On the behavior of Dempster's rule of combination

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***Abstract*—In this paper we present simple examples showing the insensitivity of Dempster's rule of combination proposed in Dempster-Shafer Theory (DST) of evidence with respect to the level of conflict between two sources of evidences. Aside famous Zadeh's example on the validity of Dempster's rule of combination, it is shown that for an infinite number of cases Dempster's rule does not respond adequately to combine sources of evidence even when the level of conflict between sources is low. For a comparison purpose, we present the solution obtained by the more efficient PCR5 fusion rule proposed originally in Dezert-Smarandache Theory (DSmT) framework.**

Keywords: Data fusion, belief function, DSmT, DST.

I. Introduction

The variety of the fusion rules dealing with imperfect and uncertain information are based on different mathematical models and on different methods for transferring the conflicting mass of belief to specific hypotheses of the frame of the problem under consideration. DST [1] has been the first mathematical theory of evidence for combining uncertain information expressed as basic belief assignments. The fusion rule proposed by Shafer in DST is Dempster's rule of combination, i.e. the normalized conjunctive rule of combination. The legitimacy and the justification of Dempster's rule for combining sources of evidence has been a source of open endless strong debates in the community of users of belief functions since the publication in 1979 of Zadeh's famous example in [4], [5]. In this paper, we present new interesting examples to show another potential problem, or at least questionable behavior, of Dempster's rule of combination. Clearly, we show that in some case Dempster's rule is unable to respond adequately to the fusion of different basic belief assignments (bba's) whatever the level of conflict is. The problem is not due to the level of conflict between sources (contrary to Zadeh's example) but it is due to the inadequate normalization step done in Dempster's rule and the way the conflicting mass is redistributed back to focal elements through this normalization step. Such very simple examples reinforce the justification for using more efficient rules of combination to circumvent such unsatisfactory behavior of Dempster's rule, or at least to use Dempster's rule with extreme caution in applications.

Dezert-Smarandache Theory (DSmT) [2] has been developed to overcome the limitations of DST and circumvent the problems of inconsistency of Dempster's rule in possibly highly conflicting fusion problems and to go beyond the limitations of Shafer's model. DSmT distinguishes three possible classes of underlying models for the frame of discernment on which the basic belief assignment is defined depending on the nature of the problem: 1) Shafer's model (as in DST) considers the frame of discernment $\Theta = \{\theta_1, \dots, \theta_n\}$ as a finite set of n exhaustive and exclusive hypotheses according to the possible solutions of the problem under consideration; 2) A free-DSm model, is an opposite to Shafer's model and consists in assuming that all elements $\theta_i, i = 1, 2, ..., n$ of Θ are not exclusive. The only requirement is for their exhaustivity. Shafer's model can be considered as the most constrained model; 3) Between Shafer's and free models, there exists a set of fusion problems represented in term of DSm hybrid models where the frame of discernment could involve both fuzzy continuous and discrete hypotheses, in accordance with the exact nature of the problem. In DST the bba's $m(.)$ are defined on the power-set 2^Θ whereas in DSmT, and when working with and underlying hybrid model for the frame, the bba's are defined on the hyper-power set D^Θ that corresponds to Dedekind's lattice. The mathematical definition of D^Θ with many detailed examples can be found in [2], Vol. 1 and is out of the scope of this paper.

The main purpose of this paper being the analysis of Dempster's rule of combination, we will work in the classical DST framework where Shafer's model is assumed valid for the frame Θ and therefore we don't need to work with hyper-power set D^Θ. More precisely, one has $D^\Theta = 2^\Theta$ when Shafer's model holds. From a frame of discernment Θ, a basic belief assignment (bba) is defined [1] as a mapping $m(.) : 2^\Theta \to [0, 1]$ associated to a given source of evidence:

$$m(\emptyset) = 0 \qquad \text{and} \qquad \sum_{X \in 2^\Theta} m(X) = 1 \tag{1}$$

