preserves the symmetry and conservation properties given by Eqs. (5)-(6), see Ref. [4] for details.

The filter is based upon the Helmholtz operator, where the boundary conditions that supplement the Navier-Stokes equations are applied to the filter too. Since solving the Helmholtz equation for $\overline{u}$ is rather expensive, we have tried to reduce the filtering costs by truncating the iterative solution method for solving the Helmholtz problem before the point of convergence is approached. In doing so, we found that two Jacobi iterations (with $\overline{u}$ = u as initial guess) suffice already. Therefore, the results shown in this section are obtained by means of two Jacobi iterations.

The least to be expected is a good prediction of the mean flow. Figure 1 shows that the symmetry-preserving regularization model $\widetilde{C}_4$ satisfies that minimal requirement already at coarse grids: 16×16×8 grid points for Re_τ = 180 and 32×32×16 for Re_τ = 395, resp. (when the filter length ϵ is about two to four times the grid width h).

One-dimensional (streamwise) energy spectra at $y^+ \approx 5$ are shown in Fig. 2. The energy spectra follow the DNS for large scales of motion, whereas a much steeper (numerically speaking: more gentle) power law is found for small scales, which is precisely what a regularization model is ought to do. Fig. 3 illustrates the convergence of the skin friction coefficient as function of ratio of the filter length ϵ to the grid width h for Re_τ=180 and Re_τ=395. Here, the reference values are depicted by the dashed line.

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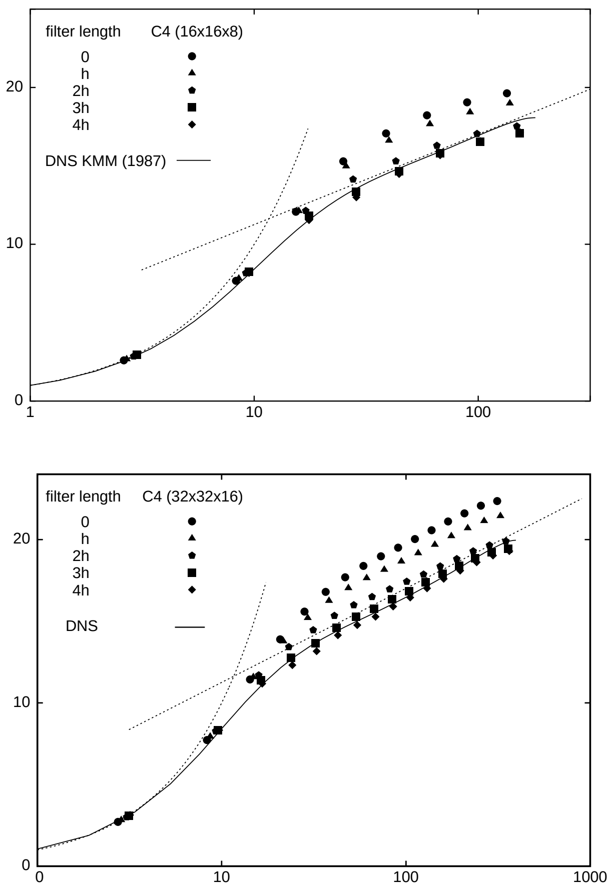

Fig. 1. Comparison of the mean velocity at Re_τ=180 (top) and Re_τ=395 (bottom).

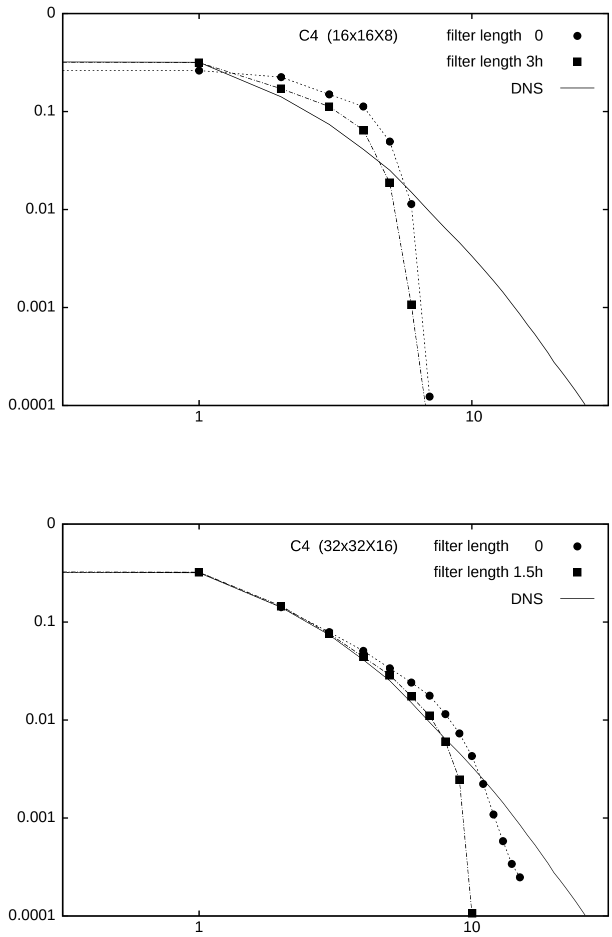

Fig. 2. One-dimensional (streamwise) energy spectra at $y^+ \approx 5$ (Re_τ=180).

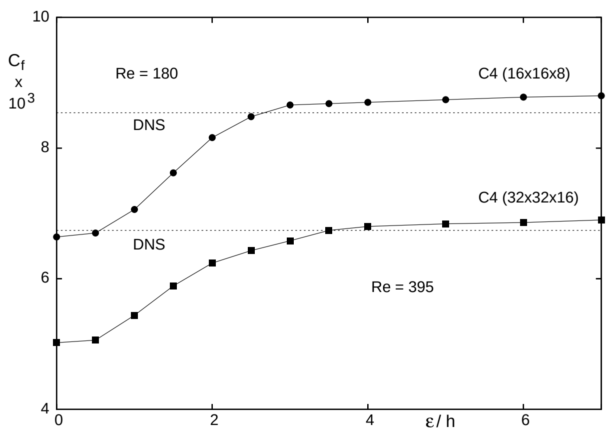

Fig. 3. Convergence of the skin friction as function of the relative filter-length for Re_τ=180 and Re_τ=395.