Time implicit high-order discontinuous Galerkin method with reduced evaluation costs

Florent Renac[1], Frederic Coquel[2] & Claude Marmignon[1]

[1]ONERA/DSNA/NUMF
[2]CMAP, Ecole Polytechnique

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ONERA Châtillon

Motivation and goal

- DG method:
  - compact stencil: well-suited for unstructured meshes, algorithm parallelization, BC application.
  - time explicit discretization: strongest CFL restriction associated to parabolic term
  - time implicit discretization: high computational cost induced by the large number of DOFs in practical app. (e.g. 3D Navier-Stokes eq.)
- Aim:
  - efficient implicit procedure for the DG method:
    - uncoupling of low order DOFs in each element
    - reconstruction of higher order DOFs
  - application to steady solution of nonlinear parabolic equations
  - numerical experiments with BR2 scheme

Bassi *et al.*, 2nd ETC, Antwerpen, 1997.

Outline

1. Equations and Numerical Approach
   - Nonlinear diffusion equation
   - Entropy solution
   - Space discretization
   - Implicit procedure
2. Numerical experiments
   - 1D nonlinear diffusion equation
   - 2D nonlinear diffusion equation
3. Summary and outlook

Outline

Nonlinear diffusion equation

model problem

Consider the elliptic problem

$$-\nabla\cdot(\mathbf{B}(\mathbf{x},u)\nabla u) = s(\mathbf{x}) \quad \text{in } \Omega\subset\mathbb{R}^d \tag{1}$$

with $\mathbf{B}(\mathbf{x},u)$ a nonlinear function of $u\in\mathbb{R}$ and BCs on $\partial\Omega=\Gamma_D\cup\Gamma_N$:

$$\begin{aligned} u &= u_D \quad \text{on } \Gamma_D \\ \nabla u\cdot\mathbf{n} &= g_N \quad \text{on } \Gamma_N \end{aligned}$$

We solve (1) with a fast time marching method:

$$u_t - \nabla\cdot(\mathbf{B}(\mathbf{x},u)\nabla u) = s(\mathbf{x}) \quad \text{in } \Omega\times(0,\infty) \tag{2}$$

and IC

$$u(\mathbf{x},0) = u_0(\mathbf{x}) \quad \text{in } \Omega$$

entropy solution

- $\mathcal{U}(u)$ is an entropy function for (2) if
  - $\mathcal{U}_{uu} > 0$
  - $\mathcal{U}$ satisfies

$$\mathbf{B}(\mathbf{x}, u) = \mathcal{U}_{uu}\mathbf{C}(\mathbf{x})$$

  where $\mathbf{C} \in \mathbb{R}^{d\times d}$ is a positive definite matrix

- Introduce the change of variable $u(v)$ s.t.

$$\mathcal{U}_{uu}u_v = 1$$

one obtains the following linear (in space) parabolic problem for v:

$$u(v)_t - \nabla \cdot (\mathbf{C}(\mathbf{x})\nabla v) = s(\mathbf{x}) \quad \text{in } \Omega \times (0, \infty)$$

Space discretization
discrete weak form

- Partition of Ω:

$$\Omega_h = \bigcup_{j=1}^{N} \Omega_j$$

- Function space of discontinuous polynomials:

$$\mathcal{V}_h^p = \{\varphi \in L^2(\Omega_h) : \ \varphi|_{\Omega_j} \in \mathcal{P}^p(\Omega_j), \ j = 1, N\}$$

with

$$\mathcal{P}^p(\Omega_j) = \{\varphi \in L^2(\Omega_j) : \ \varphi(x,y) = \sum_{l=0}^{N_p} \alpha^l \phi^l(x,y)\}$$

- We look for a numerical approximation of the solution $v_h \in \mathcal{V}_h^p$:

$$v_h(\mathbf{x}, t) = \sum_{l=1}^{N_p} \phi^l(\mathbf{x}) V_j^l(t) \quad \forall \mathbf{x} \in \Omega_j, \ t \in (0, \infty)$$

Remark: number of DOFs per discretization element:

$$N_p = \begin{cases} p+1 & \text{if} \quad d=1 \\ \frac{(p+1)(p+2)}{2} & \text{if} \quad d=2 \end{cases}$$

