This appendix is the proof of Theorem 1 of paper "M. Boyer, Half-modeling of shaping in FIFO net with network calculus. In *Proceedings of the of the 18th International Conference on Real-Time and Network Systems*, Toulouse, France, November 2010."

A Proof of Theorem 1

To prove equality (23), as presented in the sketch of proof, we first have to restrict the value domain of the θ_i variables (eq. (24)).

The second step consists in having an expression of the delay to minimise, *i.e.* that the second term of (24)) can be reduced into (25).

The last step is the application of Lemma 5, which reduces expression $\inf_{x_i \geqslant 0} \left\{ \sum_{i=1}^n x_i + \bigvee_{i=1}^n [b_i - a_i x_i]^+ \right\}$.

A.1 First step: reducing the variable range

1. We are first going to restrict the θ_1 domain, *i.e.* proving eq.(24).

   Let us first show two intermediate results.

   (a) $\forall \theta_1 \in [0, T_1 + \frac{b_1}{R_1}], \dot{\theta}_1 = T_1 + \frac{b_1}{R_1} : \beta_{R_1', T^1_{\theta_1}} \mathbb{1}_{\{>\theta_1\}} \leqslant \beta_{R_1', T^1_{\dot{\theta}_1}} \mathbb{1}_{\{>\dot{\theta}_1\}}$

   PROOF First, notice that $\theta_1 \leqslant T^1_{\theta_1} \iff \theta_1 \leqslant T_1 + \frac{b_1}{R_1}$. And if $\theta_1 \leqslant T^1_{\theta_1}$, then $\beta_{R_1', T^1_{\theta_1}} \mathbb{1}_{\{>\theta_1\}} = \beta_{R_1', T^1_{\theta_1}}$, the test term is useless, we are handling simple rate-latency $\beta_{R,T}$ functions. Furthermore, $T \leqslant T' \Rightarrow \beta_{R,T} \geqslant \beta_{R,T'}$. Since $T^1_{\theta_1}$ is a decreasing function of θ_1, we have $\forall \theta_1 \in [0, T_1 + \frac{b_1}{R_1}], T^1_{\theta_1} \leqslant T^1_{\dot{\theta}_1}$. To sum up $\forall \theta_1 \in [0, T_1 + \frac{b_1}{R_1}]$:

   $$\beta_{R_1', T^1_{\theta_1}} \mathbb{1}_{\{>\theta_1\}} = \beta_{R_1', T^1_{\theta_1}} \leqslant \beta_{R_1', T^1_{\dot{\theta}_1}} = \beta_{R_1', T^1_{\dot{\theta}_1}}$$ ∎

   (b) $g \geqslant g' \Rightarrow h(f, g * h) \leqslant h(f, g * h)$.

   PROOF We know that $f \geqslant g \geqslant g' \Rightarrow h(f, g) \leqslant h(f, g')$. We also know that convolution is isotone on wide-sense increasing functions, that is to say, $g \geqslant g' \Rightarrow g * h \geqslant g * h)$. ∎

   With previous results, we know that, for all sets of value $\theta_1, \ldots, \theta_n$, if $\theta_1 \leqslant \dot{\theta}_1 = T_1 + \frac{b_1}{R_1}$, then $h(\bigwedge_i \gamma_{r_i,b_i}, \ast_{i=1}^n \beta_{R_i', T^i_{\theta_i}} \mathbb{1}_{\{>\theta_i\}}) \geqslant h(\bigwedge_i \gamma_{r_i,b_i}, \beta_{R_1', T^1_{\dot{\theta}_1}} \mathbb{1}_{\{>\dot{\theta}_1\}} \ast_{i=2}^n \beta_{R_i', T^i_{\theta_i}} \mathbb{1}_{\{>\theta_i\}})$.

