

Half-modeling of shaping in FIFO net with network calculus

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Abstract

Shaping is a well known solution to increase performances in networks, and especially worst-case performances. In the avionic context, considering the AFDX embedded backbone, it had been shown that modelling the shaping introduced by the links leads to significant gain. This shaping have been modelled to compute local delay with network calculus in previous studies. In this paper, we try to model it for an end-to-end delay where each server is shared by two flows with a FIFO policy. Due to technical difficulties, only the shaping of one of the two flows is modelled (giving the title “half-modeling of shaping”).

1 Introduction

Shaping is a well known solution to increase performances in networks, and especially worst-case performances. In the avionic context, considering the AFDX embedded backbone, it had been shown in [1, Table 2] that modelling the shaping introduced by the links¹ on realistic configuration can lead to gain from 10% to 50%, depending on the formal method used, the data flow considered, etc.

Among other methods, network calculus [12, 13, 16, 10] can be used to compute bounds on worst-case traversal time of network (WCTT). Network calculus has a solid mathematical background (the (min,plus) algebra), is scalable (useful classes of modelling have linear algorithms [9]), and have been used to certify the A380 AFDX backbone [15, 14].

This paper aims to model in network calculus two characteristic of AFDX networks: FIFO policy and shaping introduced by links. Both have been studied independently, and the aim is to combine both. In this first work, we consider a simple topology, a sequence of servers, each one

¹Depending on authors, this impact of the links is called *grouping*, to illustrate the fact that the flows sharing a single link can be seen as a “group”, or *serialisation*, to highlight the fact that the frames on the link are transmitted in sequence. From network calculus point of view, this effect is taken into account by modelling the link as a shaper.

shared by two flows with a FIFO policy (cf Figure 1). Moreover, the modelling is incomplete: for technical reasons, when computing the WCTT of a flow, we model only the shaping on the considering flow R , not on the interfering ones R'_i (leading to the title “half-modeling of shaping”). The main results of the paper consists in solving a well-known equation related to FIFO scheduling (presented in eq (4)) on this specific topology and one shaped flow (Theorem 1).

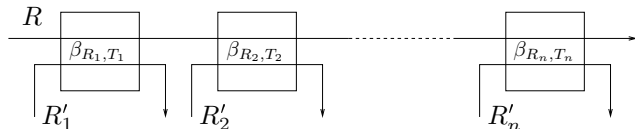


Figure 1. Tandem topology

As presented on small examples, depending on the bursts sizes, this could lead to significant gains (cf Table 1). But on realistic industrial example, the method fully modelling the shaping and neglecting the FIFO is still the better one. This encourages us to continue our works and fully combine shaping and end-to-end FIFO impact.

After a short introduction on network calculus in Section 2, related works will be presented in Section 3, which will also include a detailed presentation on the model considered in this paper (Section 3.3). The main contribution of the paper is in Section 4. Section 5 gives some numerical results on a simple example, to illustrate the gains of the new method. Section 6 concludes.

2 Network calculus

Notations \mathbb{R} denotes the set of real numbers. The minimum (resp. maximum) operator is denoted \wedge (resp. \vee). $[x]^+ \stackrel{\text{def}}{=} x \vee 0$

Here is a (very short) introduction to network calculus. The reader should refer to [12, 13] for the first works and [10, 16] for a complete presentation.

Network calculus is a theory to get deterministic upper bounds in networks. It is mathematically based on the $(\wedge, +)$ dioid.

Network calculus mainly handles non decreasing functions, null before 0 : \mathcal{F} . They are, among others, four common curves (parametrised by real positive values d, R, T, r, b) latency δ_d , rate λ_R , rate-latency $\beta_{R,T}$, token bucket $\gamma_{r,b}$, test $\mathbb{1}_{\{>T\}}$ defined by:

$$\delta_d(t) = \begin{cases} 0 & \text{if } t \leq d \\ \infty & \text{otherwise} \end{cases} \quad \mathbb{1}_{\{>T\}}(t) = \begin{cases} 1 & \text{if } t > T \\ 0 & \text{otherwise} \end{cases}$$

$$\gamma_{r,b}(t) = (rt + b)\mathbb{1}_{\{>0\}}(t) \quad \lambda_R(t) = Rt \\ \beta_{R,T} = R[t - T]^+$$

Three basic operators on \mathcal{F} are of interest, convolution $*$, deconvolution \oslash and the sub-additive closure f^* .

$$\mathcal{F} = \left\{ f : \mathbb{R} \rightarrow \mathbb{R} \mid \begin{array}{l} x < y \implies f(x) \leq f(y) \\ x < 0 \implies f(x) = 0 \end{array} \right\}$$

$$(f * g)(t) = \inf_{0 \leq u \leq t} (f(t-u) + g(u)) \quad (1)$$

$$(f \oslash g)(t) = \sup_{0 \leq u} (f(t+u) - g(u)) \quad (2)$$

$$f^* = \delta_0 \wedge f \wedge (f * f) \wedge (f * f * f) \wedge \dots \quad (3)$$

They are several well-known properties of these operators and the common curves. The convolution is associative and commutative, $\delta_t * \delta_{t'} = \delta_{t+t'}$, $f * \delta_d(t) = f(t-d)$ if $t \geq d$, 0 otherwise, $f \oslash \delta_d(t) = f(t+d)$, $\beta_{R,T} = \delta_T * \lambda_R$.