Space discretization

BR2 scheme

Numerical approximation of the weak formulation: $\forall k = 1, \dots, N_p$

$$\int_{\Omega_j} \phi^k u(v_h)_t d\mathbf{x} + \int_{\Omega_j} (\mathbf{C}\boldsymbol{\theta}_h) \cdot \nabla \phi^k d\mathbf{x} - \oint_{\partial\Omega_j} \phi^k h_v dS = \int_{\Omega_j} \phi^k s(\mathbf{x}) d\mathbf{x}$$

lifting operators

$$\boldsymbol{\theta}_h \triangleq \nabla v_h + \mathbf{R_h}$$
$$h_v = \mathbf{C}\{\nabla v_h + \mathbf{r_h^\sigma}\} \cdot \mathbf{n}$$

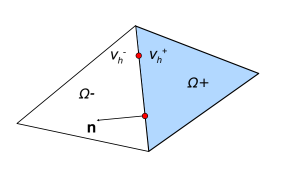
with

$$\mathbf{R_h} \triangleq \sum_{\sigma \in \partial\Omega_j} \mathbf{r_h^\sigma} \quad \forall \mathbf{x} \in \Omega_j$$

and

$$\int_{\Omega^+ \cup \Omega^-} \phi \mathbf{r_h^\sigma} dV = - \int_\sigma \{\phi\}(v_h^+ - v_h^-) \mathbf{n} dS$$

mean operator



$$\{q\} = \frac{q^+ + q^-}{2}$$

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Time discretization

backward Euler scheme

Semi-discrete equation:

$$\mathbf{M}\frac{d\mathbf{V}}{dt} + \mathbf{R}(\mathbf{V}) = 0$$

Mass matrix:

$$M_{kl} = \int_{\Omega_j} \phi^k \phi^l u_v(v_h) d\mathbf{x}$$

Backward Euler scheme:

$$\mathbf{A}(\mathbf{V}^{(n+1)} - \mathbf{V}^{(n)}) + \mathbf{R}(\mathbf{V}^{(n)}) = 0$$

with $\mathbf{V}^{(n)} = \mathbf{V}(n\Delta t)$ and $\mathbf{A}$ the implicit matrix:

$$\mathbf{A} = \frac{1}{\Delta t}\mathbf{M} + \frac{\partial \mathbf{R}}{\partial \mathbf{V}}$$

Remark: $\mathbf{A}$ is an unsymmetric real square matrix of size $N_{DOF} = N \times N_p$.
Hence matrix inversion is a $\mathcal{O}(N_{DOF}^3)$ process

Implicit procedure

evaluation cost reduction (1/2)

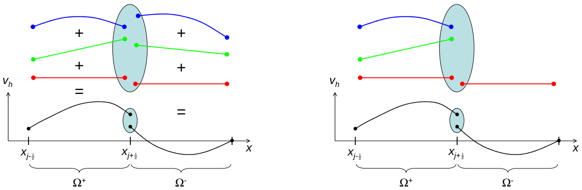
Basics:

- implicit matrix coefs represent coupling between DOFs
- coupling appears through numerical flux $\hat{f}_{v\,h}(v_h^+, v_h^-)$

Lowering coupling between modes by:

1. low order problem resolution: modes s.t. $0 \leq q \leq p_{simp}$
2. reconstruction of higher modes s.t. $p_{simp} < q \leq p$

1D example for $p = 2$ and $p_{simp} = 0$:



(a) full coupling

(b) reduced coupling

Implicit procedure
evaluation cost reduction (2/2)

Resolution algorithm:

1. low order problem resolution:

