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Estimating satellite versus debris collision probabilities via the adaptive splitting technique

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Résumé *On February 10th 2009, satellites Iridium and Cosmos collided though their configuration had been reported safe. This triggered the quest for a better assessment methodology and therefore a better collision probability estimation. As a matter of facts, rare event probability and extreme quantile estimations arise more and more frequently in the industrial and engineering worlds, be it for safety requirement or performance objectives. As usual Monte Carlo technique can not cope, a new dedicated tool is needed. Using the spacecraft collision real context as a support, the Adaptive Splitting Technique is introduced and compared with Crude Monte Carlo it is shown to outperform.*

A. INTRODUCTION

The constantly growing population of active spacecrafts now has to face the threat of colliding with the debris bequeathed by their predecessors [NAS10, Gla09] : due to their high speeds, even small left behind bolts can seriously damage a satellite.

Collision avoidance procedures were hence designed for improved satellite safety such as described in [Smi02]. Active spacecraft managing teams, when a potential collision occurs, now have to decide whether to start a collision avoidance maneuver. This extra move sure clears away the danger but also costs extra energy, thereby shortening the satellite's activity timespan. A trade-off between safety and lifetime expectation therefore has to be found.

To support the maneuver decision, a dedicated tool is the probability of collision between the debris and the satellite. A great deal of effort has been spent on how to estimate its very low value. In his work at the NASA [Cha08], Chan concludes through a numerical integration after a sequence of approximations and hypothesis. To avoid those and improve the estimation reliability, we choose the Monte Carlo method. As Crude Monte Carlo (CMC) fails to deliver when faced with a rare event, we aim at estimating this very low collision probability via a rare event probability estimation tool : the Adaptive Splitting Technique (AST) proposed by Cerou and Guyader *et alii* in [CDFG09].

In this paper, we first introduce the spacecraft collision issue via the Iridium-Cosmos case : these two satellites collided on February 10th 2009, tough their configuration did not appear troublesome [Kel09]. We then try to estimate the collision probability through Crude Monte Carlo (CMC) in section C., in vain. Next, in section D. we present the Adaptive Splitting Technique (AST), the special rare event probability estimation technique we advocate to our purpose, and show it can provide valuable accurate information in this framework.

B. SPACECRAFT ENCOUNTER AND NOISED STATE MEASUREMENT

On February 10, 2009, a commercial Iridium communications satellite and a defunct Russian satellite Cosmos collided, though their configuration was not reported dangerous [Kel09]. We will estimate in this article what was the probability it happened.

In order to understand the predicament an active spacecraft managing team can find itself in, we will describe the geometrical issue at hand and then the main source of randomness.

1. A basic geometry problem

Consider two satellites orbiting around the Earth in a Galilean frame of reference with our planet as origin and equipped with the Euclidean distance. This three-body problem will be considered a double two-body problem : each satellite interacts through gravity only with the Earth and not with the other satellite. Besides, the Earth and the satellites are

assumed to be homogeneous spheres with radii d_E, d_1 and d_2. The collision distance is therefore $d_c = d_1 + d_2$. We wonder about the relative position of the two satellites : might they collide during a given time span $I = [t_s, t_e]$?

2. A convenient model of dynamics

To keep things simple, orbital mechanics being not our topic, we will use Kepler mechanics. One can use more advanced models such as SGP4 [Miu09] if wanted. The discussed probability estimation methodologies are independent of the method.

At time t, the satellites will be represented by their states $\vec{s}_1(t)$ and $\vec{s}_2(t)$ i.e. their positions $\vec{r}_1(t)$ and $\vec{r}_2(t)$ and their speeds $\vec{v}_1(t)$ and $\vec{v}_2(t)$ such that $\vec{s}_i = (\vec{r}_i, \vec{v}_i)$.

In our setting, the speeds evolve according to the same well-known Ordinary Differential Equations defining the two body problem

$$\forall t, \frac{d\vec{v}_i}{dt} = -a\frac{\vec{r}_i}{r_i^3} \tag{1}$$

where a is a positive constant given by physics.

