A New Belief Function Based Approach for Multi-Criteria Decision-Making Support

Jean Dezert
ONERA
The French Aerospace Lab
F-91761 Palaiseau, France.
E-mail: jean.dezert@onera.fr

Deqiang Han, Hanlin Yin
Center for Information Engineering Science Research
Xi’an Jiaotong University
Xi’an, China 710049.
E-mail: deqhan@gmail.com, iverlon1987@stu.xjtu.edu.cn

Abstract—In this paper, we propose a new approach based on belief functions for multi-criteria decision-making (MCDM) support which is inspired by the technique for order preference by similarity to ideal solution (TOPSIS). This new approach, called BF-TOPSIS (Belief Function based TOPSIS), includes four distinct methods with different computational complexities. BF-TOPSIS offers the advantage of avoiding the problem of the choice of data normalization, of dealing with some missing scores, and of taking into account the reliability of each source (or criterion) that provides the scores of alternatives. We present results of BF-TOPSIS for different MCDM examples and discuss its robustness to rank reversal phenomena.

Keywords: Information fusion, multi-criteria, decision-making, belief functions, TOPSIS, MCDM, DSmT.

I. INTRODUCTION

Classical Multi-Criteria Decision-Making (MCDM) consists of choosing an alternative among a known set of alternatives based on their quantitative evaluations (numerical scores) obtained with respect to different criteria. The MCDM problem, although easily formulated, is not easy to solve because the scores are usually expressed in different (physical) units and different scales which generally necessitates a choice of a normalization step that yields many problems, e.g. rank reversal.

Many methods have been proposed in the literature to try to solve the MCDM problem [3]–[12]. Among them, the following ones have attracted great interests in the operational research community and are widely used: AHP [13], ELECTRE [14], TOPSIS [15], [16]. These methods however are not exempt of problems and none of them makes consensus in the MCDM community, see discussions in [14], [19]–[23]. More recently, COWA-ERV and Fuzzy-COWA-ERV techniques based on belief functions [17], [18] have been proposed by the authors. In 2013, a new interesting MCDM method called estimator ranking vector (ERV) obtained from a Multiple-Attribute Competition Measure Matrix (MACMM) has been proposed by Yin et al. [24]. The ERV uses the “joint” information just like Pitman’s closeness measure (PCM) to rank the performance of different estimators. As shown by the authors, the ERV method performs better than PCM and it offers the advantage to avoid data normalization, but ERV method is not exempt from rank reversal problem.

In this work, we propose a new method based on belief functions inspired by the ERV and TOPSIS methods called BF-TOPSIS which does not involve direct data normalization, and which evaluates more precisely how much better or worse an alternative is with respect to the others. The main idea is to build basic belief assignments (BBAs) directly from the available scores values that reflect the evidences supporting each alternative on one hand, and the evidences supporting its complement on the other hand. Once all BBAs are known as well as all their related scores values, e.g. rank reversal, a Multiple-Attribute Competition Measure Matrix (MACMM) is constructed of BBA for the MCDM context in section IV. Four new BF-TOPSIS methods are then detailed in section V, with some examples in section VI. Conclusions, perspectives and open challenging questions are discussed in section VII.

II. CLASSICAL MCDM PROBLEM

We consider a classical MCDM problem with a given set of alternatives $A \triangleq \{A_1, A_2, \ldots, A_M\}$ ($M > 2$), and a given set of criteria $C \triangleq \{C_1, C_2, \ldots, C_N\}$ ($N \geq 1$). Each alternative $A_i$ represents a possible choice (a possible solution to the problem) and each criterion $C_j$ represents a specific aspect of the problem. The goal is to rank the alternatives with respect to each criterion and then to rank the alternatives overall.

For example, in a car selection problem, fuel economy can be measured in miles per gallon (or in km/L), and price can be expressed in different currencies (e.g. pound sterling, US dollars, or Euros), etc.

A rank reversal is a change in the rank ordering if we change the structure of the MCDM problem by adding (or deleting) some alternatives. The rank reversal phenomenon’s appearing in most of MCDM methods is partially due to the choice of direct data normalization as explained in [1], [2], [7], [14], [19], [20], [22], [23], [31], [33].

1) For example, in a car selection problem, fuel economy can be measured in miles per gallon (or in km/L), and price can be expressed in different currencies (e.g. pound sterling, US dollars, or Euros), etc.

2) A rank reversal is a change in the rank ordering if we change the structure of the MCDM problem by adding (or deleting) some alternatives. The rank reversal phenomenon’s appearing in most of MCDM methods is partially due to the choice of direct data normalization as explained in [1], [2], [7], [14], [19], [20], [22], [23], [31], [33].

3) Analytic Hierarchy Process
4) Elimination and choice translating reality
5) Technique for order preference by similarity to ideal solution
6) Cautious Ordered Weighted Averaging with Evidential Reasoning
7) In our general context, we speak about alternatives instead of estimators.