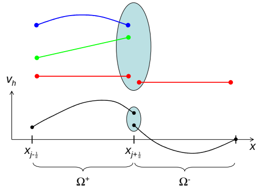
$$\tilde{\mathbf{A}}(\tilde{\mathbf{V}}^{(n+1)} - \tilde{\mathbf{V}}^{(n)}) = -\tilde{\mathbf{R}}(\mathbf{V}^{(n)})$$

   with $\tilde{\mathbf{V}} = (\mathbf{V}^0, \dots, \mathbf{V}^{p_{simp}})$

2. local reconstruction of higher modes:

$$\mathbf{V}^{q^{(n+1)}} = \mathbf{V}^q(\tilde{\mathbf{V}}^{(n+1)}, \mathbf{V}^{q^{(n)}}) \quad , \quad p_{simp} + 1 \leq q \leq p$$

1D example for $p = 2$ and $p_{simp} = 0$:



Implicit procedure

operation count

Remark: $\tilde{\mathbf{A}}$ is a square matrix of size $N_{simp} = N \times N_{p_{simp}}$:

- matrix inversion is a $\mathcal{O}(N_{simp}^3)$ process
- higher DOFs reconstruction is a $\mathcal{O}(N_{DOF}^2)$ process

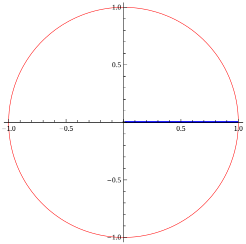
FLOPs reduction for matrix inversion:

$$\frac{N_{DOF}^3}{N_{simp}^3 + N_{DOF}^2} \sim \left(\frac{N_p}{N_{p_{simp}}}\right)^3 \qquad \text{when } N \text{ large}$$

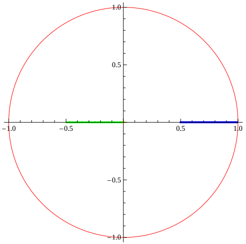
Methods comparison

Von Neumann analysis for the 1D heat equation with periodic BCs

Amplification matrix eigenspectra for $p = 1$ and $D = \frac{\nu \Delta t}{\Delta x^2} = 10^3$:



(d) BASE



(e) SIMP

BASE: full implicit matrix

SIMP: simplified implicit matrix with $p_{simp} = 0$

Outline

1D nonlinear diffusion equation

model problem

- Equation:

$$\begin{aligned}
\frac{\partial u}{\partial t} - \nu \frac{\partial^2 B(u)}{\partial x^2} &= 1 + \sin 2\pi x \quad \text{in } (0,1) \times (0,\infty) \\
u(x,0) &= u_0(x) \quad \text{on } [0,1] \\
u(0,t) &= 0 \\
u(1,t) &= \log 3
\end{aligned}$$

with $\nu > 0$ and $B(u) = e^u - 1$

$$\lim_{t\to\infty} u(x,t) = \log\left(1 + (2 + \frac{1}{2\nu})x - \frac{x^2}{2\nu} + \frac{\sin 2\pi x}{4\pi^2\nu}\right)$$

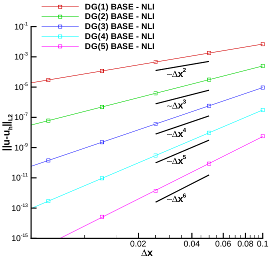
- Partition of the segment:

$$\Omega_h = \bigcup_{j=1}^{N} \Big[(j-1)\Delta x,\, j\Delta x\Big], \quad \Delta x = \frac{1}{N}$$

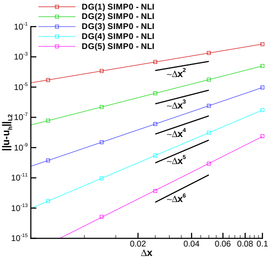
1D nonlinear diffusion equation

numerical results

Error analysis for $1 \leq p \leq 5$ and $D = 10^3$



(f) BASE



(g) SIMP0

BASE: full implicit matrix

SIMP0: simplified implicit matrix with $p_{simp} = 0$

NLI: nonlinear

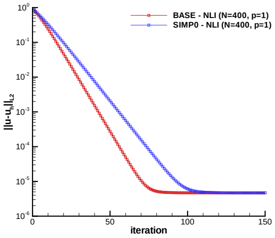
Arnold *et al.*, SIAM J. Numer. Ana., **39**(5), 2002.