A flow is represented by its cumulative function $R \in \mathcal{F}$, where $R(t)$ is the total number of bits sent by this flow up to time t . A flow R is said to have a function α as *arrival curve* iff $\forall t, s \geq 0 : R(t+s) - R(t) \leq \alpha(s)$. It means that, from any instant t , the flow R will produce at most $\alpha(s)$ new data in s time units. An equivalent condition, expressed in the $(\wedge, +)$ dioid is $R \leq R * \alpha$. If α is an arrival curve for R , also is α^* . A server offers a *service* of curve β iff for all arrival flow, the relation between the input flow R and the output flow R' , we have $R' \geq R * \beta$. In this case, $\alpha' = \alpha \oslash \beta$ is an arrival curve for R' . The delay experimented by the flow R can be bounded by the maximal horizontal difference between curves α and β , formally defined by $h(\alpha, \beta)$ (a graphical interpretation of h can be found in Figure 2).

$$h(\alpha, \beta) = \sup_{s \geq 0} (\inf \{ \tau \geq 0 \mid \alpha(s) \leq \beta(s + \tau) \})$$

These first results allow to handle linear topologies, like the one of Figure 3. Given the arrival curve α of flow R , and the services curves β, β' of network elements S, S' , we are able to get a bound² on the delay in S : $h(\alpha, \beta)$, and another for the delay in S' : $h((\alpha \oslash \beta)^*, \beta')$.

²To effectively compute these bounds, we need algorithms computing operations like $\oslash, *, h(\cdot, \cdot)$. This is the aim of [3] and its COINC [11] implementation.

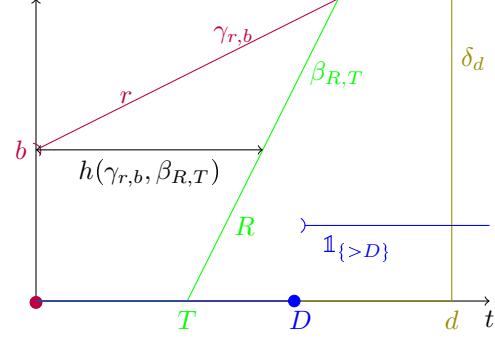


Figure 2. Common curves and delay

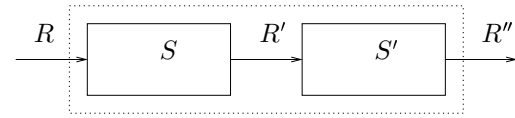


Figure 3. A flow going through two network elements in sequence

But one famous result of the network calculus is known as “pay burst only once” (PBOO). It states that a sequence of two servers, S and S' , with respective service of curve β, β' , like the one of Figure 3, can be seen as a single server with service of curve $\beta * \beta'$. It's interest comes from the fact that the end-to-end delay is lesser than the sum of local delays (*i.e.* $h(\alpha, \beta * \beta') \leq h(\alpha, \beta) + h((\alpha \oslash \beta)^*, \beta')$). And the difference can be, in practise, significant. It's popularity perhaps comes from the simplicity of the proof³.

In case of more realistic topology, when a network element is shared by different flows (like in Figure 4), with some service policy (FIFO, static priority, etc.), it also exists results to compute bounds on each flow. The idea is to extract, from an *aggregated* flow, the *residual* service offered to each flow (also known as *per-flow* service). This extraction is, at first step, local, *i.e.* on a single server. Depending on the policy, the bounds could be tight (*i.e.* it exists some configuration where the computed bound is reached), or not, or the result being up to now unknown. The second step consists in considering a complete topology, trying to get an end-to-end bound better than the sum of local delays (like the PBOO result). As for local step, depending on the policy *and* the topology, the bounds could be tight or not.

For example, with the non preemptive static priority policy, it is known⁴ that the lower priority flow has the residual

³Server S' offers a service of curve β means that $R'' \geq R' * \beta'$. The same for S : $R' \geq R * \beta'$. It implies $R'' \geq (R * \beta) * \beta'$ *i.e.* by associativity, $R'' \geq R * (\beta * \beta')$, which is the definition of a server with service $\beta * \beta'$.

⁴To be precise, there is a restriction on the flavor of service, which must be *strict*. See [16, Def 1.3.2, Cor. 6.2.1.] for details.

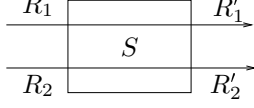


Figure 4. A single shared network element

service $\beta_L = [\beta - \alpha_H]^+$, if the network element has service curve β and the high priority flow has arrival curve α_H .

The common result for FIFO policy (the one focused in this paper) claims that, if a FIFO server (with a service of curve β) is shared by two flows R_1, R_2 (with respective arrival curves α_1, α_2), then, for each $\theta \geq 0$, each flow R_i receives the residual service β_i^θ defined by

$$\beta_i^\theta = [\beta - \alpha_j \oslash \delta_\theta]^+ \mathbb{1}_{\{>\theta\}} \quad (4)$$

with $\{i, j\} = \{1, 2\}$ (see [16, Prop 6.4.1] for details). An equivalent definition is $\beta_i^\theta(t) = [\beta(t) - \alpha_j(t - \theta)]^+ \mathbb{1}_{\{>\theta\}}$.

Notice that this result does not define a *single* residual service, but an *infinite set* of services, one for each value of θ . They all compute true bounds, but some are better than others. The issue is the choice of a good θ value.

Another result must be presented, so simple that nobody did ever present it as a result. It is used in [15], and claims that, if d denotes the delay experimented by the aggregated flow, then a FIFO server can be under-approximated by a variable delay d , and β_d is a residual service curve for each flow R_i .

$$\beta_d = \delta_{h(\beta, \alpha_1 + \alpha_2)} \quad (5)$$

3 Related works and considered model

This paper is the merging and continuation of two works. On the one hand, the one of the Computer Networking Group of the Pisa University on FIFO bounds with network calculus [18, 17, 19]. On the other hand, the one of the IRT team from Toulouse University on AFDX networks [15, ?, 14].