A MCDM problem is said classical if all criteria $C_j$ and all alternatives $A_i$ are known as well as all their related scores values $S_{ij}$. Unclassical MCDM problems refer to problems involving incomplete or qualitative information.
decision to make). In a general context, each criterion is also characterized by a relative importance weighting factor $w_j \in [0, 1]$, $j = 1, \ldots, N$ which are normalized by imposing the condition $\sum_j w_j = 1$. The set of normalized weighting factors is denoted by $w \triangleq \{w_1, w_2, \ldots, w_N\}$. The score of each alternative $A_i$ with respect to each criteria $C_j$ is expressed by a real number $S_{ij}$ called the score value of $A_i$ based on $C_j$. We denote $S$ the score matrix $M \times N$ matrix which is defined as $S \triangleq [S_{ij}]$. The MCDM problem now aims to select the best alternative $A^* \in A$ given $S$ and the weighting factors $w$ of criteria.

### III. BASICS OF THE THEORY OF BELIEF FUNCTIONS

To follow classical notations of Dempster-Shafer Theory, also called the theory of belief functions [25], we assume that the answer of the problem under concern belongs to a known (or given) finite discrete frame of discernment (FoD) $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$, with $n > 1$, and where all elements of $\Theta$ are exclusive. The set of all subsets of $\Theta$ (including empty set $\emptyset$ and $\Theta$) is the power-set of $\Theta$ denoted by $2^{\Theta}$. A basic belief assignment (BBA) associated with a given source of evidence is defined [25] as the mapping $m(\cdot) : 2^{\Theta} \to [0, 1]$ satisfying $m(\emptyset) = 0$ and $\sum_{A \subseteq 2^{\Theta}} m(A) = 1$. The quantity $m(A)$ is called the mass of $A$ committed by the source of evidence. Belief and plausibility functions are respectively defined by

$$Bel(A) = \sum_{B \subseteq A} m(B), \quad Pl(A) = 1 - Bel(\bar{A}). \tag{1}$$

If $m(A) > 0$, $A$ is called a focal element of $m(\cdot)$. When all focal elements are singletons then $m(\cdot)$ is called a Bayesian BBA [25] and its corresponding $Bel(\cdot)$ function is homogeneous to a (subjective) probability measure. The vacuous BBA, or VBBA for short, representing a totally ignorant source is expressed in same unit. The units are usually different from one criterion (i.e column) to another. For simplicity, we assume that each criterion $C_j$ expresses a benefit so that the ranking is done according to the preference the greater is better. For a mono-criteria problem and if one has no same multiple score values, it is easy to rank alternatives $A_i$ directly from the score values by descending order. In MCDM problems, the direct rankings associated with different criteria can be inconsistent (different). Therefore, efficient fusion techniques must be developed in order to provide the global ranking solution to solve the MCDM problem.

To diminish rank reversal phenomenon, we want to avoid direct data normalization [23], and of course we also want to estimate the global ranking vector drawn from the score values. For this aim, we propose to estimate the ranking vector (ERV) from all evidences that support [18] and refute [19] each alternative thanks to BBAs [25]. In our approach, the FoD is the set of alternatives, that is $\Theta \triangleq \{A_1, A_2, \ldots, A_M\}$. The construction of BBAs is based on the following theorem.

Theorem: We consider a criterion $C_j$ and a score vector $s_i = [S_{ij} S_{2j} \ldots S_{Mj}]^T$ with $S_{ij} \in \mathbb{R}$ associated to the FoD $\Theta = \{A_1, A_2, \ldots, A_M\}$. For any proposition $A_i$ of $\Theta$ and its positive and negative evidence supports defined by

$$Sup_j(A_i) \triangleq \sum_{k \in \{1, \ldots, M\} | S_{ij} \leq S_{kj}} |S_{ij} - S_{kj}| \tag{2}$$

$$Inf_j(A_i) \triangleq - \sum_{k \in \{1, \ldots, M\} | S_{kj} \geq S_{ij}} |S_{ij} - S_{kj}| \tag{3}$$

If $A_{i_{\text{max}}} \triangleq \max_i Sup_j(A_i)$ and $A_{i_{\text{min}}} \triangleq \min_i Inf_j(A_i)$ are different from zero, the following inequality holds

$$\frac{Sup_j(A_{i_{\text{max}}})}{A_{i_{\text{max}}}} \leq \frac{Inf_j(A_{i_{\text{min}}})}{A_{i_{\text{min}}}}. \tag{4}$$

In some references, the score matrix is also called the decision matrix because it is the matrix from which the decision must be taken. By convention, $x^T$ denotes the transpose of $x$.

Of course, a similar presentation can be done with the preference the lower is better if the criterion corresponds to a loss.

We call it positive evidence.

We call it negative evidence.
Proof: Given in Appendix 1.

Supj(Ai) is called the positive support of Ai because it measures how much Ai is better than other alternatives according to criterion Cj, and Infj(Ai) is called the negative support of Ai because it measures how much Ai is worse than other alternatives according to criterion Cj. The length of [0, Supj(Ai)] measures the support in favor of Ai as being the best alternative with respect to all other ones, and the length of [Infj(Ai), 0] measures the support against Ai based on the criterion Cj.