   Then, we can restrict the research of the minimum to values of $\theta \geqslant T_1 + \frac{b_1}{R_1}$, *i.e.*

   $$\inf_{\forall i \in [1,n] : \theta_i \geqslant 0} h(\bigwedge_i \gamma_{r_i,b_i}, \ast_i \beta_{R_i', T^i_{\theta_i}} \mathbb{1}_{\{>\theta_i\}}) = \inf_{\theta_1 \geqslant T_1 + \frac{b_1}{R_1}, \forall i \in [2,n] : \theta_i \geqslant 0} h(\bigwedge_i \gamma_{r_i,b_i}, \ast_i \beta_{R_i', T^i_{\theta_i}} \mathbb{1}_{\{>\theta_i\}}) \qquad (26)$$

2. We can iterate the previous step to restrict the same way domains of θ_2 to θ_n.

A.2 Expression of delay

This proof uses a different way than [18]: they are using the pseudo-inverse of the function $\ast_{i=1}^n \beta_{R_i', T^i_{\theta_i}} \mathbb{1}_{\{>\theta_i\}}$, and we are using the distributivity of the distributivity of min on delay.

We first apply (4) on each server S_i. Then, each S_i offers to flow R a residual service $\beta_i^{\theta_i} = \beta_{R_i', T^i_{\theta_i}} \mathbb{1}_{\{>\theta_i\}}$ with $R_i' = R_i - r_i'$ and $T^i_{\theta_i} = \frac{R_i T_i + b_i' - r_i' \theta_i}{R_i'}$.

Then, the delay of R is the same as if it was traversing an unique server of service curve $\beta^{\theta_1,\ldots,\theta_n} = \ast_{i=1}^n \beta_{R_i', T^i_{\theta_i}} \mathbb{1}_{\{>\theta_i\}}$ and from Lemma 3 $\beta^{\theta_1,\ldots,\theta_n} = \delta_{\sum_{i=1}^n (\theta_i \vee T^i_{\theta_i})} \wedge \bigwedge_{i=1}^n \beta_{R_i', (\sum_{i=j}^n \theta_j \vee T^j_{\theta_j}) - [\theta_i - T^i_{\theta_i}]^+}$

The horizontal delay can be decomposed using Lemma 4. $h(\bigwedge_i \gamma_{r_i,b_i}, \beta^{\theta_1,\ldots,\theta_n}) = h(\bigwedge_i \gamma_{r_i,b_i}, \delta_{\sum_{i=1}^n (\theta_i \vee T^i_{\theta_i})}) \vee \bigvee_i h(\bigwedge_i \gamma_{r_i,b_i}, \beta_{R_i', (\sum_{i=j}^n \theta_j \vee T^j_{\theta_j}) - [\theta_i - T^i_{\theta_i}]^+})$.

From Lemma 1, we have an expression for the delay. Let $k_i' = \min \{ j \mid r_j \leqslant R_i - r_i' \}$.

Then, $h(\bigwedge_i \gamma_{r_i,b_i}, \beta^{\theta_1,\ldots,\theta_n}) = \sum_{i=1}^n [\theta_i \vee T^i_{\theta_i}]^+ \vee \bigvee_i \left\{ \sum_{j=1}^n [\theta_j \vee T^j_{\theta_j}]^+ - [\theta_i - T^i_{\theta_i}]^+ + \frac{y_{k_i'}}{R_i'} - x_{k_i'} \right\}$

We are only intersted in values of $\theta_i \geqslant T_i + \frac{b_i'}{R_i}$. In this cases, $\theta_i \geqslant T^i_{\theta_i}$, $[\theta_i - T^i_{\theta_i}]^+ = \frac{R_i \theta_i - R_i T_i - b_i'}{R_i - r_i'}$ and the expression be simplified.

Then, under assumption $\theta_i \geqslant T_i + \frac{b_i'}{R_i}$ we can rewrite the delay:

$$\begin{aligned}
& h(\bigwedge_i \gamma_{r_i,b_i}, \mathop{\ast}_{i=1}^{n} \beta_{R_i', T^i_{\theta_i}} \mathbb{1}_{\{>\theta_i\}}) \\
&= \sum_{i=1}^n \theta_i \vee \bigvee_{i=1}^n \left\{ \left(\sum_{i=1}^n \theta_i \right) - \theta_i + T^i_{\theta_i} + \frac{y_{k_i'}}{R_i'} - x_{k_i'} \right\} \\
&= \sum_{j=1}^n \theta_j \vee \bigvee_{i=1}^n \left\{ \left(\sum_{i=j}^n \theta_j \right) + \frac{R_i T_i + b_i' + y_{k_i'} - R_i \theta_i}{R_i - r_i'} - x_{k_i'} \right\} \\
&= \bigvee_{i=1}^n \left\{ \left(\sum_{j=1}^n \theta_j \right) + \left[\frac{R_i T_i + b_i' + y_{k_i'} - R_i \theta_i}{R_i - r_i'} - x_{k_i'} \right]^+ \right\}
\end{aligned}$$

