Symmetry-preserving regularization of turbulent channel flow

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Navier-Stokes equations

$$\partial_t \boldsymbol{u} + \mathcal{C}(\boldsymbol{u}, \boldsymbol{u}) + \mathcal{D}(\boldsymbol{u}) + \nabla p = 0$$

$$\mathcal{C}(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} \qquad \mathcal{D}(\boldsymbol{u}) = -\Delta \boldsymbol{u}/\mathrm{Re}$$

Large-eddy simulation

$$\partial_t \overline{\boldsymbol{u}} + \mathcal{C}(\overline{\boldsymbol{u}}, \overline{\boldsymbol{u}}) + \mathcal{D}(\overline{\boldsymbol{u}}) + \nabla \overline{p} = \mathbf{model}(\overline{\boldsymbol{u}})$$

$$\mathbf{model}(\overline{\boldsymbol{u}}) \approx \mathcal{C}(\overline{\boldsymbol{u}}, \overline{\boldsymbol{u}}) - \overline{\mathcal{C}(\boldsymbol{u}, \boldsymbol{u})}$$

Regularization

$$\partial_t u + \tilde{\mathcal{C}}(u,u) + \mathcal{D}(u) + \nabla p = 0$$

Regularization model

$$\partial_t \overline{u} + \mathcal{C}(\overline{u},\overline{u}) + \mathcal{D}(\overline{u}) + \nabla \overline{p} =$$

$$\mathcal{C}(\overline{u},\overline{u}) - \overline{\tilde{\mathcal{C}}(u,u)} = \mathbf{model}(\overline{u})$$

Examples

Leray $\quad \tilde{\mathcal{C}}(u,v) = \mathcal{C}(\overline{u},v)$

NS-α $\quad \tilde{\mathcal{C}}_r(u,v) = \mathcal{C}_r(u,\overline{v}) = (\nabla \times u) \times \overline{v}$

Approximate deconvolution

$$\partial_t \overline{\boldsymbol{u}} + \overline{\mathcal{C}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}})} + \mathcal{D}(\overline{\boldsymbol{u}}) + \nabla \overline{\boldsymbol{p}} = 0$$

Approximate inverse : $\tilde{\boldsymbol{u}} = \tilde{\mathcal{F}}^{-1}\overline{\boldsymbol{u}} = \tilde{\mathcal{F}}^{-1}\mathcal{F}\boldsymbol{u} \approx \boldsymbol{u}$

Dynamics approximately deconvolved velocity:

$$\partial_t \tilde{\boldsymbol{u}} + \tilde{\mathcal{F}}^{-1}\mathcal{F}\mathcal{C}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}}) + \mathcal{D}(\tilde{\boldsymbol{u}}) + \nabla \tilde{\boldsymbol{p}} = 0$$

Regularization defines implicitly approximate deconvolution:

$$\tilde{\mathcal{C}}(\boldsymbol{u}, \boldsymbol{v}) = \tilde{\mathcal{F}}^{-1}\mathcal{F}\mathcal{C}(\boldsymbol{u}, \boldsymbol{v})$$

How to alter the non-linearity?

Preserve

- *symmetries*
- *conservation properties*
- *transformation properties*
- *Kelvin's circulation theorem*
- *Bernoulli's theorem*
- *Karman-Howarth theorem*
- *etc.*

Symmetry and conservation properties

Energy invariance

$$< \mathcal{C}(u, v), w > = - < v, \mathcal{C}(u, w) >$$

$$\Rightarrow \quad < \mathcal{C}(u, u), u > = 0$$

Enstrophy invariance (2D)

$$< \mathcal{C}(u, v), \Delta v > = < u, \mathcal{C}(\Delta v, v) >$$

$$\Rightarrow \quad < \mathcal{C}(u, u), \Delta u > = 0$$

Symmetry-preserving regularization

$$\tilde{\mathcal{C}}_n(u,v) = \mathcal{C}(u,v) + \mathcal{O}(\epsilon^n)$$

Energy invariance

$$< \tilde{\mathcal{C}}_n(u,v), w > = - < v, \tilde{\mathcal{C}}_n(u,w) >$$

Enstrophy invariance (2D)

$$< \tilde{\mathcal{C}}_n(u,v), \Delta v > = < u, \tilde{\mathcal{C}}_n(\Delta v, v) >$$