Thanks to the previous theorem, the construction of BBAs for each alternative Ai based on the score vector sj relative to a criterion Cj can be done as follows:

1. **Step 1:** For each Ai, evaluate the evidential supports Supj(Ai) and Infj(Ai) by (2) and (3).
2. **Step 2:** Compute the min value A^j_{min} of Infj(Ai), and the max value A^j_{max} of Supj(Ai), ∀i = 1, ..., M.
3. **Step 3:** We define the belief of Ai as the evidential support of hypothesis “Ai is better than its competitors Ai” by taking
   \[ Belij(Ai) = \frac{Supj(Ai)}{A^j_{max}} \]

   Similarly, we define the belief of Ai (i.e. the complement of Ai in Θ) by taking
   \[ Belij(\bar{Ai}) = \frac{Infj(Ai)}{A^j_{min}} \]

   By construction, Belij(Ai) and Belij(\bar{Ai}) belong to [0, 1], and thanks to the theorem this belief function construction is perfectly consistent. More specifically, the inequality Belij(Ai) ≤ Plij(Ai) is satisfied with the plausibility of Ai defined by [25]

   \[ Plij(Ai) = 1 - Belij(\bar{Ai}) = 1 - \frac{Infj(Ai)}{A^j_{min}} \]

   If we do not have evidential support of Ai (i.e. A^j_{max} = 0) then we take naturally Belij(Ai) = 0, and if we do not have evidential support of Ai (i.e. A^j_{min} = 0) then we take Belij(Ai) = 0. The belief interval of choice Ai based on the criterion Cj and based on the local comparisons with respect to its alternatives is given by [20]

   \[ [Belij(Ai); Plij(Ai)] = \frac{Supj(Ai)}{A^j_{max}}, 1 - \frac{Infj(Ai)}{A^j_{min}} \]

   From this belief interval, one computes it corresponding BBA mij(·) defined on the power set of Θ based on the local comparisons with respect to its alternatives by taking for \( i = 1, ..., M \)

   \[ mij(Ai) = Belij(Ai) \]
   \[ mij(\bar{Ai}) = Belij(\bar{Ai}) = 1 - Plij(Ai) \]
   \[ mij(Ai ∪ \bar{Ai}) = Plij(Ai) - Belij(Ai) \]

   At the output of step 3 of our BBA construction, one has an \( M × N \) BBA matrix \( M = [mij(·)] \), where each element \( m_{ij}(·) \) of the BBA matrix corresponds in fact to a triplet \( (m_{ij}(A_i), m_{ij}(\bar{A}_i), m_{ij}(A_i ∪ \bar{A}_i)) \) defined by the formula (9)–(11).

   For a given criterion \( C_j \), if all score values \( S_{ij} \) are the same, we get for all Ai, Supj(Ai) = Infj(Ai) = 0. Hence, no evidence support can be drawn for or against Ai, and we take Belij(Ai) = Belij(\bar{Ai}) = 0, so that Pl(Ai) = 1 and consequently \( m_{ij}(A_i) = m_{ij}(\bar{A}_i) = 0 \) and \( m_{ij}(A_i ∪ \bar{A}_i) = 1 \). In this very particular case, the fusion of the M BBA \( m_{ij}(·) \) provides a combined BBA \( m_{ij}(·) \) equals to the vacuous belief assignment over the refined frame \( Θ \), that is \( m_j(A_1 ∪ ... ∪ A_M) = 1 \) from which naturally no specific ranking can be drawn, which makes a perfect sense.

   This approach of BBA construction is very interesting for applications because it is invariant to the bias and scaling effects of score values21. Also, it allows us to model our lack of evidence (if any) with respect to an (or several) alternative(s) when their corresponding score values are missing for any reason. For example, if a numerical value \( S_{ij} \) is missing or indeterminate, then we will use the vacuous belief assignment \( m_{ij}(A_i ∪ \bar{A}_i) = 1 \) because we have no evidential support for \( A_i \) and for \( \bar{A}_i \). In our BF-TOPSIS methods presented in the next section, we will use only the score matrix and the importance weighting factor related to each criteria, but it is worth noting that one can also do a pre-processing step to discount the BBA \( m_{ij}(·) \) by a reliability factor using the classical Shafer’s discounting method if one wants to express some doubts on the reliability of \( m_{ij}(·) \), see [25] for reliability discounting formulae.

V. NEW BF-TOPSIS METHODS

In the previous Section, we have presented an appealing method (invariant to bias and scale effects on score values) to build the BBA matrix from any general score matrix. The major concern of MCDM problem is now how to deal with these elementary BBAs \( m_{ij}(·) \), and how to establish a final ranking from them? Also if possible, how to avoid the rank reversal problem?