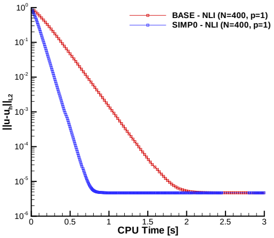
1D nonlinear diffusion equation

numerical results

Convergence analysis ($N = 400$, $p = 1$, $D = 10^3$)

Speedup $\simeq 2.25$



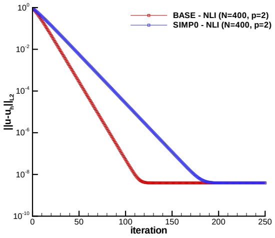


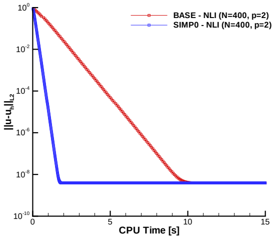
BASE: full implicit matrix

SIMP0: simplified implicit matrix with $p_{simp} = 0$

1D nonlinear diffusion equation
numerical results

Convergence analysis ($N = 400,\ p = 2,\ D = 10^3$)
Speedup $\simeq 5$





BASE: full implicit matrix

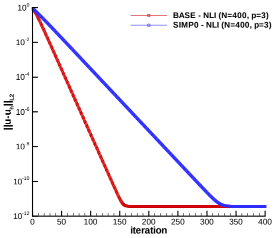
SIMP0: simplified implicit matrix with $p_{simp} = 0$

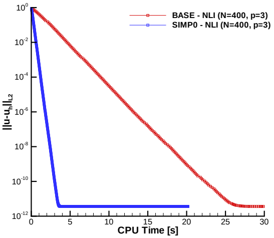
1D nonlinear diffusion equation

numerical results

Convergence analysis ($N = 400,\ p = 3,\ D = 10^3$)

Speedup $\simeq 7.5$



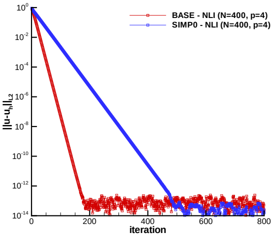


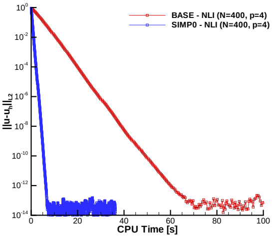
BASE: full implicit matrix

SIMP0: simplified implicit matrix with $p_{simp} = 0$

1D nonlinear diffusion equation
numerical results

Convergence analysis ($N = 400$, $p = 4$, $D = 10^3$)
Speedup $\simeq 10$





BASE: full implicit matrix

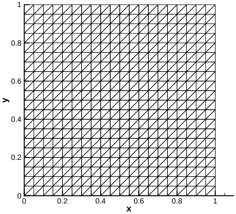
SIMP0: simplified implicit matrix with $p_{simp} = 0$

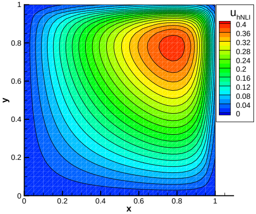
2D nonlinear diffusion equation

model problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \nabla \cdot (\mathbf{B}(u)\nabla u) &= s(x,y) \quad \text{in } \Omega \times (0,\infty) \\ u(x,y,0) &= 0 \quad \text{in } \Omega \\ u(x,y,t) &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

with $\mathbf{B}(u) = \nu e^{u}\mathbf{I}$ with $\nu > 0$

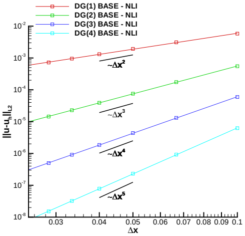




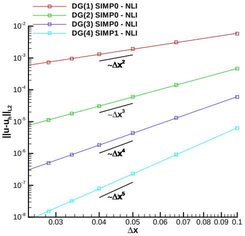
2D nonlinear diffusion equation

numerical results

Error analysis for $1 \leq p \leq 4$ and $D = 10^3$



(p) BASE



(q) SIMP

BASE: full implicit matrix

SIMP0: simplified implicit matrix with $p_{simp} = 0$

SIMP1: simplified implicit matrix with $p_{simp} = 1$ $(p = 4)$

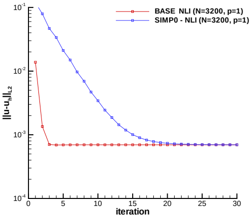
Arnold *et al.*, SIAM J. Numer. Ana., **39**(5), 2002.