This ordinary differential equation (*ode*) is analytically solved in many textbooks and its solution depends continuously and in a bijective fashion on the given so called initial conditions : its value $\vec{s}_i^m$ at t_i^m, the measurement time, through Φ the ode's resolvant i.e. its solution map :

$$i \in \{1,2\}, \forall t \in I, \vec{s}_i = \Phi(\vec{s}_i^m, t_i^m, t) \tag{2}$$

At this point, there is a natural way to clear out the collision issue using

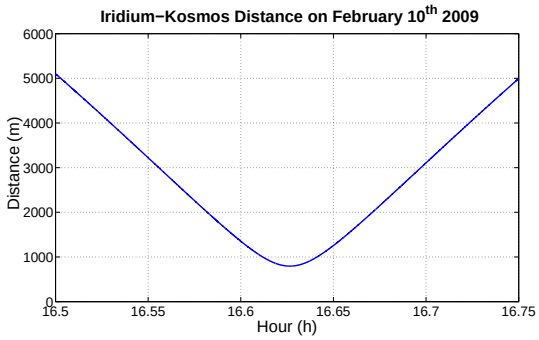
$$\delta = \min_{t \in I}\{\|\vec{r}_2 - \vec{r}_1\|(t)\} \tag{3}$$

$t \in I \mapsto \|\vec{r}_2 - \vec{r}_1\|(t)$ experimental convexity, figure 1, makes δ available through numerical optimisation and its associated test :

$$\xi(\vec{s}_1^m, t_1^m, \vec{s}_2^m, t_2^m, I) = \begin{cases} 1 & \text{if} \quad \delta \leq d_c \\ 0 & \quad\quad \text{otherwise} \end{cases} \tag{4}$$

eventually closes the deal.

Fig. 1 – *Iridium-Comos distance on collision day according to TLE.*



Things would be all that easy and deterministic, had randomness not barged in.

3. Random measurements lead to uncertainty

In real life, the states are not monitored around-the-clock : they are merely measured from times to times, by a radar. The Two Line Elements (TLE) provided by NORAD sum up this information and feed the models with the $(\vec{s}_i^m, t_i^m)$ pairs. However, TLEs are inaccurate and their inaccuracy is unknown. This uncertainty is our very issue. To cope with this and to better reflect the reality, we added independent, identically distributed (*iid*) noises $\vec{\epsilon}_1$ and $\vec{\epsilon}_2$, with density $f_{\vec{\epsilon}}$, to the models' inputs s_i^m.

$$\begin{gathered} \vec{E} = \begin{pmatrix} \vec{\epsilon}_1 \\ \vec{\epsilon}_2 \end{pmatrix} \\ f_{\vec{E}}(\vec{e}) = f_{\vec{\epsilon}}(\vec{\varepsilon}_1) \times f_{\vec{\epsilon}}(\vec{\varepsilon}_2) \end{gathered} \tag{5}$$

The collision issue can not be answered in a cut-and-dried way anymore as it has to be rephrased in a probabilistic fashion itself : what is the probability of collision between the two satellites ?

Via the random counterpart of our deterministic geometrical problem

$$\begin{gathered} i \in \{1,2\}, \forall t \in I, \vec{S}_i = \Phi(\vec{s}_i^m + \vec{\epsilon}_i, t_i^m, t) = (\vec{R}_i(t), \vec{V}_i(t)) \\ \Delta = \min_{t \in I}\{\|\vec{R}_2 - \vec{R}_1\|(t)\} \\ \Xi = \xi(\vec{s}_1^m + \vec{\epsilon}_1, t_1^m, \vec{s}_2^m + \vec{\epsilon}_2, t_2^m, I) \end{gathered} \tag{6}$$

this question is equivalently stated as

$$\mathbb{P}[\{\text{ The satellites collide during } I\}] = \mathbb{E}[\Xi] \tag{7}$$

We now face a plain expectation estimation problem.

C. THE CRUDE MONTE CARLO

As an explicit analytical way to calculate $\mathbb{E}[\Xi]$ is unlikely to be found, one will most likely make do with an estimation. Crude Monte Carlo (CMC) is a very convenient and reliable way to reach it, if the sought probability is not too low, indeed. CMC can not handle rare event probability estimation efficiently.