The elements of the power set having a strict positive mass of belief are called focal elements of $m(.)$. The set of all focal elements is called the core of $m(.)$ and is denoted $\mathcal{K}$. The measures of credibility and plausibility of any proposition $X \in 2^\Theta$ are defined from $m(.)$ by

$$\text{Bel}(X) \triangleq \sum_{\substack{Y \subseteq X \\ Y \in 2^\Theta}} m(Y) \tag{2}$$

$$\text{Pl}(X) \triangleq \sum_{\substack{Y \cap X \neq \emptyset \\ Y \in 2^\Theta}} m(Y) \tag{3}$$

A. Dempster's fusion rule

Dempster's rule of combination, also called Dempster-Shafer's (DS) rule since it was proposed by Shafer in his mathematical theory of evidence [1], is a normalized conjunctive operation. Based on Shafer's model of the frame, Dempster's rule for two sources is defined by $m_{DS}(\emptyset) = 0$, and $\forall X \in 2^{\Theta} \setminus \{\emptyset\}$ by

$$m_{DS}(X) = \frac{m_{12}(X)}{1 - m_{12}(\emptyset)} \tag{4}$$

where

$$m_{12}(X) \triangleq \sum_{\substack{X_1, X_2 \in 2^{\Theta} \\ X_1 \cap X_2 = X}} m_1(X_1) m_2(X_2) \tag{5}$$

corresponds to the conjunctive consensus on X between the two sources. The *degree of conflict* between the sources is defined by

$$K_{12} \triangleq m_{12}(\emptyset) = \sum_{\substack{X_1, X_2 \in 2^{\Theta} \\ X_1 \cap X_2 = \emptyset}} m_1(X_1) m_2(X_2) \tag{6}$$

The mass of conflict is distributed to all meaningful propositions (i.e. the non-empty elements belonging to the intersection of the cores of $m_1(.)$ and $m_2(.)$) of the power set through a simple normalization procedure[1] (with the division by $1 - m_{12}(\emptyset)$, assuming $m_{12}(\emptyset) < 1$). As already pointed out by Zadeh's [4], this rule has a very questionable behavior when $K_{12} \rightarrow 1$ because Dempster's rule can reflect the minority of opinions and moreover it is insensitive to inputs values as shown in [2], p. 114. That is why we have proposed to use the PCR5 fusion rule developed originally in DSmT framework [2]. It has been proved that PCR5 does not suffer of such unexpected behavior even in highly conflicting situations, but at the price of higher complexity for its implementation with respect to the complexity of Dempster's rule.

B. PCR5 fusion rule

The idea behind the Proportional Conflict Redistribution rule no. 5 [2] (Vol. 2) is to transfer conflicting masses (total or partial) proportionally to non-empty sets involved in the model according to all integrity constraints. The general principle of PCR rules is then to : *1*) calculate the conjunctive rule of the belief masses of sources; *2*) calculate the total or partial conflicting masses; *3*) redistribute the conflicting mass (total or partial) proportionally on non-empty sets involved in the model according to all integrity constraints. Under Shafer's model[2] of the frame Θ, the PCR5 combination rule for only two sources of information is defined as: $m_{PCR5}(\emptyset) = 0$ and $\forall X \in 2^{\Theta} \setminus \{\emptyset\}$

$$m_{PCR5}(X) = m_{12}(X) + \sum_{\substack{Y \in 2^{\Theta} \setminus \{X\} \\ X \cap Y = \emptyset}} [\frac{m_1(X)^2 m_2(Y)}{m_1(X) + m_2(Y)} + \frac{m_2(X)^2 m_1(Y)}{m_2(X) + m_1(Y)}] \tag{7}$$

where all denominators are different from zero. All sets involved in the formula are in canonical form. All denominators are different from zero. If a denominator is zero, that fraction is discarded. No matter how big or small is the conflicting mass, PCR5 mathematically does a better redistribution of the conflicting mass than Dempster's rule since PCR5 goes backwards on the tracks of the conjunctive rule and redistributes the partial conflicting masses only to the sets involved in the conflict and proportionally to their masses put in the conflict, considering the conjunctive normal form of the partial conflict. PCR5 is quasi-associative and preserves the neutral impact of the vacuous belief assignment.