Our first idea to solve this MCDM problem was to use directly some rules of combination, mainly Dempster’s rule [25], or PCR6 rule [26] to combine all the BBAs \( m_{ij}(·) \) together to obtain a global basic belief assignment \( m(·) \) from which a final ranking of alternatives would be drawn. Unfortunately, such a very simple idea fails to provide correct ranking result, even in the simplest mono-criterion case (see the example 1 in the next section), because all the BBAs \( m_{ij}(·) \) for a given criterion \( C_j \) are not independent since they are built on same set of score values \( \{S_{ij}, i = 1, ..., M\} \), so that DS and PCR6 rules theoretically should not be used. Of course, such a global fusion approach would also require too high computational complexity and resources to solve high dimension MCDM problems.

21More specifically, if for a given criterion \( C_j \), the score values \( S_{ij} \) of alternatives are replaced by \( S’_{ij} = a·S_{ij} + b \), with a scale factor \( a > 0 \) and a bias \( b ∈ R \), then the corresponding BBAs \( m_{ij}(·) \) and \( m’_{ij}(·) \) built by our method are equal.
Therefore, in this paper, we propose four MCDM methods inspired by the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) that uses belief masses \( m_{ij}(\cdot) \) as defined in the previous section. We call them the Belief-based TOPSIS methods, and we denote them as BF-TOPSIS1, BF-TOPSIS2, BF-TOPSIS3 and BF-TOPSIS4. Our four BF-TOPSIS methods are somehow inspired by ERV method \([24]\) which has proved that data normalization can be avoided for MCDM, and also by the TOPSIS \([15, 16]\) method for borrowing the idea of using the best and worst ideal solutions. Our BF-TOPSIS methods are however totally new in the way of processing information to obtain the final ranking of alternatives. Our BF-TOPSIS methods present different complexity of implementation and robustness to rank reversal. These methods are detailed below:

A. BF-TOPSIS1 method

**Step 1:** From the score matrix \( S \), compute BBAs \( m_{ij}(A_i) \) \( m_{ij}(A_i \cup A_i) \) using (9)–(11).

**Step 2:** For each alternative \( A_i \), compute the Belief Interval-based Euclidean distance\(^{22}\) \( d_{BI}^E(m_{ij}, m_{ij}^{best}) \) (defined in [43]) between \( m_{ij}(\cdot) \) and the best ideal BBA defined by \( m_{ij}^{best}(A_i) \approx 1 \), and the distances \( d_{BI}^E(m_{ij}, m_{ij}^{worst}) \) between \( m_{ij}(\cdot) \) and the worst ideal BBA defined by \( m_{ij}^{worst}(A_i) \approx 1 \).

**Step 3:** Compute the weighted average of \( d_{BI}^E(m_{ij}, m_{ij}^{best}) \) values with relative importance weighting factor \( w_j \) of criteria \( C_j \). Similarly, compute the weighted average of \( d_{BI}^E(m_{ij}, m_{ij}^{worst}) \) values. More specifically, compute

\[
d^{\text{best}}(A_i) \triangleq \sum_{j=1}^{N} w_j \cdot d_{BI}^E(m_{ij}, m_{ij}^{best}) \tag{12}
\]

\[
d^{\text{worst}}(A_i) \triangleq \sum_{j=1}^{N} w_j \cdot d_{BI}^E(m_{ij}, m_{ij}^{worst}) \tag{13}
\]

**Step 4:** The relative closeness of the alternative \( A_i \) with respect to ideal best solution \( A^{\text{best}} \) is then defined by

\[
C(A_i, A^{\text{best}}) \triangleq \frac{d^{\text{worst}}(A_i)}{d^{\text{worst}}(A_i) + d^{\text{best}}(A_i)} \tag{14}
\]

Because \( d^{\text{best}}(A_i) \geq 0 \) and \( d^{\text{worst}}(A_i) \geq 0 \), then \( C(A_i, A^{\text{best}}) \in [0, 1] \). If \( d^{\text{best}}(A_i) = 0 \), it means that alternative \( A_i \) coincides with the ideal best solution and thus \( C(A_i, A^{\text{best}}) = 1 \) (the relative closeness of \( A_i \) with respect to \( A^{\text{best}} \) is maximal). Contrariwise, if \( d^{\text{worst}}(A_i) = 0 \), it means that alternative \( A_i \) coincides with the ideal worst solution and thus \( C(A_i, A^{\text{best}}) = 0 \) (the relative closeness of \( A_i \) with respect to \( A^{\text{best}} \) is minimal).

**Step 5:** (Preference ranking) The set of alternatives is preference ranked according to the descending order of \( C(A_i, A^{\text{best}}) \in [0, 1] \), where a larger \( C(A_i, A^{\text{best}}) \) value means a better alternative (or higher preference).