1. A basic set of tools and notations

CMC's estimators for Ξ's expectation $\mathbb{E}[\Xi]$ and variance $\mathbb{V}[\Xi]$ are defined respectively as μ, the empirical mean and σ^2, the empirical variance of n *iid* tests, $\Xi^i, i \in \{1, \cdots, n\}$:

$$\mu(\Xi, n) \equiv \frac{1}{n}\sum_{i=1}^{n} \Xi^i \tag{8}$$

$$\sigma^2(\Xi, n) \equiv \frac{1}{n}\sum_{i=1}^{n} (\Xi^i - \mu(\Xi, n))^2 \tag{9}$$

As the empirical mean of *iid* random variables, $\mu(\Xi, n)$ is a random variable as well and we hope

its variance is as small as possible with respect to its estimated mean. To measure this, m *iid* $\mu(\Xi, n)$ throws are made in order to calculate $\mu(\Xi, n)$'s empirical relative deviation (*Erd*) estimator ρ

$$\rho(\mu, \Xi, n, m) \equiv \frac{\sigma(\mu(\Xi, n), m)}{\frac{1}{m}\sum_{i=1}^{m} \mu(\Xi, n)^i} \quad (10)$$

as a way to measure its accuracy as the ratio of its standard deviation over its empirical mean.

2. Why CMC can not cope with rare events

Using the independence of the Ξ^i and the fact that $\Xi^2 = \Xi$ for it is a binary 1 or 0 mapping, one can write the following easily.

$$\mathbb{E}[\mu(\Xi, n)] = \mathbb{E}[\Xi] \quad (11)$$

$$\mathbb{V}[\mu(\Xi, n)] = \frac{\mathbb{E}[\Xi](1 - \mathbb{E}[\Xi])}{n} \quad (12)$$

If the sought probability $\mathbb{E}[\Xi]$ is 10^{-4}, to be accurate up to a tenth of the real value i.e. to have $\sqrt{\mathbb{V}[\mu(\Xi, n)]}/\mathbb{E}[\Xi] \leq 10^{-1}$, almost 10^6 points are needed!

Most of the time though, in a real context, there is no way one can generate that many samples. Yet, this degree of accuracy is becoming a standard, for amounts of money at stake are huge (the Iridium satellite program is worth at least 200 M$) and safety standards more and more demanding.

D. THE ADAPTIVE SPLITTING TECHNIQUE

So as to estimate this unlikely collision probability, a rare event dedicated technique is needed.

The Splitting Technique (ST) is a rare event dedicated technique, a good introduction to which can be found in [LLLT09] and [LEG08]. Just as Importance Sampling, an other rare events technique, it aims at approximating the optimal probability measure change with respect to the sought probability. However, while the former requires a prior knowledge about the measure change to be designed before starting the estimation, as can seen in [Den01], [BS99] or [Zha96], the latter builds it as it goes, gathering valued information on the way.

Our transfer function δ being not much short from a black box, in order not to perform quite intractable analysis or integration, we chose a special Monte Carlo technique that would adapt to the information gathered on the fly : the Adaptive Splitting Technique (AST) as explained in [CG07a, JMl10]for the static case and in [CG07b] for a Markov process case.

1. An intuitive approach to AST

The basic idea here is *divide and conquer* : instead of estimating the very low probability directly, the work is divided in estimating a sequence of easier probabilities and eventually calculating the sought value as a plain product. This is the very purpose of this Bayesian formulation of our problem.

$$\mathbb{P}[\Delta \leq d_c] = \prod_{i=1}^{K} \mathbb{P}[\Delta \leq l_i | \Delta \leq l_{i-1}] \quad (13)$$

where $l_0 = \infty \geq l_1 \geq \cdots \geq l_K = d_c$ form a decreasing sequence of thresholds to be defined later. Hopefully, we have just reformulated our hard to estimate expectation as the product of easy to estimate conditional expectations!

The Adaptive Splitting Technique (AST) is a way of making this wish come true in an iterative three step way. Start with a sample of *iid* throws of Δ known to be under threshold l_i. With $i = 1$, this is only performing a plain CMC simulation. Then, and until the threshold is less than d_c, do as follows :

1. Define l_{i+1} as a well chosen empirical p_{i+1}-quantile of the current sample.
2. Resample uniformly among the realisations under the new threshold.
3. Use the selected points to sample new points conditionally to being under l_{i+1}.