II. A COUNTER-EXAMPLE TO DEMPSTER'S RULE

Here we point out a new (counter) example showing the inadequate behavior of Dempster's rule. This example is not related with the level of conflict between sources. In this example the level of conflict can be chosen at any low value and Dempster's rule is not responding adequately to the combination of different bba's since it provides always same results which is not a good expected behavior for a good fusion rule for applications. Stated otherwise, whatever the (strictly positive) level of conflict is, Dempster's rule gives always same result which is very surprising and counter-intuitive.

Example 1: Let's consider the following frame $\Theta = \{A, B, C\}$ with Shafer's model. We consider the following two bba's associated with two distinct bodies of evidence as follows with $0 < a < 1$ and $0 < b < 1$:

Focal elem. \ bba's	$m_1(.)$	$m_2(.)$
A	a	0
$A \cup B$	$1-a$	0
C	0	b
$A \cup B \cup C$	0	$1-b$

Table I
INPUT BBA'S $m_1(.)$ AND $m_2(.)$

Using the conjunctive consensus operator, one gets:

$$m_{12}(A) = a(1-b)$$
$$m_{12}(A \cup B) = (1-a)(1-b)$$
$$K_{12} = m_{12}(\emptyset) = m_1(A)m_2(C) + m_1(A \cup B)m_2(C) = b$$

Applying Dempster's rule by normalizing by $1 - K_{12} = 1 - b$, one gets

$$m_{DS}(A) = a(1-b)/(1-b) = a$$
$$m_{DS}(A \cup B) = (1-a)(1-b)/(1-b) = 1-a$$

Clearly, one sees in this example that:

[1]When the sources are in total contradiction, i.e. when $m_{12}(\emptyset) = 1$, then Dempster's rule cannot be used since one has a division by zero in (4).

[2]We consider only Shafer's model here to make the comparison with Dempster's rule results.

1) $m_2(.)$ plays here the same role as the vacuous belief assignment represented by $m_v(A \cup B \cup C) = 1$ since one gets finally $m_{DS}(.) = m_1(.)$.
2) Dempster's rule is not sensitive (i.e. adequately responding) to the values of the bba $m_2(.)$ because the result is independent of the input parameter b. This is not very satisfactory because Dempster's rule doesn't capture efficiently the impact of the level of conflict between two sources due to the inadequate normalization procedure.

If now we apply PCR5 rule of combination in this example, we obtain the following result:

$$m_{PCR5}(A) = a(1-b) + \frac{a^2 b}{a+b}$$
$$m_{PCR5}(A \cup B) = (1-a)(1-b) + \frac{(1-a)^2 b}{1-a+b}$$
$$m_{PCR5}(C) = \frac{ab^2}{a+b} + \frac{(1-a)b^2}{1-a+b}$$

It can be easily verified that $m_{PCR5}(.)$ is normalized. We see clearly that PCR5 does react more efficiently to the variations of the inputs bba's contrary to Dempster's rule. When b tends to zero, $m_{PCR5}(.)$ tends to $m_1(.)$ which is normal since $m_2(.)$ coincides with the vacuous belief assignment. When $b = 1$, then one combines the bba's of Table II

Focal elem. \ bba's	$m_1(.)$	$m_2(.)$
A	a	0
$A \cup B$	$1-a$	0
C	0	1

Table II
INPUT BBA'S $m_1(.)$ AND $m_2(.)$

Because of the principle of proportional redistribution of the masses of partial conflicts to the focal elements involved into them, one will obtain in this special case

$$m_{PCR5}(A) = \frac{a^2}{1+a}$$
$$m_{PCR5}(A \cup B) = \frac{(1-a)^2}{2-a}$$
$$m_{PCR5}(C) = \frac{a}{1+a} + \frac{1-a}{2-a}$$

The figures 1-3 show the evolution of fusion results. The figure 1 shows the mass of belief committed to A, the figure 2 the mass of $A \cup B$ and the figure 3 the mass of C obtained with Dempster's rule and PCR5 rules for couples (a, b) of input parameters varying in $[0, 1]$. One sees clearly the non-responding of Dempster's rule to the variation of the input parameter b involved in $m_2(.)$.