with variable substitution $\theta'_i = \theta_i - (T_i + \frac{b'_i}{R_i})$ it becomes

$$= \bigvee_{i=1}^{n} \left\{ \left(\sum_{j=1}^{n} \theta'_j + T_j + \frac{b'_j}{R_j} \right) + \left[\frac{y_{k'_i} - R_i\theta'_i}{R_i - r'_i} - x_{k'_i} \right]^+ \right\}$$
$$= \left(\sum_{j=1}^{n} T_j + \frac{b'_j}{R_j} \right) + \bigvee_{i=1}^{n} \left\{ \left(\sum_{j=1}^{n} \theta'_j \right) + \left[\frac{y_{k'_i} - R_i\theta'_i}{R_i - r'_i} - x_{k'_i} \right]^+ \right\}$$

Then, looking for the inf of such expression leads to:

$$\inf_{\forall i \in [1,n]: \theta_i \geqslant 0} h \left(\bigwedge_i \gamma_{r_i, b_i}, \mathop{\ast}_{i=1}^{n} \beta_{R'_i, T^i_{\theta_i}} \mathbb{1}_{\{>\theta_i\}} \right) = \left(\sum_{i=1}^{n} T_i + \frac{b'_i}{R_i} \right) + \inf_{\forall i \in [1,n]: \theta'_i \geqslant 0} \bigvee_{i=1}^{n} \left\{ \left(\sum_{j=1}^{n} \theta'_j \right) + \left[\frac{y_{k'_i} - R_i\theta'_i}{R_i - r'_i} - x_{k'_i} \right]^+ \right\}$$

A.3 Last step

Here comes the last proof: giving an analytical solution to the inf term in (25). Abstracting from the specific network calculus problem, we are solving

$$\inf_{x_i \geqslant 0} \left\{ \sum_{i=1}^{n} x_i + \bigvee_{i=1}^{n} [b_i - a_i x_i]^+ \right\} \tag{27}$$

with $a_i = \frac{R_i}{R_i - r'_i} \geqslant 1$ and $b_i = \frac{y_{k'_i} - (R_i - r'_i)x_{k'_i}}{R_i - r'_i} \geqslant 0$ ([6]).

Then, using Lemma 5, we can compute the infimum of this expression. This is a generalisation of what is done in [18], where all b_i have the same value.

So, using Lemma 5, if $\sum_{i=1}^{n} \frac{R_i - r'_i}{R_i} \geqslant 1$, then, the delay value is

$$\sum_{i=1}^{n} \frac{y_{k'_i} - (R_i - r'_i)x_{k'_i}}{R_i} \tag{28}$$

Otherwise, the index must be re-ordered. Let $p : [1,n] \mapsto [1,n]$ be a permutation such that $\left(b_{p(i)}\right)$ is a wide-sense increasing sequence. Let $q \stackrel{\text{def}}{=} \min\left\{ p(i) \middle| 1 \geqslant \sum_{j=1}^{n-p(i)+1} \frac{R_i - r'_i}{R_i} \right\}$. In this case, the delay value is

$$\sum_{p(i) \leqslant k} \frac{y_{k'_i} - (R_i - r'_i)x_{k'_i}}{R_i} + \frac{y_{k'_q} - (R_q - r'_q)x_{k'_q}}{R_q - r'_q} \left(1 - \sum_{p(i) \leqslant k} \frac{R_i - r'_i}{R_i} \right)$$

A.4 A usefull Lemma on specific inf

Lemma 5 ($\inf_{x_i \geqslant 0} \left\{ \sum_{i=1}^{n} x_i + \bigvee_{i=1}^{n} [b_i - a_i x_i]^+ \right\}$) *Let be $\forall i \in [1,n] : b_i \geqslant 0, a_i \geqslant 1$. We can assume, without loss of generality, that the b_i are sorted in non decreasing order ($b_i \leqslant b_{i+1}$). Then*

$$\inf_{x_i \geqslant 0} \left\{ \sum_{i=1}^{n} x_i + \bigvee_{i=1}^{n} [b_i - a_i x_i]^+ \right\} = \begin{cases} \sum_{i=1}^{n} \frac{b_i}{a_i} & \textit{if } \sum_{i=1}^{n} \frac{1}{a_i} \leqslant 1 \\ \sum_{j=k}^{n} \frac{b_j}{a_j} + b_k \left(1 - \sum_{j=k}^{n} \frac{1}{a_j} \right) & \textit{otherwise} \end{cases} \tag{29}$$

with $k \stackrel{\text{def}}{=} \max\left\{ i \middle| 1 \geqslant \sum_{j=i}^{n} \frac{1}{a_j} \right\}$. □