3.1 LUB from Computer Networking Group, Pisa University

As presented in section 2, to handle real case studies, with servers shared by different flows, with network calculus, the first step is to handle locally a service policy, and the second is the handling of a complete topology. From this point of view, the work presented in [18, 17, 19], called LUB (least upper bound), is the generalisation of equation (4), trying to compute an end-to-end delay better than the sum of local delays. Considering more and more complex topologies, they consider the possible ways to chose

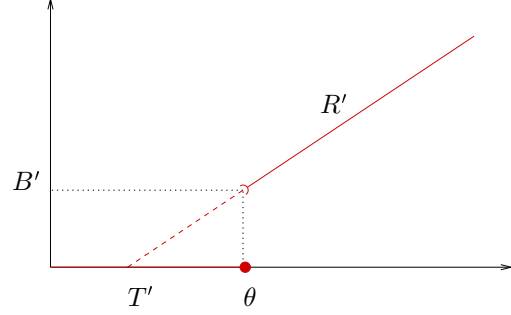


Figure 5. Local residual service

the θ values on each server, the ways to combine local computation, to get better bounds. Some common restriction are done on the class of functions used: arrival curves are token bucket $(\gamma_{r,b})$ and service curves are rate-latency $(\beta_{R,T})$.

In [18], they are considering what they call *tandem* topologies, *i.e.* a sequence of servers, with the flow of interest traversing all servers, and sharing each server with an interfering flow, like in Figure 1. This is also the kind of topology considered in this paper. The idea is, at each node S_i , to compute the residual service with (4), parametrised by a value θ_i , to get a parametrised expression of the end-to-end delay bound (to benefit of the PBOO result), and to minimise this expression. In [17], this approach is generalised by considering sink-tree networks, and in [19] generic topologies (without cyclic dependencies in data flows).

Let's go to mathematical details of [18].

The service at each node S_i has a curve β_{R_i, T_i} , and each interfering flow has arrival curve $\gamma_{r'_i, b'_i}$. For each $\theta_i \geq 0$, the residual service can be computed using (4).

$$\beta_i^{\theta_i} = [\beta_{R_i, T_i} - \gamma_{r'_i, b'_i} \oslash \delta_{\theta_i}]^+ \mathbb{1}_{\{>\theta_i\}} = \beta_{R'_i, T'_i} \mathbb{1}_{\{>\theta_i\}} \quad (6)$$

with $R'_i = R_i - r'_i$ and $T'_i = \frac{R_i T_i + b_i - r_i \theta_i}{R_i - r'_i}$. But this expression can be seen as the convolution of a delay and a token bucket

$$\beta_i^{\theta_i} = \beta_{R'_i, T'_i} \mathbb{1}_{\{>\theta_i\}} = \delta_{\theta_i} * \gamma_{R'_i, B'_i} \quad (7)$$

with $B'_i = R_i(\theta_i + T_i) + b'_i$, as illustrated in Figure 5.

The end-to-end residual service is the convolution of the local residual services $\beta_{R'_1, T'_1} \mathbb{1}_{\{>\theta_1\}} * \beta_{R'_2, T'_2} \mathbb{1}_{\{>\theta_2\}} * \dots * \beta_{R'_n, T'_n} \mathbb{1}_{\{>\theta_n\}} = \delta_{\sum_i \theta_i} * (\gamma_{R'_1, B'_1} \wedge \dots \wedge \gamma_{R'_n, B'_n})$. It can be seen as the convolution of a delay and the minimum of affine functions (cf. Figure 6). Such a curve will be called “pseudoaffine curve” in [17]. Computing the end-to-end delay bound consists in computing the horizontal deviation $h(\gamma_{r,b}, \delta_{\sum_i \theta_i} * (\gamma_{R'_1, B'_1} \wedge \dots \wedge \gamma_{R'_n, B'_n}))$.

Last step consists in minimising the previous expression, which implies to compute the following infimum, with $b_i \geq$

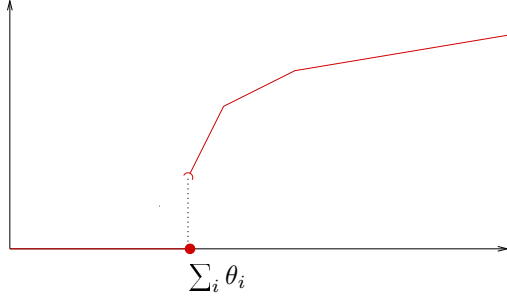


Figure 6. End-to-end residual service

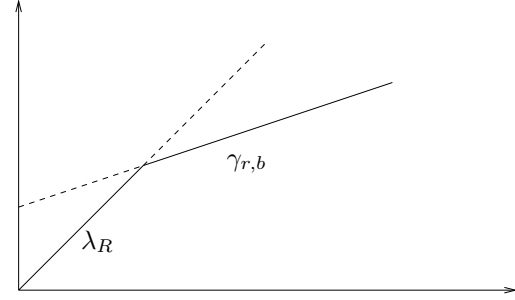


Figure 7. Shaped token bucket

0 and $a_i \geq 1$.

$$\inf_{x_i \geq 0} \left\{ \sum_i x_i \wedge \bigwedge_i [b - a_i x_i]^+ \right\} \quad (8)$$

3.2 IRT, Toulouse University

In [15], the approach is different: only local delays are computed, there is no attempt of modelling the PBOO principle. The global delay of each server is computed, *i.e.* all flows incoming into a server are aggregated (the aggregated arrival curve is simply the sum of individual arrival curve), and the delay experimented by this aggregated flow is computed. Then, the delay computed for a flow is simply the sum of the aggregated delay of each server crossed by the flow.