III. INFINITE CLASS OF COUNTER-EXAMPLES

The previous example showing the inadequate behavior of Dempster's rule is not unique and actually there exists an infinity of cases where this non-responding behavior occurs with Dempster's rule as we prove in this section.

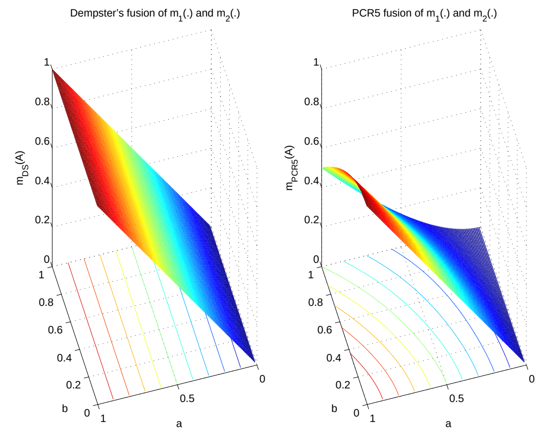

Figure 1. $m_{DS}(A)$ and $m_{PCR5}(A)$ for $a, b \in [0, 1]$.

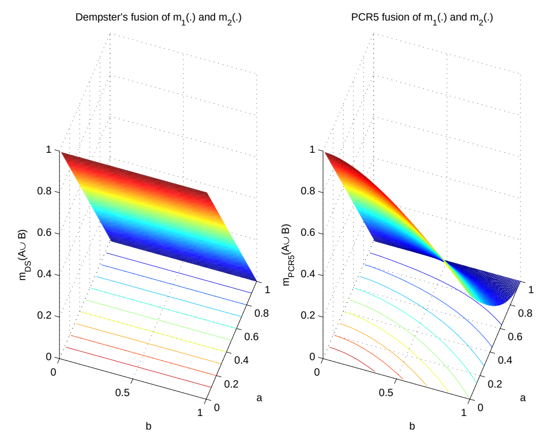

Figure 2. $m_{DS}(A \cup B)$ and $m_{PCR5}(A \cup B)$ for $a, b \in [0, 1]$.

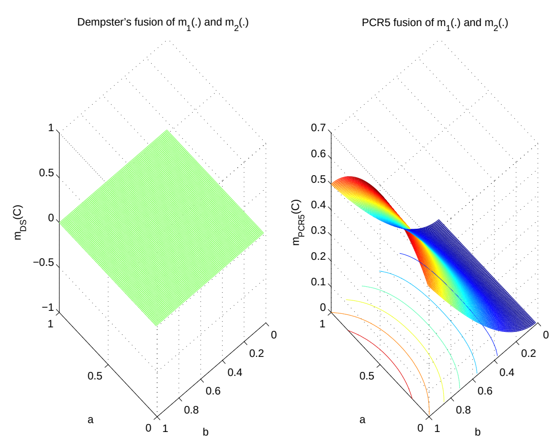

Figure 3. $m_{DS}(C)$ and $m_{PCR5}(C)$ for $a, b \in [0, 1]$.

Example 2: Let's modify a little bit the previous example by still considering $\Theta = \{A, B, C\}$ with Shafer's model and by taking the two non-Bayesian bba's given in Table III where $a \in [0, 1]$ and $b_1, b_2 > 0$ such that $b_1 + b_2 \in [0, 1]$.

Focal elem. \ bba's	$m_1(.)$	$m_2(.)$
A	a	0
$A \cup B$	$1-a$	b_1
C	0	$1-b_1-b_2$
$A \cup B \cup C$	0	b_2

Table III
INPUT BBA'S $m_1(.)$ AND $m_2(.)$

Using the conjunctive operator, one gets:

$$\begin{aligned} m_{12}(A) &= a(b_1+b_2) \\ m_{12}(A \cup B) &= (1-a)(b_1+b_2) \\ K_{12} = m_{12}(\emptyset) &= 1-b_1-b_2 \end{aligned}$$

which yields after the normalization by $1-K_{12} = b_1+b_2$,

$$\begin{aligned} m_{DS}(A) &= m_{12}(A)/(1-K_{12}) = a \\ m_{DS}(A \cup B) &= m_{12}(A \cup B)/(1-K_{12}) = 1-a \end{aligned}$$

Clearly, Dempster's rule is still insensitive to $m_2(.)$ for this new example because whatever the different values of input paremeters b_1 and b_2 are, the rule provides exactly the same result, i.e. $m_{DS}(.) \equiv m_1(.)$ even if $m_2(.)$ doesn't correspond to the vacuous belief basic assignment. It can be easily verified in this second example (and for any cases actually) that PCR5 does respond efficiently for combining these two bba's.