This lemma have been solved in [18] in a more specific way:

$$\inf_{\tau'_i \geqslant 0} \left\{ \sum_i \tau'_i + \bigvee_i \left[\frac{\sigma - \rho^i \tau'_i}{\rho^i - \rho_i} \right]^+ \right\} \tag{30}$$

with $\sigma > 0$, $\rho^i \geqslant \rho_i \geqslant 0$. This problem can be also written, in a more general way $\inf_{\tau'_i \geqslant 0} \left\{ \sum_i \tau'_i + \bigvee_i \left[\frac{u - v_i \tau'_i}{w_i} \right]^+ \right\}$ with $u > 0$ $v_i, w_i \geqslant 0$. This problem is equivalent to (29), with $u = 1$, $b_i = \frac{1}{w_i}$, $a_i = \frac{v_i}{w_i}$, except the case $b_i = 0$. So, we redo the proof to see if the null case could makes a difference. It does not. Here is the proof.

In the following, $\bar{x}$ will denotes a vector of n values $(x_1, \ldots, x_n)$, and $f(\bar{x}) = f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i + \bigvee_{i=1}^{n} [b_i - a_i x_i]^+$. We are looking for the minimum for f on $\mathbb{R}^n_{\geqslant 0}$.

Except for step 1, this proof is a clone of the one of [18], generalised to handle different values of b_i.

1. $$\inf_{0 \leqslant x_i} \left\{ \sum_{i=1}^{n} x_i + \bigvee_{i=1}^{n} [b_i - a_i x_i]^+ \right\} = \inf_{0 \leqslant x_i \leqslant \frac{b_i}{a_i}} \left\{ \sum_{i=1}^{n} x_i + \bigvee_{i=1}^{n} (b_i - a_i x_i) \right\}$$

[6] The fact that theses b_i are non negative come from the definitions of k'_i: it is the smaller index such that the slopes before $x_{k'_i}$ is greater than $(R_i - r'_i)$. It implies that $y_{k'_i} \geqslant (R_i - r'_i)x_{k'_i}$

PROOF They are two contradictory terms in this expression: $\sum_{i=1}^{n} x_i$ increases with any x_i, and terms $b_i - a_i x_i$ decreases. More formally, for any vector of value $\bar{x} = (x_1, \ldots, x_n)$ with some $x_i \geqslant \frac{b_i}{a_i}$, then, it exist $\bar{x}' = (x_1, \ldots, x_{i-1}, \frac{b_i}{a_i}, x_{i+1}, \ldots, x_n)$ such that $f(\bar{x}') \leqslant f(\bar{x})$. ■

We have done exactly the same kind of proof for (24).

In [18], this is done in a different way: they first study the case $0 \leqslant x_i \leqslant \frac{b_i}{a_i}$ (which is "Case 1: $0 \leqslant \rho^i \tau_i' \leqslant \sigma$ for any i.")

2. $\inf_{0 \leqslant x_i \leqslant \frac{b_i}{a_i}} f(\bar{x}) = \bigwedge_{j=1}^{n} \inf_{\bar{x} \in X_j} \sum_{i=1}^{n} x_i + b_j - a_j x_j$
with $X_j \overset{\text{def}}{=} \left\{ \bar{x} \in \mathbb{R}^n_{\geqslant 0} \,\middle|\, \forall i : x_i \leqslant \frac{b_i}{a_i} \text{ and } b_i - a_i x_i \leqslant b_j - a_j x_j \right\}$