Symmetry-preserving regularizations

$$\tilde{\mathcal{C}}_2(u,v) = \overline{\mathcal{C}(\overline{u},\overline{v})}$$

$$\tilde{\mathcal{C}}_4(u,v) = \mathcal{C}(\overline{u},\overline{v}) + \overline{\mathcal{C}(\overline{u},v')} + \overline{\mathcal{C}(u',\overline{v})}$$

$$\tilde{\mathcal{C}}_6(u,v) = \mathcal{C}(\overline{u},\overline{v}) + \mathcal{C}(\overline{u},v') + \mathcal{C}(u',\overline{v}) + \overline{\mathcal{C}(u',v')}$$

Energy, enstrophy (2D) and helicity are conserved

$\Rightarrow$ *Unconditional stable in energy-norm; 2D: enstrophy-norm*

Vortex stretching mechanism

$$\partial_t \boldsymbol{\omega} + \tilde{\mathcal{C}}_n(\boldsymbol{u}, \boldsymbol{\omega}) + \mathcal{D}(\boldsymbol{\omega}) = \tilde{\mathcal{C}}_n(\boldsymbol{\omega}, \boldsymbol{u})$$

$$\tilde{\mathcal{C}}_2(\boldsymbol{\omega}, \boldsymbol{u}) = \overline{\overline{S}\,\overline{\boldsymbol{\omega}}}$$

$$\tilde{\mathcal{C}}_4(\boldsymbol{\omega}, \boldsymbol{u}) = \overline{S}\,\overline{\boldsymbol{\omega}} + \overline{\overline{S}\boldsymbol{\omega}' + S'\overline{\boldsymbol{\omega}}}$$

$$\tilde{\mathcal{C}}_6(\boldsymbol{\omega}, \boldsymbol{u}) = \overline{S}\,\overline{\boldsymbol{\omega}} + \overline{S}\boldsymbol{\omega}' + S'\overline{\boldsymbol{\omega}} + \overline{S'\boldsymbol{\omega}'}$$

Navier-Stokes

$$\mathcal{C}(\boldsymbol{\omega}, \boldsymbol{u}) = \overline{S}\,\overline{\boldsymbol{\omega}} + \overline{S}\boldsymbol{\omega}' + S'\overline{\boldsymbol{\omega}} + S'\boldsymbol{\omega}'$$

Triadic interactions

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{|\boldsymbol{k}|^2}{\mathrm{Re}}\right) \hat{\boldsymbol{u}}_{\boldsymbol{k}} + \mathcal{C}_{4,\boldsymbol{k}}(\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}) = \mathbf{0}$$

Helmholtz filter || $\alpha^2 = \epsilon^2/24$

$$\mathrm{i}\Pi(\boldsymbol{k}) \sum_{\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{k}} \hat{\boldsymbol{u}}_{\boldsymbol{p}} \boldsymbol{q} \hat{\boldsymbol{v}}_{\boldsymbol{q}} \, \frac{1+\alpha^2(|\boldsymbol{k}|^2+|\boldsymbol{p}|^2+|\boldsymbol{q}|^2)}{(1+\alpha^2|\boldsymbol{k}|^2)(1+\alpha^2|\boldsymbol{p}|^2)(1+\alpha^2|\boldsymbol{q}|^2)}$$

Large eddies || *local interactions*

$$\mathrm{i}\Pi(\boldsymbol{k}) \sum_{\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{k}} \hat{\boldsymbol{u}}_{\boldsymbol{p}} \boldsymbol{q} \hat{\boldsymbol{v}}_{\boldsymbol{q}} \left(1 + \mathcal{O}(\alpha^4)\right)$$

Numerical simulation method

Finite-volume discretization (4th-order)

$$\Omega \frac{\mathrm{d}\boldsymbol{u}_h}{\mathrm{d}t} + \boldsymbol{C}_n(\boldsymbol{u}_h)\,\boldsymbol{u}_h + \boldsymbol{D}\boldsymbol{u}_h - \boldsymbol{G}\boldsymbol{p}_h = \mathbf{0}$$