If we apply the PCR5 fusion in this example, the proportional redistribution of the mass of the partial conflict $m_1(A)m_2(C) = a(1-b_1-b_2)$ is done by

$$\frac{x_A}{m_1(A)} = \frac{x_C}{m_2(C)} = \frac{m_1(A)m_2(C)}{m_1(A)+m_2(C)} = \frac{a(1-b_1-b_2)}{a+1-b_1-b_2}$$

whence $x_A = \frac{a^2(1-b_1-b_2)}{a+1-b_1-b_2}$ and $x_C = \frac{a(1-b_1-b_2)^2}{a+1-b_1-b_2}$; and similarly the proportional redistribution of the mass of the partial conflict $m_1(A \cup B)m_2(C) = (1-a)(1-b_1-b_2)$ is done by

$$\frac{y_{A \cup B}}{m_1(A \cup B)} = \frac{y_C}{m_2(C)} = \frac{m_1(A \cup B)m_2(C)}{m_1(A \cup B)+m_2(C)}$$

whence $y_{A \cup B} = \frac{(1-a)^2(1-b_1-b_2)}{1-a+1-b_1-b_2}$ and $y_C = \frac{(1-a)(1-b_1-b_2)^2}{1-a+1-b_1-b_2}$.

Therefore with PCR5, one gets a fusion result that does react efficiently to the values of all the masses of focal elements of each source since one has

$$\begin{aligned} m_{PCR5}(A) &= m_{12}(A) + x_A \\ &= a(b_1+b_2) + \frac{a^2(1-b_1-b_2)}{a+1-b_1-b_2} \\ m_{PCR5}(A \cup B) &= m_{12}(A \cup B) + y_{A \cup B} \\ &= (1-a)(b_1+b_2) + \frac{(1-a)^2(1-b_1-b_2)}{2-a-b_1-b_2} \\ m_{PCR5}(C) &= x_C + y_C \\ &= \frac{a(1-b_1-b_2)^2}{a+1-b_1-b_2} + \frac{(1-a)(1-b_1-b_2)^2}{2-a-b_1-b_2} \end{aligned}$$

Example 3: Let's modify again a little bit the previous example by still considering $\Theta = \{A, B, C\}$ with Shafer's model and by taking the two non-Bayesian bba's given in Table IV where $a_1, a_2 > 0$ such that $a_1 + a_2 \in [0, 1]$ and $b_1, b_2 > 0$ such that $b_1 + b_2 \in [0, 1]$.

Focal elem. \ bba's	$m_1(.)$	$m_2(.)$
A	a_1	0
B	a_2	0
$A \cup B$	$1-a_1-a_2$	b_1
C	0	$1-b_1-b_2$
$A \cup B \cup C$	0	b_2

Table IV
INPUT BBA'S $m_1(.)$ AND $m_2(.)$

Using conjunctive operator, one gets:

$$\begin{aligned} m_{12}(A) &= a_1(b_1+b_2) \\ m_{12}(B) &= a_2(b_1+b_2) \\ m_{12}(A \cup B) &= (1-a_1-a_2)(b_1+b_2) \\ K_{12} = m_{12}(\emptyset) &= 1-b_1-b_2 \end{aligned}$$

which yields after the normalization by $1-K_{12} = b_1+b_2$,

$$\begin{aligned} m_{DS}(A) &= m_{12}(A)/(1-K_{12}) = a_1 \\ m_{DS}(B) &= m_{12}(B)/(1-K_{12}) = a_2 \\ m_{DS}(A \cup B) &= m_{12}(A \cup B)/(1-K_{12}) = 1-a_1-a_2 \end{aligned}$$

One sees that also in such very simple example 3, Dempster's rule is insensitive to the input parameters b_1 and b_2 of $m_2(.)$ because they are automatically simplified through the normalization procedure used in Dempster's formula. More generally, an infinite class of counter-examples based on a generalization of this type of examples can be easily constructed and therefore the following theorem holds.