When the threshold is lower than d_c, conclude that

$$\mathbb{P}[\Delta \leq d_c] \approx \prod_{i=1}^{K-1} p_i \times p^{\bullet}_{K-1} \quad (14)$$

where $p^{\bullet}_{K-1}$ is the estimated probability of collision given that the minimum relative distance is less than l_{K-1}.

Though biased, slightly as it uses classical linear statistics to estimate quantiles [CVC88], AST seems a good choice to estimate rare event probabilities.

Its three points now have to be formally detailed so as to gain real understanding of the technique.

2. AST's three main points detailed

The three main points of the AST are now going to be detailed. For a complete description, one should read [CDFG09].

Choice of the empirical quantile's level According to [Lag06], all quantile levels should be equal, say to p, in order to minimise the estimator's variance. Nowadays however, there is no theoretical answer to what is the optimal value of p. A very low p will lead to a little variance but requires many thresholds and therefore many points [GHML10]. If p is high, less points will be needed, but the increased interdependence of the points will inflate the variance. $0.75 \leq p \leq 0.8$ seems to be a good choice.

Resampling under the threshold Once the threshold is set, points above it are discarded. To replenish our sample set, according to being under the threshold, there is no perfect solution, unless we can generate directly a *iid* sample set according to the conditional law, which is unlikely. A most natural way of coping is duplicating selected samples, say choosing the points to duplicate according to *iid* uniform throws. Thus, the set grows back to its original size. However, points are independent not identically distributed anymore (*iid*) but correlated identically distributed (*cid*). This is an issue as it increases the estimators variance. The last step addresses this point.

Variety to reduce variance Right after duplication, points are identically distributed according to the conditional law but not independent as some are copies of one another and others may share a common ancestor. We hence have a triple objective :

1. Increase variety in the set : there must be no pair of identical points.
2. Respect the probability law : it must be kept through the changes to be done.
3. Decrease interdependence : the correlation between points and there ancestors must be reduced to a minimum.

To this purpose, we will use a $f_{\vec{E}}$-reversible Markov kernel $M(\cdot,\cdot)$.

$M(\cdot,\cdot) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is mapping such that

$$\forall x \in \mathbb{R}^d,\ M(x,\cdot) : \mathbb{R}^d \to \mathbb{R} \text{ is a density function.} \tag{15}$$

$\forall x \in \mathbb{R}^d$, $M(x,\cdot)$ stands as a x-specific random way to propose another $\mathbb{R}^d$ point. This will provide adapted variety associating to any available point x an other one y chosen according to $M(x,\cdot)$.

Let us now impose a constraint on M so as to respect the probability law. M is said to be a $f_{\vec{E}}$-reversible Markov kernel if

$$\forall (x,y) \in \mathbb{R}^d \times \mathbb{R}^d,\ f_{\vec{E}}(x)M(x,y) = f_{\vec{E}}(y)M(y,x) \tag{16}$$

This equation is known to physicists as the detailed balance equation. It means that if from a $f_{\vec{E}}$ set, you use M to generate another, then

- the new set is distributed according to $f_{\vec{E}}$ as well : this is the invariance property.
- statistically, no one can say which set generated the other : this is the reversibility property.

Invariance would do the trick on its own if we only had to deal with $f_{\vec{E}}$ and not its associated conditionals :

$$f_{\vec{E}|i} = \frac{\mathbf{1}_{\Delta \le l_i} f_{\vec{E}}}{\int \mathbf{1}_{\Delta \le l_i} f_{\vec{E}}} \tag{17}$$

To avoid looking for many $f_{\vec{E}|i}$-invariant Markov kernels, we use M's reversibility property in the following way, assuming $X_i \sim f_{\vec{E}|i}$. First, let Y_i be $M(X_i,\cdot)$'s proposal i.e. throw. Then set

$$\psi_i(X_i) = \begin{cases} Y_i & \text{if } \Delta(Y_i) \le l_i \\ X_i & \text{otherwise} \end{cases} \tag{18}$$

Thanks to reversibility, ψ_i is distributed according to $f_{\vec{E}|i}$ as well. We now can grow and inflate a $f_{\vec{E}|i}$-sample set thanks to the $f_{\vec{E}|i}$ available points via a $f_{\vec{E}}$-reversible Markov kernel. The last remaining issue is correlation.