Convection is skew-symmetric

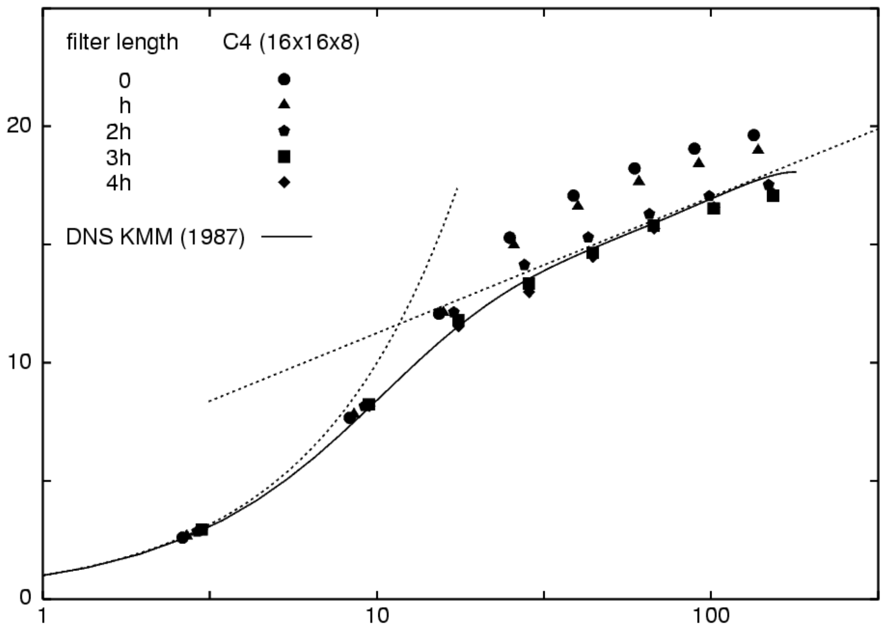
$$\boldsymbol{C}_n(\boldsymbol{u}_h) + \boldsymbol{C}_n^{\mathrm{T}}(\boldsymbol{u}_h) = \mathbf{0}$$

Unconditional stable:

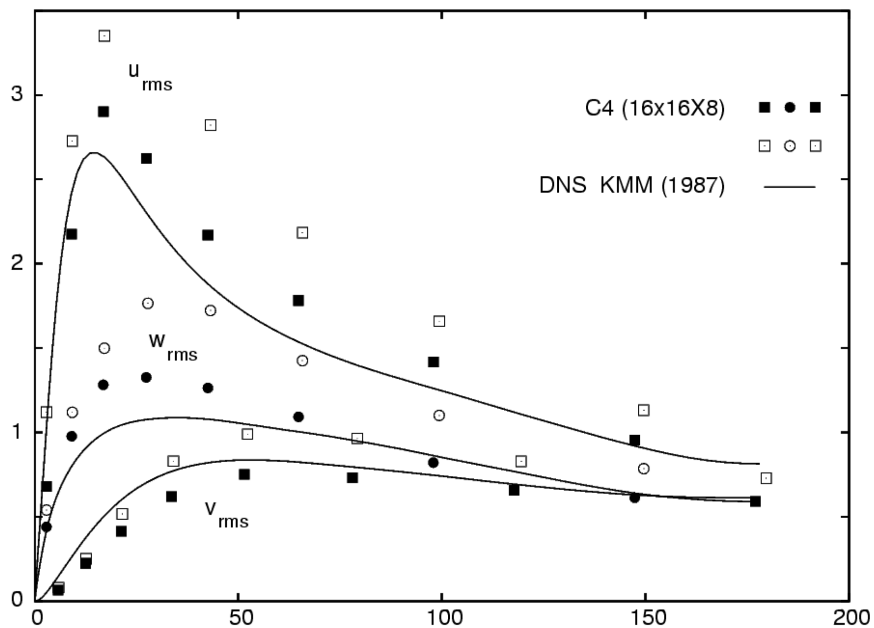
$$\frac{\mathrm{d}}{\mathrm{d}t}||\boldsymbol{u}_h||^2 = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}_h \cdot \Omega \boldsymbol{u}_h) = -\boldsymbol{u}_h \cdot (\boldsymbol{D} + \boldsymbol{D}^{\mathrm{T}})\boldsymbol{u}_h \leq \mathbf{0}$$