The contribution of [15], in addition to modelling the AFDX network in network calculus, is to model the shaping introduced by the links (called “grouping” in [15, 14, 1]). On a sender, whatever the applicative burst is, the throughput outgoing of the server is limited by the throughput of the link (100Mb/s in common AFDX). Another point of view, looking at trajectories of frames, is that links *serialise* packets. In network calculus modelling, it means that an applicative token bucket $\gamma_{r,b}$ is shaped by the bit-rate of the link λ_R , leading to a two-slopes affine arrival curve, as presented in Figure 7. But in switches, we have to consider the sum of flow incoming from different links, which is no more two-slope affine curve, as presented in Figure 8. It leads to general concave piecewise linear function (CPL).

Let’s go to mathematical details.

Each incoming flow f_i of a server S has arrival curve α_i , in the CPL class. The server as a service of curve β_S . The aggregated arrival curve $\alpha_S = \sum_i \alpha_i$ is still a CPL. The delay bound computed is $d_S = h(\beta_S, \sum_i \alpha_i)$. Because of FIFO policy, δ_{d_S} is a service curve for each flow (cf eq (5)), and each f'_i , output corresponding to the f_i input has $(\alpha_i \oslash \delta_{d_S})^*$ as an arrival curve, which is still a CPL.

The delay experimented by a flow is then the sum of the d_S delays of all the S crossed servers.

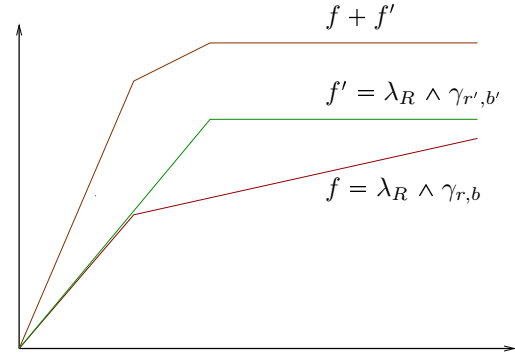


Figure 8. Sum of shaped token bucket gives general CPL curve

This approach does not consider the PBOO principle, but models correctly the shaping introduced by the links. This shaping can increase the performances of the analysis up to 40%. And even when considering another theory, based on trajectories, modelling of shaping leads to 5%–50% gain [1].

3.3 System model

This work is a direct continuation and join of the works presented in previous sections 3.1 and 3.2. We are considering exactly the same kind of topology than the one of [18], illustrated in Figure 1, but with a more general class of arrival curve: the considered flow has a CPL arrival curve, but the interfering flows still have token-bucket $\gamma_{r'_i, b'_i}$ arrival curves.

4 Computing the minimal bound

To begin, let us introduce some notations on the kind of manipulated curves. A concave piecewise linear (CPL) function is said to be under normal form if the terms are sorted by decreasing slopes, and there is no useless term (cf the *non-redundant* property of pseudoaffine curves in [19]).

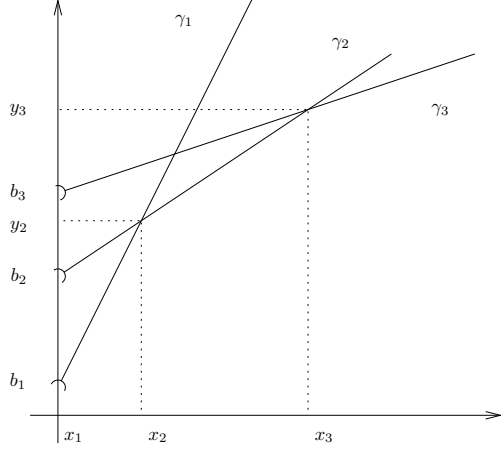


Figure 9. Normal form of CPL

Definition 1 (Normal form of CPL) Let

$\gamma_{r_1, b_1}, \dots, \gamma_{r_n, b_n}$ be a set of token-bucket functions. Let γ_i denotes γ_{r_i, b_i} . The term $\bigwedge_{i=1}^n \gamma_i$ is said to be *under normal form of minimum of γ functions*, iff there is no useless constraint (9) and the γ_i are sorted by decreasing rate (10).

$$\forall i, \exists t_i > 0, \forall j \neq i : \gamma_i(t_i) < \gamma_j(t_i) \quad (9)$$

$$i < j \Rightarrow r_i > r_j \quad (10)$$

If $\bigwedge_{i \in [1, n]} \gamma_{r_i, b_i}$ is under normal form, the sequence x_1, \dots, x_{n+1} of intersection points, and y_1, \dots, y_{n+1} the intersection values are formally defined by:

$$\begin{cases} x_1 = 0, y_1 = b_1 \\ \gamma_i(x_i) = \gamma_{i+1}(x_i) = y_k & \text{for } 1 \leq i \leq n \\ x_{n+1} = y_{n+1} = \infty \end{cases} \quad (11) \quad \square$$

It should be obvious that each CPL has a normal form. To effectively compute it, an algorithm is given in [8].

An example of such a set and related definitions is shown in Figure 9.