B. BF-TOPSIS2 method

Steps 1, 2 and 5 are the same as in BF-TOPSIS1. Only steps 3 and 4 differ as follows:

**Step 3:** For each criteria \( C_j \), compute the relative closeness of the alternative \( A_i \) w.r.t. ideal best solution \( A^{\text{best}} \) by

\[
C_j(A_i, A^{\text{best}}) \triangleq \frac{d_{BI}^E(m_{ij}, m_{ij}^{best})}{d_{BI}^E(m_{ij}, m_{ij}^{worst}) + d_{BI}^E(m_{ij}, m_{ij}^{best})} \tag{15}
\]

**Step 4:** The relative closeness of the alternative \( A_i \) with respect to ideal best solution \( A^{\text{best}} \) is then defined by the weighted average of \( C_j(A_i, A^{\text{best}}) \) that is

\[
C(A_i, A^{\text{best}}) \triangleq \sum_{j=1}^{N} w_j \cdot C_j(A_i, A^{\text{best}}) \tag{16}
\]

C. BF-TOPSIS3 method

This third method is more complicate to implement because it requires the direct fusion of \( N \) BBAs \( m_{ij}(\cdot) \) for a given alternative index \( i \) with the PCR6 rule of combination \([26, 32]\). The Step 1, 4 and 5 are the same as in BF-TOPSIS1. Only the Step 2 and 3 differ as follows:

**Step 2:** For each alternative \( A_i \) and for the set of BBAs \( m_{ij}(\cdot) \) and criteria importance factors \( w_j \), compute with the PCR6 combination rule\(^{25}\) \([32]\), the fused BBA \( m_i^{\text{PCR6}}(\cdot) \).

**Step 3:** Compute the Belief Interval-based Euclidean distances (see [43]) \( d_{BI}^E(m_{i}^{\text{PCR6}}, m_{i}^{best}) \) between \( m_{i}^{\text{PCR6}}(\cdot) \) and the ideal best BBA defined by \( m_{i}^{best}(A_i) \approx 1 \), and the distances \( d_{BI}^E(m_{i}^{\text{PCR6}}, m_{i}^{worst}) \) between \( m_{i}^{\text{PCR6}}(\cdot) \) and the ideal worst BBA defined by \( m_{i}^{worst}(A_i) \approx 1 \). More specifically, compute

\[
d^{\text{best}}(A_i) \triangleq d_{BI}^E(m_{i}^{\text{PCR6}}, m_{i}^{best}) \tag{17}
\]

\[
d^{\text{worst}}(A_i) \triangleq d_{BI}^E(m_{i}^{\text{PCR6}}, m_{i}^{worst}) \tag{18}
\]

D. BF-TOPSIS4 method

BF-TOPSIS4 method is similar to BF-TOPSIS3 except that we use the more complicate ZPCR6 fusion rule which is a modified version of PCR6 rule taking into account Zhang’s degree of intersection of focal elements in the conjunctive consensus operator. ZPCR6 rule is explained in details with examples in \([34]\).

VI. EXAMPLES

In this section, we provide the results of our new BF-TOPSIS methods when applied to different examples.

A. Example 1 (Mono-criterion)

Let’s consider a criterion \( A_i \), \( (i = 1, \ldots, 7) \) with the following corresponding score values \( s_j = [10, 20, -5, 0, 100, -11, 0]^T \). The direct ranking with the preference “greater is better” yields\(^{25}\) \( A_5 \succ A_2 \succ A_1 \succ (A_4 \sim A_7) \succ A_3 \succ A_6 \). Because \( A_4 \) and \( A_7 \) have same

\(^{22}\) The justification of this distance with respect to other existing ones (Jousselme’s, Tessem’s, etc.) has been given in \([43]\).

\(^{23}\) The choice of PCR6 rule instead of DS rule for taking into account importance discounting has been justified in \([32]\). The weighted average fusion rule has been also tested but it can provide rank reversal.

\(^{24}\) In this example \( j = 1 \) because we consider a mono-criterion example.

\(^{25}\) where the symbol \( \succ \) means better than (or is preferred to).
score values, then both ranking vectors $r_j = [5, 2, 1, 4, 7, 3, 6]$ and $r'_j = [5, 2, 1, 4, 7, 3, 6]$ are admissible ranking solutions. In applying formulas (9)–(11), we get the following set of BBAs:

$$m_{ij}(A_1) = 0.9595, m_{ij}(A_2) = 0.1809, m_{ij}(A_3) = 0.0102, m_{ij}(A_4) = 0.0273, m_{ij}(A_5) = 0.0000, m_{ij}(A_6) = 0.0273, m_{ij}(A_7) = 0.6606, 0.2921$$

It can be verified that the combination of these BBAs by Dempster's rule yields $m_{DS}(A_5) = 1$ from which no ranking of alternatives can be inferred, but $A_2$ is the best one. The combination of these BBAs by PCR6 yields $m_{PCR6}(A_1) = 0.0063$, $m_{PCR6}(A_2) = 0.0255$, $m_{PCR6}(A_4) = 0.0014$, $m_{PCR6}(A_5) = 0.9665$ and $m_{PCR6}(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_6 \cup A_7) = 0.0003$. If we sort the alternatives by decreasing order of belief or plausibility values computed from $m_{PCR6}()$, we will get the following preferences order $A_5 \succ A_2 \succ A_1 \succ A_4 \succ (A_3 \sim A_6 \sim A_7)$, which is better than the preferences order result obtained with Dempster’s rule. Unfortunately, it is still not fully consistent with the direct ranking.