At this point we have a *cid* sample set and we want to decrease the correlation between points. The iterative nature of the algorithm is such that points have a common past, a history, a genealogy. That, at the end of the day, translates into increased estimator variance. We have no theoretically proven way to avoid that yet. The intuition is that using functional composition [1] i.e. $\psi_i^{\circ(\omega)}$, leads to lower and lower variance as ω increases. It was shown in [TIE94] that under mild conditions, $\omega > 1$ cannot increase variance and might even help. One can hence iterate ψ_i at will or based on a stopping time e.g. until 90% of the points moved from their original position.

3. The AST algorithm

Let us now state the AST algorithm.

Algorithm 1 (Adaptive Splitting Technique)**.** *So as to estimate $\mathbb{P}[\Delta \le d_c]$, proceed as follows.*

1. *Set $\kappa = 1$.*
2. *Generate η iid throws of Δ.*
3. *Calculate the empirical p-quantile l_κ.*
4. *While $l_\kappa \ge d_c$, do :*
   - *(a) Select throws under l_κ and discard others.*
   - *(b) Replace discarded points resampling uniformly with replacement among selected points.*
   - *(c) Apply $\psi_\kappa^{\circ(\omega)}$ to all the points.*
   - *(d) Increment κ by one : $\kappa = \kappa + 1$.*
5. *Estimate $p^\bullet_{\kappa-1}$.*
6. *Conclude that $\mathbb{P}[\Delta \le d_c] \approx p^{\kappa-1} \times p^\bullet_{\kappa-1}$*

η, p, ψ and ω are fixed before hand. The number of quantiles i.e. κ's ultimate value, is random. The AST estimator will be denoted

$$\nu(d_c, \eta, p, M, \omega) \tag{19}$$

and the total number of generated points is

$$N = \eta \times (1 + ((\kappa - 1) \times \omega)) \tag{20}$$

1. Notation convention : $f^{\circ(n+1)} = f^{\circ(n)} \circ f$ and $f^{\circ(0)} = Id$.

E. EXPERIMENTAL RESULTS

Let us now proceed to the application, describing the noise model, the actual AST parameters, especially the Markov kernel tuning, and comparing the results delivered by CMC and AST. Eventually, some insight about the AST results sensibility with respect to its parameters will be given.

Refer to table 1 for the definition of $f_{\vec{\epsilon}}$, $M(x, dy)$ and the tuning parameter α.

1. Markov kernel tuning

Given a point $X \sim \mathcal{N}(0_6, D^2)$, the Markov kernel proposal is

$$Y \sim \frac{\alpha X + W}{\sqrt{1+\alpha^2}} \text{ where } W \text{ and } X \text{ are } iid \tag{21}$$

To choose $\alpha > 0$, there is no theoretical result yet. According to [CDFG09], one should make big steps at the beginning and make smaller and smaller steps, as the thresholds l_i decrease. The chosen heuristic is setting initially $\alpha = 1$ and adapt it in the course of the algorithm :

$$\alpha = \begin{cases} \alpha \times 1.1 & \text{if over } 50\% \text{ of accepted transitions} \\ \alpha / 1.1 & \text{if under } 50\% \text{ of accepted transitions} \end{cases} \tag{22}$$

We experimentally found out it leads to estimates with lesser variance than fixing α.

2. Comparison between CMC and AST

Using parameters in table 1, the results [2] in table 2 presented in figures 2 and 3 were obtained.