Lemma 1 (Horizontal dev. between CPL and rate-latency)

Let $\bigwedge_i \gamma_{r_i, b_i}$ be a CPL under normal form, and R, T , two non negative reals, with $r_n \leq R$. Then, the horizontal deviation can be computed

$$h\left(\bigwedge_i \gamma_{r_i, b_i}, \beta_{R, T}\right) = T + \frac{y_k}{R} - x_k \quad (12)$$

with $k = \min \{i \mid r_i \leq R\}$. \square

PROOF Graphically (see Figure 10), the maximal distance must be computed at one y_i level. Going from the origin, while the rates r_i are greater than R , the horizontal deviation grows, up to x_k , which is, by definition, the first slope lesser than R . \blacksquare

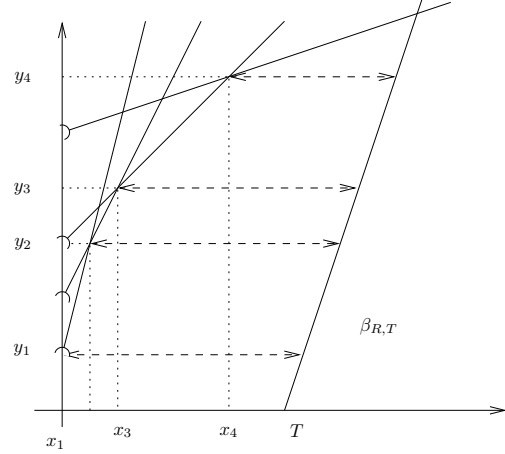


Figure 10. Horizontal deviation between CPL and rate-latency

To make the different proofs in this paper, we manipulate a lot of $\beta_{R, T} \mathbb{1}_{\{> T'\}}$ related expressions (going from eq. (4), cf Figure 6). Here are expressed different points of view on such expressions. They are of interest because of the different properties on the involved operators.

Lemma 2 (Some point of view on $\beta_{R, T} \mathbb{1}_{\{> T'\}}$ expressions)

Let R, T, D, r, b be some positive reals and $f \in \mathcal{F}$ an increasing function. Then

$$f \mathbb{1}_{\{> T'\}} = \delta_{T'} \wedge f \quad (13)$$

$$\beta_{R, T} \mathbb{1}_{\{> D\}} = \begin{cases} \beta_{R, T} & \text{if } T \geq D \\ \delta_D * \gamma_{R, R(D-T)} & \text{if } T < D \end{cases} \quad (14)$$

$$= \delta_{D \vee T} * \gamma_{R, R[D-T]^+} \quad (15)$$

$$\delta_D * \gamma_{r, b} = \begin{cases} \delta_D \wedge \gamma_{r, b-rD} & \text{if } b \geq rD \\ \delta_D \wedge \beta_{r, D-\frac{b}{r}} & \text{if } b \leq rD \end{cases} \quad (16) \quad \square$$

Interest of eq (13) comes from the fact that, in $(\wedge, +)$ algebra, we have more results on minimum than on product. In particular, $\beta_{R, T} \mathbb{1}_{\{> T'\}} = \delta_{T'} \wedge \beta_{R, T}$.

The idea behind expression (14) was somehow already present in [18, Eq. (12)], and generalised with the notion of *pseudoaffine curve* in [17, Eq. (6)]. Equation (15) is a new point of view that avoid some ‘‘per case’’ manipulations.

PROOF Expression (13) is obvious.

To prove the expression (14), just have a look on Figure 5, illustrated with $D = \theta > T$. If $D \leq T$: the $\mathbb{1}_{\{> D\}}$ term has no effect.

To go on from (14) up to (15), just remark that $\beta_{R, T}$ can

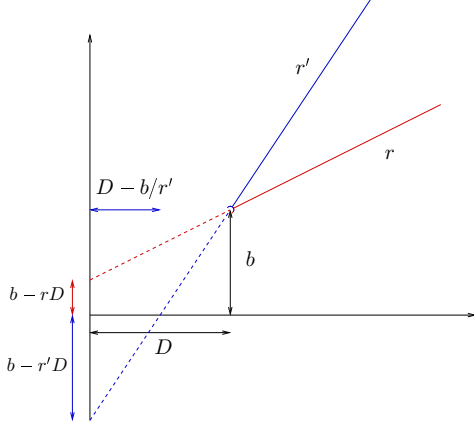


Figure 11. Shifted γ can be expressed as a minimum

also be written $\beta_{R,T} = \delta_T * \gamma_{R,0}$. Then, we have

$$\beta_{R,T} \mathbb{1}_{\{>D\}} = \begin{cases} \delta_T * \gamma_{R,0} & = \delta_{D \vee T} * \gamma_{R,R[D-T]^+} \\ & \text{if } T \geq D \\ \delta_D * \gamma_{R,R(D-T)} & = \delta_{D \vee T} * \gamma_{R,R[D-T]^+} \\ & \text{if } T < D \end{cases}$$

Looking at Figure 11 gives eq (16). Shifting a $\gamma_{r,b}$ function of D to the right is like, either truncating a γ function, or a β function, depending on the value $b - rD$. ■

Lemma 3 (Convolution of truncated beta [18]) *Let R, T, D, R', T', D' be six positive reals with $T < D$ and $T' < D'$. Then*

$$\beta_{R,T} \mathbb{1}_{\{>D\}} * \beta_{R',T'} \mathbb{1}_{\{>D'\}} = \delta_{D+D'} * \gamma_{R,R(D-T)} * \gamma_{R',R'(D'-T')} \quad (17)$$

It can be generalised to any number of function and any values of T_i, D_i .

$$\begin{aligned} & \bigstar_{i=1}^n \beta_{R_i, T_i} \mathbb{1}_{\{>D_i\}} \\ &= \delta_{\sum_{j=1}^n (D_j \vee T_j)} * \bigwedge_{i=1}^n \gamma_{R_i, R_i [D_i - T_i]^+} \end{aligned} \quad (18)$$

$$= \delta_{\sum_{j=1}^n (D_j \vee T_j)} \wedge \bigwedge_{i=1}^n \beta_{R_i, (\sum_{j=1}^n D_j \vee T_j) - [D_i - T_i]^+} \quad (19) \quad \square$$

Equation (18) comes from [18, Proof of Theorem 2], and expression (19) is another point of view on the same equation.

Both expressions (18) and (19) are of interest.