Turbulent channel flow

$Re_\tau = 180$ $\tilde{c}_4$ 16x16x8 grid points



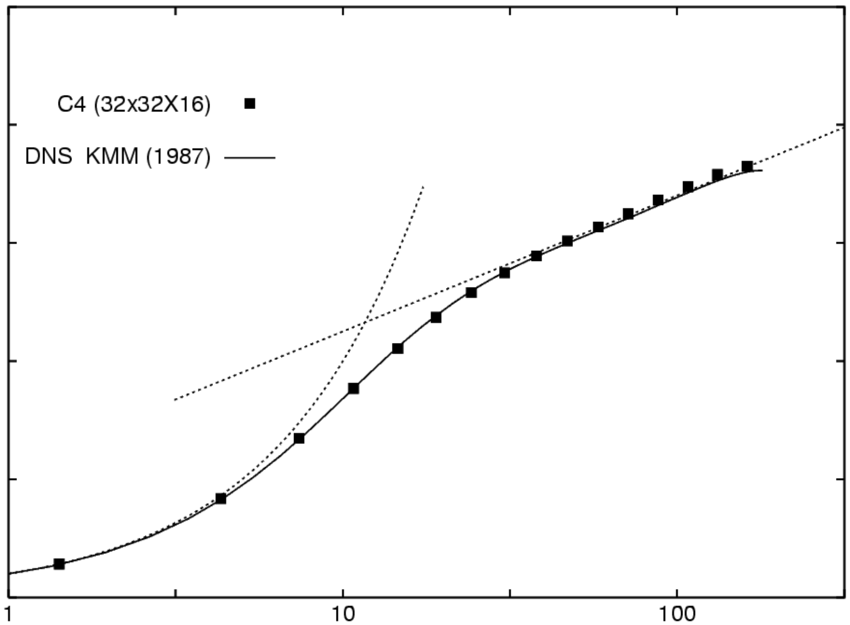
mean flow



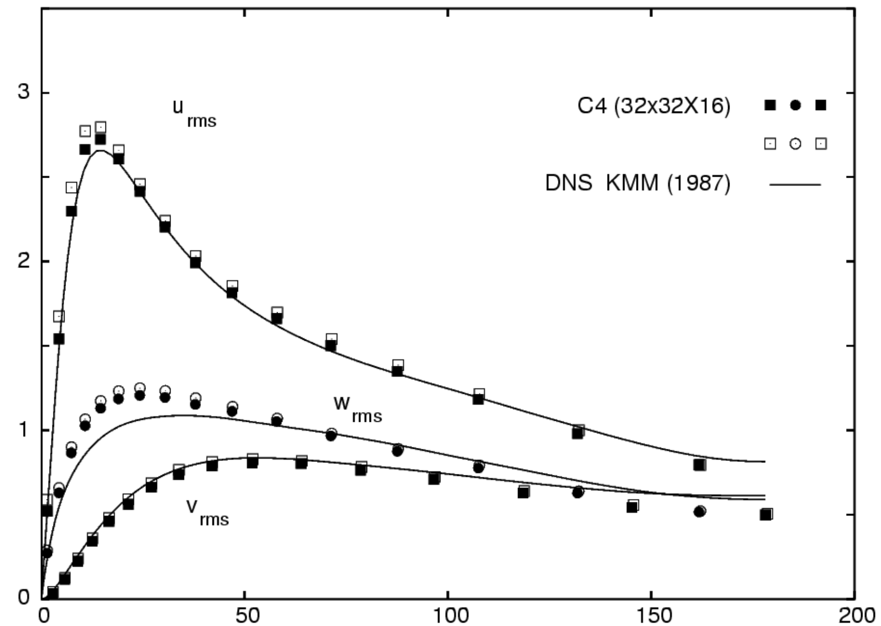
turbulence intensities

Turbulent channel flow

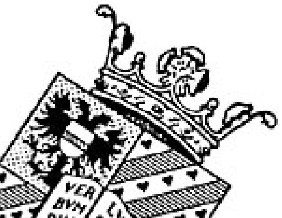
$Re_{\tau} = 180$ $\tilde{c}_4$ 32x32x16 grid points