This example shows that Dempster’s and PCR6 rules cannot infer the correct ranking even in a simple mono-criterion example. The main reason is because the BBAs to combine are not built from independent sources of evidence, so that Dempster’s and PCR6 rules should not apply. This simple example motivates the development of our new BF-TOPSIS methods based on BF and TOPSIS for MCDM.

Using BF-TOPSIS method, we get the following distance values, and the relative closeness measures:

<table>
<thead>
<tr>
<th>BBAs CONSTRUCTED FROM SCORE VALUES.</th>
<th>$m_{ij}(A_1)$</th>
<th>$m_{ij}(A_2)$</th>
<th>$m_{ij}(A_3)$</th>
<th>$m_{ij}(A_1 \cup A_3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>0.9595</td>
<td>0.1809</td>
<td>0.0102</td>
<td>0.3809</td>
</tr>
<tr>
<td>$A_2$</td>
<td>0.1809</td>
<td>0.1418</td>
<td>0.0403</td>
<td></td>
</tr>
<tr>
<td>$A_3$</td>
<td>0.0102</td>
<td>0.8115</td>
<td>0.1783</td>
<td></td>
</tr>
<tr>
<td>$A_4$</td>
<td>0.0273</td>
<td>0.6806</td>
<td>0.2921</td>
<td></td>
</tr>
<tr>
<td>$A_5$</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$A_6$</td>
<td>0.0273</td>
<td>0.6806</td>
<td>0.2921</td>
<td></td>
</tr>
</tbody>
</table>

In sorting $C(A_i, A_{best})$ values by the descending order, we get the correct preferences order

$$A_5 \succ A_2 \succ A_1 \succ (A_4 \sim A_7) \succ A_3 \succ A_6.$$  

Let's modify a little bit this example by changing the score of $A_2$ from 100 to 21 (21 is now very close to the score of $A_2$). So, if we start with modified score values $s_j = [10, 20, -5, 0, 21, -11, 0]^T$, the result of the direct ranking method remains unchanged with respect to previous

one because $A_5$ still has the highest score among all scores of alternatives. In applying BF-TOPSIS methods, we get now

$$A_5 \succ A_2 \succ A_1 \succ (A_4 \sim A_7) \succ A_3 \succ A_6.$$  

Of course $A_5$ is still the best alternative to select because $C(A_5, A_{best}) = 1$, but $A_2$ can now be considered also as very close to the best solution also (with good confidence because $C(A_2, A_{best}) = 0.9472$), which makes perfectly sense in this modified example. So these new BF-TOPSIS methods provide the correct preferences order in the mono-criterion case, and they are able to capture somehow how much we must be confident in the ranks of the ranking result.

### B. Example 2 (Non informative case in mono-criterion)

If all score values are equal, then one gets from the BBA construction $m_{ij}(A_i \cup A_i) = 1$ for each alternative index value $i$. Consequently with BF-TOPSIS methods, all relative closeness values $C(A_i, A_{best})$ are the same, and therefore no specific choice of a specific alternative with respect to the other ones can be drawn, which is perfectly normal in such degenerate (non informative) case.

### C. Example 3 (Multi-criteria and rank reversal)

This interesting example is drawn from [22] (Table 7, p. 1224). We consider the following set of alternatives (the FoD) $\Theta = \{A_1, A_2, A_3, A_4, A_5\}$, four criteria with importance weighting vector $w = [1/6, 1/3, 1/3, 1/6]$, and the score matrix

$$S = \begin{bmatrix} 26 & 42 & 43 & 70 \\ 25 & 50 & 45 & 80 \\ 28 & 45 & 50 & 75 \\ 24 & 40 & 47 & 100 \\ 30 & 30 & 45 & 80 \end{bmatrix}$$

As shown in [22] (Table 9, p. 1226) and in the following tables, the (classical) TOPSIS method suffers from rank reversal. As seen in next Tables, BF-TOPSIS1 and BF-TOPSIS2 give same preference order results and they also suffer from rank reversal in this example. BF-TOPSIS3 and BF-TOPSIS4 preserve the preference order (yielding no rank reversal in this example). It is worth noting that the simple weighted BBA averaging rule instead of PCR6 and ZPCR6 rules has also been tested, and it gives same result as with BF-TOPSIS1 (i.e., rank reversal).
D. Example 4 (Multi-criteria for car selection)

Let’s consider a more concrete car selection problem. We consider a set of four cars \{A_1, A_2, A_3, A_4\} as follows:

- \(A_1 = \text{TOYOTA YARIS 69 VVT-i Tendance}\);
- \(A_2 = \text{SUZUKI SWIFT MY15 1.2 VVT So’City}\);
- \(A_3 = \text{VOLKSWAGEN POLO 1.0 60 Confortline}\);
- \(A_4 = \text{OPEL CORSA 1.4 Turbo 100 ch Start/Stop Edition}\);

We consider the following five criteria for making the choice of the best car to buy:

- \(C_1\) is the price (in €);
- \(C_2\) is fuel consumption (in L/km);
- \(C_3\) is the CO2 emission (in g/km);
- \(C_4\) is the fuel tank volume (in L);
- \(C_5\) is the trunk volume (in L);