Expression (18) clearly expresses the fact that we have a concave function “shifted” by a delay $\sum_{i=1}^n D_i$. Expression (19) is interesting when trying to compute the delay, with

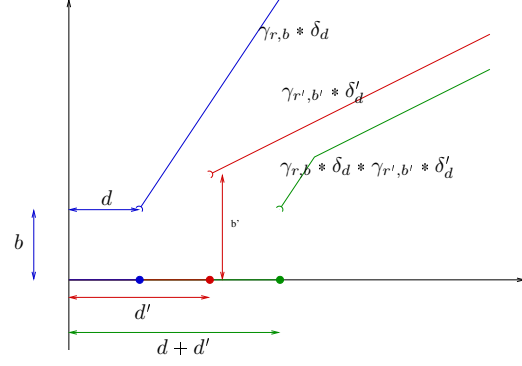


Figure 12. Convolution of two truncated beta functions

help of Lemma 4 (which states that for wide-sense increasing functions $h(f, g \wedge h) = h(f, g) \vee h(f, h)$).

PROOF

- *Proof of (17):* The proof of (17) is obvious, using equation (14) (transformation into convolution of delay and γ function) and basic properties of convolution ($\delta_d * \delta'_d = \delta_{d+d'}$; cf Figure 12).

$$\begin{aligned} & \beta_{R,T} \mathbb{1}_{\{>D\}} * \beta_{R',T'} \mathbb{1}_{\{>D'\}} \\ &= \delta_D * \gamma_{R,R(D-T)} * \delta_{D'} * \gamma_{R',R'(D'-T')} \\ &= \delta_{D+D'} * \gamma_{R,R(D-T)} * \gamma_{R',R'(D'-T')} \end{aligned}$$

- *Proof of (18):* To prove (18), we can generalise (17) to the equation (15), and the fact that convolution and minimum are equals on star-shaped functions [16, Th 3.1.6].

$$\begin{aligned} \bigstar_{i=1}^n \beta_{R_i, T_i} \mathbb{1}_{\{>D_i\}} &= \delta_{\sum_{i=1}^n (D_i \vee T_i)} * \bigstar_{i=1}^n \gamma_{R_i, R_i [D_i - T_i]^+} \\ &= \delta_{\sum_{i=1}^n (D_i \vee T_i)} * \bigwedge_{i=1}^n \gamma_{R_i, R_i [D_i - T_i]^+} \end{aligned}$$

- *Proof of (19):* Going from (18) to (19) is done using distributivity of min over deconvolution, and (16).

The first step uses the distributivity: $f * (g \wedge h) = (f * g) \wedge (f * h)$:

$$\begin{aligned} & \delta_{\sum_{i=1}^n (D_i \vee T_i)} * \bigwedge_{i=1}^n \gamma_{R_i, R_i [D_i - T_i]^+} = \\ & \bigwedge_{i=1}^n \left(\delta_{\sum_{j=1}^n (D_j \vee T_j)} * \gamma_{R_i, R_i [D_i - T_i]^+} \right) \end{aligned} \quad (20)$$

The second step uses (16). We have to compare $b = R_i[D_i - T_i]^+$ and $rD = R_i \sum_{i=j}^n (D_j \vee T_j)$.

$$\begin{aligned} \sum_{j=1}^n (D_j \vee T_j) &\geq \sum_{j=1}^n D_j \geq D_i \geq [D_i - T_i]^+ \\ \Rightarrow R_i \sum_{j=1}^n (D_j \vee T_j) &\geq R_i [D_i - T_i]^+ \end{aligned}$$

We are in the case $rD \geq b$ that is to say $\delta_D * \gamma_{r,b} = \delta_D \wedge \beta_{r,D-\frac{b}{r}}$. That is to say:

$$\begin{aligned} &\delta_{\sum_{j=1}^n (D_j \vee T_j)} * \gamma_{R_i, R_i [D_i - T_i]^+} \\ &= \delta_{\sum_{j=1}^n (D_j \vee T_j)} \wedge \beta_{R_i, \sum_{j=1}^n (D_j \vee T_j) - [D_i - T_i]^+} \end{aligned} \quad (21)$$

Reporting (21) into (20) leads to:

$$\begin{aligned} &\delta_{\sum_{j=1}^n (D_j \vee T_j)} * \bigwedge_{i=1}^n \gamma_{R_i, R_i [D_i - T_i]^+} \\ &= \bigwedge_{i=1}^n \left(\delta_{\sum_{j=1}^n (D_j \vee T_j)} \wedge \beta_{R_i, \sum_{j=1}^n (D_j \vee T_j) - [D_i - T_i]^+} \right) \\ &= \delta_{\sum_{i=j}^n (D_j \vee T_j)} \wedge \bigwedge_{i=1}^n \left(\beta_{R_i, \sum_{j=1}^n (D_j \vee T_j) - [D_i - T_i]^+} \right) \blacksquare \end{aligned}$$

Let now present one useful lemma for computing our delay.

Lemma 4 (Horizontal delay with min of functions) *Let f, g, g' be three wide-sense increasing functions. Then,*

$$h(f, g \wedge g') = h(f, g) \vee h(f, g') \quad (22)$$

□

PROOF We are going to use two results: the first is the expression of $h(\cdot, \cdot)$ in term of deconvolution and the second is the left distributivity of max on deconvolution $(f \vee g) \otimes h = (f \otimes g) \vee (f \otimes h)$, and some trivial relationship on non-decreasing functions (like $(g \wedge h)^{-1} = g^{-1} \vee h^{-1}$ or $(g \vee g') \circ f = (g \circ f) \vee (g' \circ f)$).

$$\begin{aligned} h(f, g \wedge g') &= (((g \wedge g')^{-1} \circ f) \otimes \lambda_1)(0) \\ &= (((g^{-1} \vee g'^{-1}) \circ f) \otimes \lambda_1)(0) \\ &= (((g^{-1} \circ f) \vee (g'^{-1} \circ f)) \otimes \lambda_1)(0) \\ &= (((g^{-1} \circ f) \otimes \lambda_1) \vee ((g'^{-1} \circ f) \otimes \lambda_1))(0) \\ &= h(f, g) \vee h(f, g') \quad \blacksquare \end{aligned}$$

Here comes the main result of the paper, the local application of eq. (4) on each server, parametrised by a set of $\theta_1, \dots, \theta_n$ values, and the expression of the least solution.