It this example, in can be easily verified that PCR5 fusion provides the following result:

$$\begin{aligned} m_{PCR5}(A) &= a_1(b_1+b_2) + \frac{{a_1}^2(1-b_1-b_2)}{a_1+1-b_1-b_2} \\ m_{PCR5}(B) &= a_2(b_1+b_2) + \frac{{a_2}^2(1-b_1-b_2)}{a_2+1-b_1-b_2} \\ m_{PCR5}(C) &= \frac{a_1(1-b_1-b_2)^2}{a_1+1-b_1-b_2} + \frac{a_2(1-b_1-b_2)^2}{a_2+1-b_1-b_2} \\ &\quad + \frac{(1-a_1-a_2)(1-b_1-b_2)^2}{2-a_1-a_2-b_1-b_2} \\ m_{PCR5}(A \cup B) &= (1-a_1-a_2)(b_1+b_2) \\ &\quad + \frac{(1-a_1-a_2)^2(1-b_1-b_2)}{2-a_1-a_2-b_1-b_2} \end{aligned}$$

This result shows that PCR5 does react efficiently to the masses of all the focal elements of $m_2(.)$. It can be verified also that $m_{PCR5}(.)$ is normalized of course.

Theorem 1: Let's consider the frame of discernment $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$ with $n \geq 3$ and satisfying Shafer's model and a given element, say θ_i of Θ. If $m_1(.)$ is a dogmatic bba[3] having the core $\mathcal{K}_1 = \{\theta_j$ for some indexes $j \neq i, X\}$ where X denotes the disjunction of all θ_j (i.e. a partial ignorance), and if $m_2(.)$ is a non-dogmatic bba having the core $\mathcal{K}_2 = \{\theta_i, X, I_t\}$ where $I_t = \theta_1 \cup \theta_2 \cup \dots \cup \theta_i \cup \theta_j \cup \dots \cup \theta_n$ is the total ignorance, then Dempster's rule doesn't respond adequately to the fusion of sources since it is insensitive to $m_2(.)$, and one gets $m_{DS}(.) = m_1(.)$ whatever the masses of focal elements of $m_2(.)$ are.

Proof: We need to compute Dempster's fusion result for the combination of the following two normalized bba's satisfying theorem's conditions:

Focal elem. \ bba's	$m_1(.)$	$m_2(.)$
$\theta_j, j \neq i$	> 0	0
X_j	> 0	> 0
θ_i	0	> 0
I_t	0	> 0

Table V
INPUT BBA'S $m_1(.)$ AND $m_2(.)$

Applying directly the conjunctive operator yields

$$\begin{aligned}
m_{12}(\theta_j) &= m_1(\theta_j)[m_2(X) + m_2(I_t)] \\
m_{12}(X) &= m_1(X)[m_2(X) + m_2(I_t)] \\
K_{12} = m_{12}(\emptyset) &= m_2(\theta_i)[\sum_j m_1(\theta_j) + m_1(X)] \\
&= m_2(\theta_i) = 1 - m_2(X) - m_2(I_t)
\end{aligned}$$

After the normalization by $1 - K_{12} = m_2(X) + m_2(I_t)$ one gets:

$$\begin{aligned}
m_{DS}(\theta_j) &= m_{12}(\theta_j)/(1 - K_{12}) = m_1(\theta_j) \\
m_{DS}(X) &= m_{12}(X)/(1 - K_{12}) = m_1(X)
\end{aligned}$$

Hence $m_{DS}(.) = m_1(.)$ which completes the proof.

Note: Of course, the previous theorem can be a little bit relaxed by not forcing X to be a focal element of $m_1(.)$ and of $m_2(.)$. Actually, even if $m_1(X) = 0$ or/and $m_2(X) = 0$ are chosen in the previous proof, then one still gets $m_{DS}(.) = m_1(.)$. This corresponds to the case of Example 1.