The score matrix \(S = [s_{ij}]\) is built from information extracted from car-makers technical characteristics available on the world wide web\(^{27}\). For the chosen cars, the corresponding score matrix is given by:

\[
S = \begin{bmatrix}
15000 & 4.3 & 99 & 42 & 737 \\
15290 & 5.0 & 116 & 42 & 892 \\
15350 & 5.0 & 114 & 45 & 952 \\
15490 & 5.3 & 123 & 45 & 1120
\end{bmatrix}
\]

For criteria \(C_1, C_2\) and \(C_3\) smaller is better. For criteria \(C_4\) and \(C_5\) larger is better. To make the preference order homogeneous in the score matrix, we multiply values of columns \(C_1, C_2\) and \(C_3\) by -1 so that our MCDM problem is described by a modified score matrix with homogeneous preference order ("larger is better") for each column.

For simplicity, the importance \(imp(C_j)\) of each criteria \(C_j\) takes a value in \{1, 2, 3, 4, 5\}, where 1 means the least important, and 5 means the most important. In this example we take \(imp(C_1) = 5, imp(C_2) = 4, imp(C_3) = 4, imp(C_4) = 1\) and \(imp(C_5) = 3\) which means that the price of a car (criteria \(C_1\)) is the most important criteria for us, and the volume of fuel tank (criteria \(C_4\)) is the least important one. From these importance values and after normalization, we get the following vector of relative weights of criteria:

\[
w = \begin{bmatrix}
\frac{5}{17} & \frac{4}{17} & \frac{4}{17} & \frac{1}{17} & \frac{3}{17}
\end{bmatrix}
\]

Intuitively, based on the score matrix \(S\) and importance of criteria, the choice of car \(A_1\) is anticipated to be the best choice because the three most important criteria meet clearly their best values for car \(A_1\). If we apply the classical TOPSIS [15], [16], one gets \(A_4 > A_1 > A_3 > A_2\), that is \(A_4\) would be the best car to buy, whereas \(A_2\) would be the worst one. This result is quite surprising and counter-intuitive because in this very simple and concrete example \(A_1\) should have been selected as the best choice without ambiguity by any rational decision-maker. With BF-TOPSIS methods (1, 2, 3 and 4) we get a more satisfactory preference order \(A_1 > A_3 > A_2 > A_4\), which is also what we get with AHP method [13], or with the SAW (Simple Additive Weighting) method [3], [22] in this example. The unexpected classical TOPSIS results is due to the choice of normalization of scores [15], and the problem can be (not always) circumvented by changing the normalization procedure, see discussion in [22]. The choice of a normalization procedure for TOPSIS is always an open challenging question. This problem of choice of normalization is avoided in BF-TOPSIS methods because no direct score normalization is necessary for its implementation.

E. Example 5 (Multi-criteria for student evaluation)

We consider four students (i.e., alternatives \(A_1, A_2, A_3\) and \(A_4\)) and ten attributes (i.e., criteria \(C_j, j = 1, \ldots, 10\)) with equal attribute’s weighting factor \(w_j = 1/10\) (\(j = 1, \ldots, 10\)), and the scores given in table below.

<table>
<thead>
<tr>
<th>Attributes (scores) of the students.</th>
<th>(A_1)</th>
<th>(A_2)</th>
<th>(A_3)</th>
<th>(A_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_1) Math</td>
<td>90</td>
<td>80</td>
<td>70</td>
<td>60</td>
</tr>
<tr>
<td>(C_2) Arts</td>
<td>90</td>
<td>80</td>
<td>70</td>
<td>60</td>
</tr>
<tr>
<td>(C_3) English</td>
<td>90</td>
<td>80</td>
<td>70</td>
<td>60</td>
</tr>
<tr>
<td>(C_4) Geography</td>
<td>90</td>
<td>80</td>
<td>70</td>
<td>60</td>
</tr>
<tr>
<td>(C_5) Physics</td>
<td>90</td>
<td>80</td>
<td>70</td>
<td>75</td>
</tr>
<tr>
<td>(C_6) Music</td>
<td>90</td>
<td>80</td>
<td>70</td>
<td>95</td>
</tr>
<tr>
<td>(C_7) History</td>
<td>80</td>
<td>90</td>
<td>70</td>
<td>85</td>
</tr>
<tr>
<td>(C_8) Chemistry</td>
<td>80</td>
<td>90</td>
<td>70</td>
<td>85</td>
</tr>
<tr>
<td>(C_9) Biology</td>
<td>80</td>
<td>90</td>
<td>70</td>
<td>85</td>
</tr>
<tr>
<td>(C_{10}) Long jump</td>
<td>3.5m</td>
<td>3.7m</td>
<td>4.0m</td>
<td>3.6m</td>
</tr>
</tbody>
</table>

At first, we only rank the comprehensive quality of the first three students \(A_1, A_2\) and \(A_3\). The tables VIII and IX present ranking vectors (with three digits approximation), and preference orders obtained with Multiple-Attribute Competition Measure matrix\(^{28}\) of ERV method [24], and the BF-TOPSIS methods.