Theorem 1 (End-to-end delay in simple FIFO topology)

Let R be a flow with a CPL function (under normal form of Definition 1) $\bigwedge_i \gamma_{r_i, b_i}$ as arrival curve. This flow goes through a sequence of servers S_1, \dots, S_n with respective service curve $\beta_{R_1, T_1}, \dots, \beta_{R_n, T_n}$. Each server S_i is also crossed by a flow R'_i with arrival curve $\gamma_{r'_i, b'_i}$, as illustrated on Figure 1. The policy of each server is FIFO. Then, the end-to-end delay of flow R is bounded by:

$$\begin{aligned} &\inf_{\forall i \in [1, n]: \theta_i \geq 0} h \left(\bigwedge_i \gamma_{r_i, b_i}, \bigast_{i=1}^n \beta_{R_i - r'_i, T_{\theta_i}} \mathbb{1}_{\{> \theta_i\}} \right) \quad (23) \\ &= \begin{cases} \sum_{i=1}^n \left(T_i + \frac{b'_i}{R_i} \right) + \sum_{i=1}^n \frac{y_{k'_i} - (R_i - r'_i)x_{k'_i}}{R_i} \\ \sum_{i=1}^n \left(T_i + \frac{b'_i}{R_i} \right) + \sum_{p(i) \leq k} e_i + e_q \left(1 - \sum_{p(i) \leq k} \frac{R_i - r'_i}{R_i} \right) \end{cases} \\ &\hspace{15em} \text{if } \sum_{i=1}^n \frac{R_i - r'_i}{R_i} \leq 1 \\ &\hspace{15em} \text{otherwise} \end{cases}$$

with $T_{\theta_i} = \frac{R_i T_i + b'_i - r'_i \theta_i}{R_i - r'_i}$, $k'_i = \min \{j | r_j \leq R_i - r'_i\}$, $e_i \stackrel{\text{def}}{=} \frac{y_{k'_i} - (R_i - r'_i)x_{k'_i}}{R_i - r'_i}$, $p : [1, n] \mapsto [1, n]$ a permutation such that the sequence $(e_{p(i)})$ is wide-sense increasing, and $q \stackrel{\text{def}}{=} \min \{p(i) | 1 \geq \sum_{j=1}^{n-p(i)+1} \frac{R_i - r'_i}{R_i}\}$. □

PROOF Here is a sketch of proof. By lack of space, the full proof can be downloaded as an appendix on the author WEB page. A previous version can also be found in [5].

Since this work is a generalisation of [18] (it solves $\inf_{\forall i \in [1, n]: \theta_i \geq 0} h(\gamma_{r, b}, \bigast_{i=1}^n \beta_{R_i - r'_i, T_{\theta_i}} \mathbb{1}_{\{> \theta_i\}})$) the organisation of the proof is the same. In order to be self-sufficient, the proof will be completely done, but we try to highlight the common part and our specific contribution.

The first step consist in restricting the value domain of θ .

$$\begin{aligned} &\inf_{\forall i \in [1, n]: \theta_i \geq 0} h \left(f, \bigast_{i=1}^n \beta_{R_i - r'_i, T_{\theta_i}} \mathbb{1}_{\{> \theta_i\}} \right) = \\ &\inf_{\forall i \in [1, n]: \theta_i \geq T_i + \frac{b'_i}{R_i}} h \left(f, \bigast_{i=1}^n \beta_{R_i - r'_i, T_{\theta_i}} \mathbb{1}_{\{> \theta_i\}} \right) \end{aligned} \quad (24)$$

It comes from the fact that, if $\theta_i < T_i + \frac{b'_i}{R_i}$, then, $\beta_{R_i - r'_i, T_{\theta_i}} \mathbb{1}_{\{> \theta_i\}} = \beta_{R_i - r'_i, T_{\theta_i}^i}$, i.e. the test term has no effect. And, in this case, increasing the θ_i increases the value of the $\beta_{R_i - r'_i, T_{\theta_i}^i}$ term. This step was done in [18].

This allows to make a variable substitution, $\theta'_i = \theta_i + T_i + \frac{b'_i}{R_i}$.

The second step uses the fact that $h(f, g \wedge g') = h(f, g) \vee h(f, g')$, and some rewriting used in Lemma 2

to transform the convolution in minimum. It leads to term

$$\sum_{i=1}^n \left(T_i + \frac{b'_i}{R_i} \right) + \inf_{\substack{\forall i \in [1, n] \\ \theta'_i \geq 0}} \left\{ \sum_{i=0}^n \theta'_i + \bigvee_{i=1}^n \left[\frac{y_{k'_i} - R_i \theta'_i}{R_i - r'_i} - x_{k'_i} \right]^+ \right\} \quad (25)$$

The proof in [18] gives the expression of the delay $h(\cdot, \cdot)$ by a computation of the inverse of the convolution term. We chose a more algebraic way, to use distributivity and reduce the problem to the computation of delay between a CPL and a rate-latency curve (Lemma 1).