Generalization of Theorem 1

Theorem 1 can also be easily extended by considering for $m_2(.)$, not only the focal elements I_t, X and θ_i, but also I_t, X and any θ_{i_1}, θ_{i_2}, $\dots$, θ_{i_k} and their possible partial ignorances which yields to Theorem 2 stated as follows:

[3]By definition [3], a dogmatic bba doesn't admit Θ as a focal element, i.e. the total ignorance has a zero mass of belief.

Theorem 2 (Generalization of Theorem 1): Let's consider: 1) the frame of discernment $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$ with $n \geq 3$ and satisfying Shafer's model; 2) a given non-empty (proper) subset of Θ denoted $\Theta' = \{\theta_{i_k}, i_k \in \{1, 2, \dots, n\}\}$. If $m_1(.)$ is a dogmatic bba having the core $\mathcal{K}_1 = \{\theta_j \in \{\Theta \setminus \Theta'\}, X\}$ where X denotes the disjunction of all θ_j (i.e. a partial ignorance), and if $m_2(.)$ is a non-dogmatic bba having the core $\mathcal{K}_2 = \{I_t, X, 2^{\Theta'} \setminus \{\emptyset\}\}$, then Dempster's rule doesn't respond adequately to the fusion of sources since it is insensitive to $m_2(.)$, and one gets $m_{DS}(.) = m_1(.)$ whatever the masses of focal elements of $m_2(.)$ are.

Proof: Applying directly the conjunctive operator yields

$$\begin{aligned}
m_{12}(\theta_j) &= m_1(\theta_j)[m_2(X) + m_2(I_t)] \\
m_{12}(X) &= m_1(X)[m_2(X) + m_2(I_t)] \\
K_{12} = m_{12}(\emptyset) &= [\sum_{Y \in 2^{\Theta'}} m_2(Y)][\sum_j m_1(\theta_j) + m_1(X)] \\
&= \sum_{Y \in 2^{\Theta'}} m_2(Y) = 1 - m_2(X) - m_2(I_t)
\end{aligned}$$

After normalization by $1 - K_{12} = m_2(X) + m_2(I_t)$ one gets:

$$\begin{aligned}
m_{DS}(\theta_j) &= m_{12}(\theta_j)/(1 - K_{12}) = m_1(\theta_j) \\
m_{DS}(X) &= m_{12}(X)/(1 - K_{12}) = m_1(X)
\end{aligned}$$

Hence $m_{DS}(.) = m_1(.)$ which completes the proof.

Note: If we want, we can also relax a little this theorem by not forcing X to be a focal element of $m_1(.)$ and of $m_2(.)$, and also not to force all the elements of $2^{\Theta'} \setminus \{\emptyset\}$ to be focal elements of $m_2(.)$ as shown in the next example.

Example 4: Let's take $\Theta = \{A, B, C, D, E, F\}$ satisfying Shafer's model and the two normalized non-Bayesian bba's given in Table VI, where at least one $a_i > 0$ and one $b_i > 0$.

Focal elem. \ bba's	$m_1(.)$	$m_2(.)$
A	a_1	0
B	a_2	0
$A \cup B$	$1 - a_1 - a_2$	b_1
C	0	b_2
D	0	b_3
$D \cup F$	0	b_4
$C \cup E \cup F$	0	b_5
$A \cup B \cup C \cup D \cup E \cup F$	0	$1 - \sum_{i=1}^{5} b_i$