\(^{27}\)http://www.choisir-sa-voiture.com

\(^{28}\)We assume here that all attributes (criteria) have the same weighting factor 1/10, see [24] for details.
If we rank the comprehensive quality of the four students A1, A2, A3 and A4, we get the following results:

<table>
<thead>
<tr>
<th>Methods</th>
<th>Ranking vectors</th>
<th>Preferences orders</th>
</tr>
</thead>
<tbody>
<tr>
<td>ERV</td>
<td>[0.748, 0.636, 0.248, 0.386]</td>
<td>A2 ≻ A1 ≻ A4 ≻ A3</td>
</tr>
<tr>
<td>BF-TOPSIS1</td>
<td>[0.729, 0.594, 0.100]</td>
<td>A1 ≻ A2 ≻ A3</td>
</tr>
<tr>
<td>BF-TOPSIS2</td>
<td>[0.731, 0.597, 0.100]</td>
<td>A1 ≻ A2 ≻ A3</td>
</tr>
<tr>
<td>BF-TOPSIS3</td>
<td>[0.803, 0.736, 0.100]</td>
<td>A1 ≻ A2 ≻ A3</td>
</tr>
<tr>
<td>BF-TOPSIS4</td>
<td>[0.803, 0.736, 0.100]</td>
<td>A1 ≻ A2 ≻ A3</td>
</tr>
</tbody>
</table>

In this example, one sees that ERV, BF-TOPSIS3 and BF-TOPSIS4 methods suffer from rank reversal, whereas BF-TOPSIS1 and BF-TOPSIS2 do not suffer from rank reversal.

VII. CONCLUSIONS

In this work, we have presented four new MCDM methods (BF-TOPSIS1–BF-TOPSIS4) inspired by TOPSIS and based on belief functions. We have shown that it is possible to establish basic belief assignments (BBAs) from any score values with respect to a criterion to get a mass for each alternative, its complement and uncertainty. With BF-TOPSIS methods, we can compute from these BBAs the relative closeness of each alternative to the ideal best and worst solutions for establishing the ranking to get the final preference order of alternatives. These new methods do not need direct score value normalization, and they can deal with any real score values at same time (negative, zero or positive). If necessary, these methods can also deal easily with missing score values and with the reliability of the sources as well. They are invariant to scales and bias effects in score values. As all other MCDM methods developed so far, these new BF-TOPSIS methods suffer from rank reversal in some cases, which are very difficult to anticipate. As shown in our examples, some BF-TOPSIS methods are robust to rank reversal in some cases, while others are robust in other cases. If rank reversal is really crucial in the MCDM problem under concern, we recommend to test the panel of BF-TOPSIS methods, and choose the least complex one which is robust to rank reversal. BF-TOPSIS1 and BF-TOPSIS2 are easy to implement, whereas BF-TOPSIS3 and BF-TOPSIS4 are more complicate and time consuming. In future research work, we would like to establish the theoretical conditions that any MCDM method must satisfy to not suffer from rank reversal. This remains a fundamental open challenging question for operational research and information fusion communities.
The upper bound of (26) can be rewritten as
\[
\sum_{k=m+1}^{n} (B_k - B_k) = \sum_{k=m+1}^{n} (B_k - B_k + B_M) = \sum_{k=m+1}^{n} (B_k - B_k) - \sum_{k=m+1}^{n} (B_k - B_M)
\]
\[
= (M - s) \cdot c' - a' \cdot d - a' \cdot (M - s) \cdot d
\]
where \(a' = \sum_{k=m+1}^{n} (B_k - B_M)\), and \(c' = B_k - B_M\). Because \((M - s) \cdot c' - a' \geq 0, (M - s) \cdot d - a' > 0, d > c', a' > 0\) and \(s > 0\), then one has the following inequalities satisfied
\[
a' \cdot d \geq a' \cdot c' \Rightarrow -a' \cdot d \leq -a' \cdot c' \Rightarrow (M - s) \cdot d - a' \cdot c' \leq (M - s) \cdot d - a' \cdot c' \Rightarrow (M - s) \cdot c' - a' \cdot d \leq (M - s) \cdot c' - a' \cdot d \Rightarrow (M - s) \cdot c' - a' \cdot d \leq (M - s) \cdot c' - a' \cdot d
\]
Therefore, \(F_2 \leq \frac{c'}{d}\), or equivalently
\[
\sum_{k=m+1}^{n} (B_k - B_k) \leq B_k - B_M
\]
From inequalities (25) and (29), we finally get
\[
F_1 + F_2 \leq (B_k - B_k - B_k - B_M) = 1
\]
Therefore, the inequality (21) is satisfied, and consequently inequality (19) is also satisfied, which completes the proof of the theorem.

REFERENCES
[41] J. Ding, D. Han, Y. Yang, Novel Instant-Runoff Ranking Fusion Approaches, 2015 IEEE Int. Conf. on Multisensor Fusion and Integration for Intelligent Systems (MFI), San Diego, CA, USA, Sept 14–16, 2015.