mean flow



turbulence intensities



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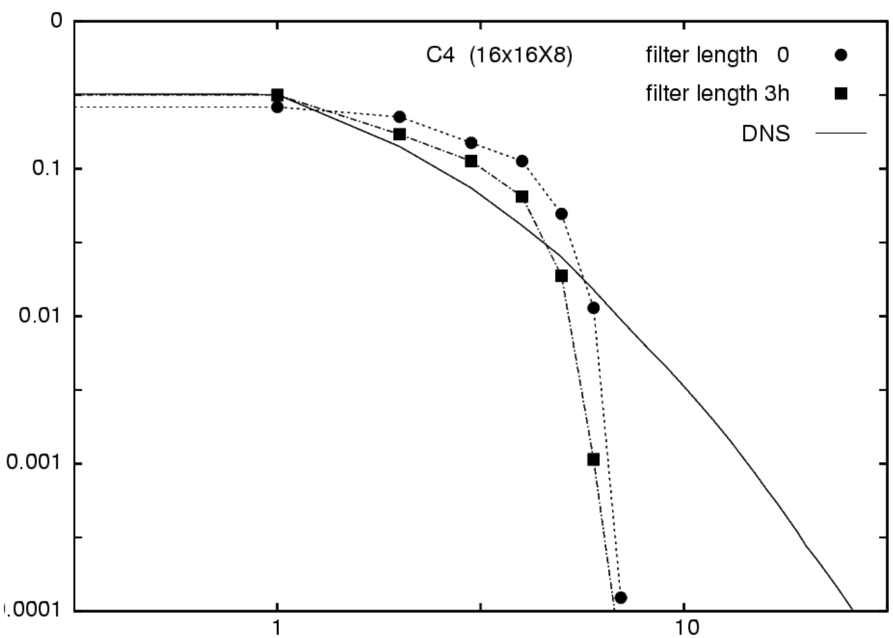
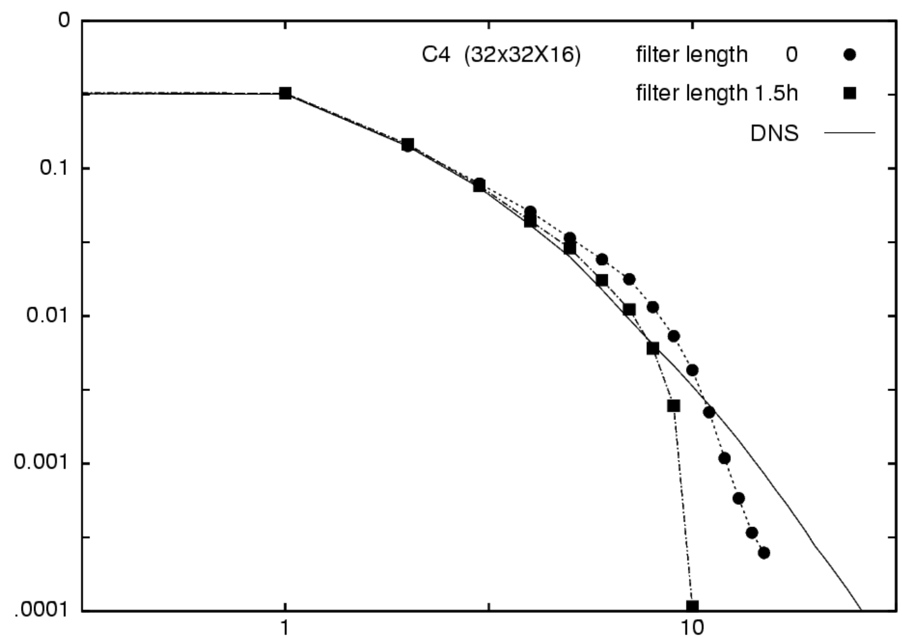
Turbulent channel flow

16x16x8 grid points



energy spectrum

32x32x16 grid points



energy spectrum

Concluding remarks

- *symmetry-preserving smoothing of non-linearity*

  yields unconditional stable simulation shortcut

- *successfully tested for turbulent channel flow*

- *more thorough investigations and comparisons*

  need be carried out