Fig. 2 – *CMC results in table 1 framework. The estimation cost was 300000 simulations exactly. Mean estimate is on the left and Empirical relative deviation on the right.*

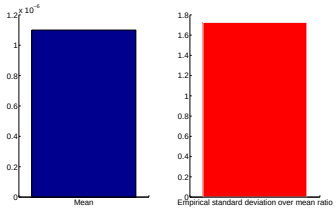
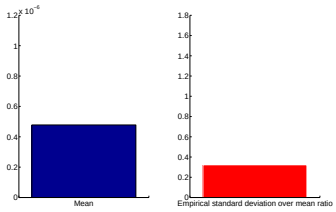


Fig. 3 – *AST results in table 1 framework. The estimation cost was* $309060(1 \pm 2\%)$ *simulations. Mean estimate is on the left and Empirical relative deviation on the right.*



They show that AST estimated more accurately than CMC as its estimators has a way lesser relative variance as it was *divided by 5*, and for a very similar cost as it consumed on average the same amount of points.

3. AST sensitivity

To test AST's sensitivity to its parameters, we did a few changes with respect to the original protocol. Only these changes are detailed. All the results are summed up in table 3.

Experiments $(1, 2)$ **and** $(4, 5)$ When ω is doubled, twice as much points are used, as expected, and relative spread decreases a bit, as hoped. The same change in the estimated probability occurred in both pair. This suggests that ω has important impact on the estimate.

Experiments $(1, 4)$**,** $(2, 5)$ **and** $(3, 6)$ When η is doubled, twice as much points are used, as expected and relative spread ratios are around $\sqrt{2}$. This is quite similar to the CMC case and suggests a $1/\sqrt{\eta}$ convergence rate.

Experiments $(1, 3)$**,** $(4, 6)$ When p is reduced by a third, which means in our case that the number of points discarded when setting the next quantile is doubled and twice as less quantiles is needed, the point consumption is cut by half, the estimated probability is divided by five and relative spread increases. Intuitively, there is trade off between points and accuracy to be found.

F. CONCLUSION

We introduced the Adaptive Splitting Technique (AST), a rare event probability estimation technique, and compared it with Crude Monte Carlo in a satellite collision context. The new technique provides estimates with lesser relative variance and consumes less points than the usual Monte-Carlo method : relative variance was reduced by half.

Though AST still needs some theoretical developments to help tuning its parameters and giving some theoretical results, it is already applicable when dealing with Gaussian based randomness. AST can already help with a fairly wide class of industrial or engineering issues when estimating rare event probabilities. Besides it can estimate extreme quantiles as well in an as easy fashion. It furthermore requires no hypothesis with respect to the transfer function to be ran : it can operate with and adapt to any black box mapping. This new tool is a valuable Monte Carlo asset. We will hopefully develop these theoretical aspects in our future works.

2. Erd stands for Empirical relative deviation *i.e.* empirical standard deviation over mean ratio as explained at 10.

G. TABLES

Tab. 1 – *Parameters*

TLEs		January 13th 2009
D	$=$	$10^2 Diag(4,4,10,4,4,7)$
$f_{\bar{\epsilon}}$	$\sim$	$\mathcal{N}(0_6, D^2)$
$M(x,dy)$	$\sim$	$\mathcal{N}(\frac{\alpha x}{\sqrt{1+\alpha^2}}, \frac{1}{1+\alpha^2}D^2)$
$d_c = 100$	$n = 3 \cdot 10^5$	$m = 100$
$\eta = 1250$	$p = 0.75$	$\omega = 5$

Tab. 2 – *AST-CMC comparison*

	Mean estimate	*Erd*	Mean simulation number	*Erd*
CMC	$1.1 \cdot 10^{-6}$	1.7258	300000	0
AST	$4.78 \cdot 10^{-7}$	0.3232	309060	0.0232

Tab. 3 – *AST sensitivity test results. In all cases, 100 iid estimations were done with* $d_c = 100$.

Experiment	1	2	3	4	5	6
η	1250	1250	1250	2500	2500	2500
p	0.75	0.75	0.50	0.75	0.75	0.50
ω	5	10	5	5	10	5
$\mu(N)/10^5$	3.10	5.96	1.40	6.21	11.85	2.77
$\rho(N) \times 10^2$	2.32	2.24	2.86	1.65	1.6153	2.01
$\mu(\nu) \times 10^7$	4.78	7.88	0.91	4.35	7.95	0.90
$\rho(\nu) \times 10$	3.32	3.07	3.81	2.24	2.03	2.96

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