PROOF Another definition of X_j could be $\left\{ \bar{x} \,\middle|\, \forall i : x_i \leqslant \frac{b_i}{a_i} \text{ and } b_j - a_j x_j = \bigvee_{i=1}^{n} a_i - b_i x_i \right\}$. It obviously comes that $\bigcup_{j=1}^{n} X_j = \left\{ \bar{x} \in \mathbb{R}^n_{\geqslant 0} \,\middle|\, \forall i : x_i \leqslant \frac{b_i}{a_i} \right\}$. Then

$$\begin{aligned} &\inf_{0 \leqslant x_i \leqslant \frac{b_i}{a_i}} \left\{ \sum_{i=1}^{n} x_i + \bigvee_{i=1}^{n} (b_i - a_i x_i) \right\} \\ &= \bigwedge_{j=1}^{n} \inf_{\bar{x} \in X_j} \left\{ \sum_{i=1}^{n} x_i + \bigvee_{i=1}^{n} (b_i - a_i x_i) \right\} \\ &= \bigwedge_{j=1}^{n} \inf_{\bar{x} \in X_j} \left\{ \sum_{i=1}^{n} x_i + b_j - a_j x_j \right\} \end{aligned}$$ ■

The X_j are just a renaming of the T_j of [18].

3. $\inf_{\bar{x} \in X_j} \left\{ \sum_{i=1}^{n} x_i + b_j - a_j x_j \right\} = \inf_{0 \leqslant x_j \leqslant \frac{b_i}{a_i}} \sum_{i=1}^{n} \left[\frac{b_i - b_j + a_j x_j}{a_i} \right]^+ + b_j - a_j x_j$

PROOF Consider x_j as a constant. This function increases with any x_i ($i \neq j$). Then, its $\inf$ is reached with the smallest values of each x_i in the interval range.

$$\begin{aligned} & 0 \leqslant b_i - a_i x_i \leqslant b_j - a_j x_j \\ \iff & -b_i \leqslant -a_i x_i \leqslant b_j - b_i - a_j x_j \\ \iff & \frac{b_i}{a_i} \geqslant x_i \geqslant \frac{-b_j + b_i + a_j x_j}{a_i} \end{aligned}$$

We also known that $x_i \geqslant 0$. Then, the minimal value for each x_i is $\left[\frac{b_i - b_j + a_j x_j}{a_i} \right]^+$, and each $x_i \neq x_j$ can be replaced by this value.

$$\inf_{\bar{x} \in X_j} \left\{ \sum_{i=1}^{n} x_i + b_j - a_j x_j \right\} = \inf_{0 \leqslant x_j \leqslant \frac{b_i}{a_i}} \sum_{\substack{i=1 \\ i \neq j}}^{n} \left[\frac{b_i - b_j + a_j x_j}{a_i} \right]^+ + x_j + b_j - a_j x_j$$

Notice that $x_j = \frac{b_j - b_j + a_j x_j}{a_j}$, then it comes.

$$\inf_{0 \leqslant x_j \leqslant \frac{b_i}{a_i}} \sum_{i=1}^{n} \left[\frac{b_i - b_j + a_j x_j}{a_i} \right]^+ + b_j - a_j x_j$$ ■

4. $\inf_{0 \leqslant x_i} \left\{ \sum_{i=1}^{n} x_i + \bigvee_{i=1}^{n} [b_i - a_i x_i]^+ \right\} = \inf_{0 \leqslant x \leqslant b_n} \sum_{i=1}^{n} \left[\frac{b_i - x}{a_i} \right]^+ + x$

PROOF In the previous steps, we have reduced the problem $\inf_{0 \leqslant x_i} \left\{ \sum_{i=1}^{n} x_i + \bigvee_{i=1}^{n} [b_i - a_i x_i]^+ \right\}$ into $\bigwedge_{j=1}^{n} \inf_{0 \leqslant x_j \leqslant \frac{b_j}{a_j}} \sum_{i=1}^{n} \left[\frac{b_i - b_j + a_j x_j}{a_i} \right]^+ + b_j - a_j x_j$

Let us redefine the independent variable x_j into $x_j' = b_j - a_j x_j$. We have $0 \leqslant x_j \leqslant \frac{b_j}{a_j} \iff 0 \geqslant -a_j x_j \geqslant -b_j \iff b_j \geqslant x_j' \geqslant 0$. Then

$$\bigwedge_{j=1}^{n} \inf_{0 \leqslant x_j \leqslant \frac{b_j}{a_j}} \sum_{i=1}^{n} \left[\frac{b_i - b_j + a_j x_j}{a_i} \right]^+ + b_j - a_j x_j = \bigwedge_{j=1}^{n} \inf_{0 \leqslant x_j' \leqslant b_j} \sum_{i=1}^{n} \left[\frac{b_i - x_j'}{a_i} \right]^+ + x_j'$$