Table VI
INPUT BBA'S $m_1(.)$ AND $m_2(.)$

Using conjunctive operator, one gets:

$$m_{12}(A) = a_1(b_1 + (1 - \sum_{i=1}^{5} b_i)) = a_1(1 - \sum_{i=2}^{5} b_i)$$
$$m_{12}(B) = a_2(b_1 + (1 - \sum_{i=1}^{5} b_i)) = a_2(1 - \sum_{i=2}^{5} b_i)$$
$$m_{12}(A \cup B) = (1 - a_1 - a_2)(b_1 + (1 - \sum_{i=1}^{5} b_i)) = (1 - a_1 - a_2)(1 - \sum_{i=2}^{5} b_i)$$
$$K_{12} = m_{12}(\emptyset) = 1 - b_1 - (1 - \sum_{i=1}^{5} b_i) = \sum_{i=2}^{5} b_i$$

which yields after the normalization by $1-K_{12} = 1-\sum_{i=2}^{5} b_i$,

$$m_{DS}(A) = m_{12}(A)/(1 - K_{12}) = a_1$$
$$m_{DS}(B) = m_{12}(B)/(1 - K_{12}) = a_2$$
$$m_{DS}(A \cup B) = m_{12}(A \cup B)/(1 - K_{12}) = 1 - a_1 - a_2$$

One sees that Dempster's rule is insensitive to the input parameters of $m_2(.)$ because they are automatically simplified through the normalization procedure used in Dempster's formula.

If we apply PCR5, one will get for this example (the verification is left to the reader):

$$m_{PCR5}(A) = a_1(1 - \sum_{i=2}^{5} b_i) + a_1^2 \sum_{i=2}^{5} \frac{b_i}{a_1 + b_i}$$
$$m_{PCR5}(B) = a_2(1 - \sum_{i=2}^{5} b_i) + a_2^2 \sum_{i=2}^{5} \frac{b_i}{a_2 + b_i}$$
$$m_{PCR5}(A \cup B) = (1 - a_1 - a_2)(1 - \sum_{i=2}^{5} b_i) + (1 - a_1 - a_2)^2 \sum_{i=2}^{5} \frac{b_i}{1 - a_1 - a_2 + b_i}$$
$$m_{PCR5}(C) = \frac{a_1 b_2^2}{a_1 + b_2} + \frac{a_2 b_2^2}{a_2 + b_2} + \frac{(1 - a_1 - a_2) b_2^2}{1 - a_1 - a_2 + b_2}$$
$$m_{PCR5}(D) = \frac{a_1 b_3^2}{a_1 + b_3} + \frac{a_2 b_3^2}{a_2 + b_3} + \frac{(1 - a_1 - a_2) b_3^2}{1 - a_1 - a_2 + b_3}$$
$$m_{PCR5}(D \cup F) = \frac{a_1 b_4^2}{a_1 + b_4} + \frac{a_2 b_4^2}{a_2 + b_4} + \frac{(1 - a_1 - a_2) b_4^2}{1 - a_1 - a_2 + b_4}$$
$$m_{PCR5}(C \cup E \cup F) = \frac{a_1 b_5^2}{a_1 + b_5} + \frac{a_2 b_5^2}{a_2 + b_5} + \frac{(1 - a_1 - a_2) b_5^2}{1 - a_1 - a_2 + b_5}$$

One can see again that PCR5 rule does react efficiently to the masses of all focal elements of the sources in the combination whereas Dempster's is unable to take properly into account the mass of focal elements of $m_2(.)$ since it works as if $m_2(.)$ was equal to the vacuous belief assignment $m_v(\Theta) = 1$. This is clearly what we consider as a very not satisfactory behavior of Dempster's rule.

IV. Conclusion

In this paper, we have introduced interesting counter-examples that show what we consider as a very unsatisfactory behavior of Dempster's rule of combination because Dempster's rule is insensitive to the second source, even if this second source is different from the vacuous belief assignment. The problem of Dempster's rule pointed out here doesn't come from the level of conflict between sources to combine (as in famous original Zadeh's example), but it comes only from the intrinsic structure of focal elements of sources and the inadequate normalization procedure. These examples are as important as Zadeh's example and show clearly that Dempster's rule can be non responding adequately to different bba's inputs in some cases and not necessarily only when sources are highly conflicting. Therefore, we consider that Dempster's rule must always be used with extreme caution in applications and we recommend to replace Dempster's rule by a more efficient rule of combination whenever possible. The PCR5 fusion rule appears to be a very good candidate for the combination of bodies of evidence because it has a more rational behavior and it also reduces the uncertainty of the combined basic belief assignment thanks to the powerful proportional conflict redistribution principle.

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