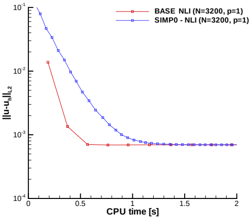
2D nonlinear diffusion equation

numerical results

Convergence analysis ($N = 3200$, $p = 1$, $D = 10^3$)

Speedup $\simeq 0.5$





BASE: full implicit matrix

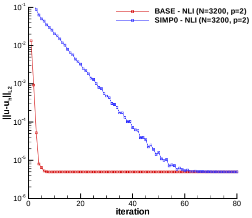
SIMP0: simplified implicit matrix with $p_{simp} = 0$

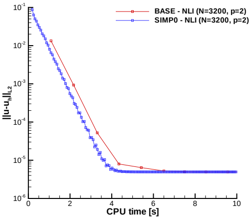
2D nonlinear diffusion equation

numerical results

Convergence analysis ($N = 3200,\ p = 2,\ D = 10^3$)

Speedup $\simeq 1.5$





BASE: full implicit matrix

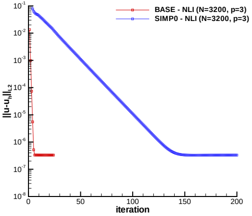
SIMP0: simplified implicit matrix with $p_{simp} = 0$

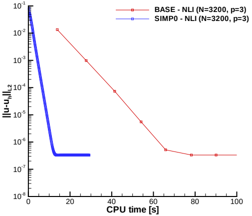
2D nonlinear diffusion equation

numerical results

Convergence analysis ($N = 3200,\ p = 3,\ D = 10^3$)

Speedup $\simeq 4$





BASE: full implicit matrix

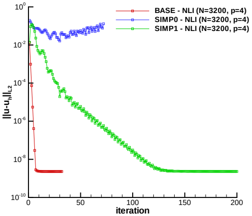
SIMP0: simplified implicit matrix with $p_{simp} = 0$

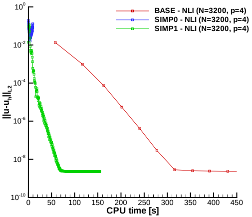
2D nonlinear diffusion equation

numerical results

Convergence analysis ($N = 3200,\ p = 4,\ D = 10^3$)

Speedup $\simeq 4.5$





BASE: full implicit matrix

SIMP0: simplified implicit matrix with $p_{simp} = 0$

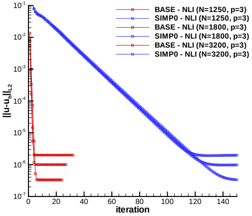
SIMP1: simplified implicit matrix with $p_{simp} = 1$

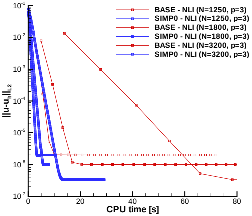
2D nonlinear diffusion equation

numerical results

Mesh size effect ($p = 3,\ D = 10^3$)

N	1250	1800	3200
Speedup	2.5	3	4





BASE: full implicit matrix

SIMP0: simplified implicit matrix with $p_{simp} = 0$

Outline

Summary and outlook

- Derived a fast time implicit discretization procedure for high-order DG method:
  - mode uncoupling between adjacent elements
  - implicit treatment for low order modes
  - local reconstruction of high order modes
- Application to nonlinear 1D & 2D scalar equations:
  - convergence acceleration
  - speedup increases with p and N
- Outlook:
  - independent of the DG method (sucessfully applied to BR1 scheme)
  - storage reduction with matrix-free reconstruction
  - application to hyperbolic problems
  - extension to systems of conservation laws

Bassi & Rebay, J. Comp. Phys., **131**(2), 1997.
T. Dairay, MSc Thesis (supervisor: F. Renac), ONERA, 2010.

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Thank you for your attention