But there is no need to keep n different variables x_j', since they is no dependency between them: they appear into different subterms. Let us denote simply $x = x_j'$. The term then becomes

$$\bigwedge_{j=1}^{n} \inf_{0 \leqslant x \leqslant b_j} \sum_{i=1}^{n} \left[\frac{b_i - x}{a_i} \right]^+ + x$$

It can be re-written as $\bigwedge_{j=1}^{n} \inf_{0 \leqslant x \leqslant b_j} f(x)$ with $f(x) \overset{\text{def}}{=} \sum_{i=1}^{n} \left[\frac{b_i - x}{a_i} \right]^+ + x$ function. This is always the same expression which is studied, on intervals $[0, b_i]$. But, the infimum is of course meet on the larger interval, *i.e.* $\bigwedge_{j=1}^{n} \inf_{0 \leqslant x \leqslant b_j} f(x) = \inf_{0 \leqslant x \leqslant \bigvee_{i=1}^{n} b_j} f(x) = \inf_{0 \leqslant x \leqslant b_n} f(x)$ ■

We now have a piecewise linear continuous function, whose minimum must be found.

5. $\inf_{0 \leqslant x \leqslant b_n} \sum_{i=1}^{n} \left[\frac{b_i - x}{a_i}\right]^+ + x =$

$$\begin{cases} \sum_{i=1}^{n} \frac{b_i}{a_i} & \text{if } \sum_{i=1}^{n} \frac{1}{a_i} \leqslant 1 \\ \sum_{j=k}^{n} \frac{b_j}{a_j} + b_k \left(1 - \sum_{j=k}^{n} \frac{1}{a_j}\right) & \text{otherwise} \end{cases}$$

with $k \stackrel{\text{def}}{=} \max \left\{ i \,\middle|\, 1 \geqslant \sum_{j=i}^{n} \frac{1}{a_j} \right\}$.

PROOF We have to find the minimum on $[0, b_n]$ of the piecewise linear function $f(x) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \left[\frac{b_i - x}{a_i}\right]^+ + x$. The x term is increasing, and each $\left[\frac{b_i - x}{a_i}\right]^+$ is decreasing up to b_i. Let us define $b_0 = 0$.

On each interval $[b_{i-1}, b_i]$, the function value is $\sum_{j=i}^{n} \frac{b_j - x}{a_j} + x$, and its slope is $1 - \sum_{j=i}^{n} \frac{1}{a_j}$.

If $\sum_{j=1}^{n} \frac{1}{a_j} \leqslant 1$, then, the function is wide sense increasing, and the minimum is reached at $x = 0$ (and the value is $\sum_{i=1}^{n} \frac{b_i}{a_i}$).

Otherwise, the functions is decreasing, up to an inflexion point b_k where is began to increase (we know that is can not be always decreasing: on last interval, $[b_{n-1}, b_n]$ the slope value is $1 - \frac{1}{a_n}$, and all $a_i \geqslant 1$.

Let be $k \stackrel{\text{def}}{=} \max \left\{ i \,\middle|\, 1 \geqslant \sum_{j=i}^{n} \frac{1}{a_j} \right\}$. Then, the function f reaches its minimum at b_k, and its value is

$$\begin{aligned} \sum_{j=k}^{n} \frac{b_j - b_k}{a_j} + b_k &= \sum_{j=k}^{n} \frac{b_j}{a_j} + b_k \left(1 - \sum_{j=k}^{n} \frac{1}{a_j}\right) \\ &= \sum_{j=k}^{n} \frac{b_j}{a_j} + b_k \left(1 - \sum_{j=k}^{n} \frac{1}{a_j}\right) \blacksquare \end{aligned}$$

It should be noticed that, with convention $a_0 = \infty$, $b_0 = 0$, both formulas for cases $\sum_{j=1}^{n} \frac{1}{a_j} \leqslant 1$ and $\sum_{j=1}^{n} \frac{1}{a_j} > 1$ are equivalents.