To solve this expression, we have to minimise an expression $\inf_{x_i \geq 0} \left\{ \sum_{i=1}^n x_i + \bigvee_{i=1}^n [b_i - a_i x_i]^+ \right\}$ with $b_i \geq 0$ and $a_i \geq 1$. This work has been done in [18] (with a little restriction, $b_i > 0$). We just have to reuse the result. ■

5 Example and comparison

Since the equations are not very intuitive, we have done some experiment to get a global idea of the weakness and strong points of the method. The computations have been done with the prototype NC-maude [7, 4, 6].

Let us consider the configuration presented in Figure 1: a flow crossing three servers, sharing each server with a interfering flow.

To simplify the interpretation, consider that all servers and all interfering flows have the same characteristics. Each server has service of curve $\beta_{R,T}$, the considered flow has arrival curve $\lambda_R \wedge \gamma_{r,b}$ and the interfering flows have arrival curve $\lambda_R \wedge \gamma_{r',b'}$. We have considered 16 possible values for these parameters. For each configurations, we have computed the delay for crossing two and three servers, with three methods: LUBD (presented in section 3.1), local delay with shaping (presented in section 3.2) and our end-to-end computation with half modelling of shaping.

In the LUBD experiment, the shaping introduced by the link (the λ_R term) can not be modelled: the considered flow (resp. interfering flows) has arrival curve $\gamma_{r,b}$ (resp. $\gamma_{r',b'}$).

In the local delay with shaping, the considered flow has arrival curve $\lambda_R \wedge \gamma_{r,b}$ and the interfering flows have arrival curves $\lambda_R \wedge \gamma_{r',b'}$.

In our half modeling of shaping, the considered flow has arrival curve $\lambda_R \wedge \gamma_{r,b}$ but the interfering flows have arrival curves $\gamma_{r',b'}$.

Before all analyse, keep in mind that, in each server, there is an intrinsic delay of $T = 1$ unit. It means that, whatever the input traffic and computation method are, the delay for crossing two servers is at least 2, and at least 3 for three servers.

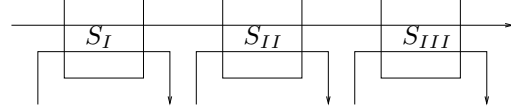


Figure 13. Test configuration

The four firsts topologies, configurations 1 up to 4, have been generated to see the influence of the different parameters, with some “normalised” values.

In the first configuration, all terms have value 1, except for the rates, which are equals to one third, for a global load $\rho = \frac{2}{3}$.

In configuration 2, burst of the interfering flow b' have been increased to 5 compared to configuration 1, in configuration 3, the crossing flow burst is increased to 5. On the opposite, in configuration 4, both bursts have been decreased to $\frac{1}{5}$.

What can be learn from these examples will be generalised in all the following configuration.

First learning it that our method, which is a direct generalisation of the LUB method, is always better. Second, the relative gain decreases with the length of the path and the absolute gain is constant. Third, the gain of course depends on the burst of the crossing flow, which is not shaped in the LUB method.

Relative performances of the new method w.r.t the local shaping depends of course of the impact of the burst of the interfering flow, whose shaping is not modelled by the new method. The new method can be worst than the other one (21% worst for the configuration 1 for the delay on two servers). But the local shaping can not benefit from the PBOO principle, and the longest the path is, the better the relative performance of the new method is.

The configurations 5 to 8 are the same than configurations 1 to 4, except that the system rate have been multiplied by 5, leading to a global load of 13%, which is more common in embedded systems than the 67% one. In this case, the long term rate of the flows becomes negligible, and the best method is two times the local shaping one, and when the new one is better, the gain is really small, and when it is worst, it really is ($\approx 50\%$).

But a burst value equals to the rate value is not very realistic situation. On a 10Mb/s system, it means that is could exist a burst of 10Mb, *i.e.* 1.25Mo, *i.e.* more than 833 frames of 1500 octets (maximal size of an Ethernet frame) at a single instant. Then, configurations 9 to 11 are copies of the configurations 1 to 4 with a burst divided by 10 (considering burst of 83 maximal size Ethernet frames). In this case, the new method gain is between 12% and 22%.

But configurations 9 to 12 still have a load of 67%. Dividing this load by 2 gives configuration 13 to 16. In this case, the main part of the delay comes from the intrinsic de-

Table 1. Comparing methods on topology from Figure 13

| Configurations | | | | | | | | | | | | | | | | |
|--|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| Conf | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| R | 1 | 1 | 1 | 1 | 5 | 5 | 5 | 5 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| T | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| r | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |
| b | 1 | 1 | 5 | $\frac{1}{5}$ | 1 | 1 | 5 | $\frac{1}{5}$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{5}{10}$ | $\frac{1}{50}$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{5}{10}$ | $\frac{1}{50}$ |
| r' | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |
| b' | 1 | 5 | 1 | $\frac{1}{5}$ | 1 | 5 | 1 | $\frac{1}{5}$ | $\frac{1}{10}$ | $\frac{5}{10}$ | $\frac{1}{10}$ | $\frac{1}{50}$ | $\frac{1}{10}$ | $\frac{5}{10}$ | $\frac{1}{10}$ | $\frac{1}{50}$ |
| $\rho = \frac{r+r'}{R}$ | 67% | 67% | 67% | 67% | 13% | 13% | 13% | 13% | 67% | 67% | 67% | 67% | 33% | 33% | 33% | 33% |
| Delay R crossing $S_I; S_{II}$ | | | | | | | | | | | | | | | | |
| LUB | 5.50 | 13.50 | 11.50 | 2.70 | 2.61 | 4.21 | 3.47 | 2.12 | 2.35 | 3.15 | 2.95 | 2.07 | 2.32 | 3.12 | 2.80 | 2.06 |
| Loc. Shap. | 5.41 | 10.50 | 9.75 | 2.81 | 2.43 | 2.62 | 2.54 | 2.09 | 2.49 | 3.12 | 2.92 | 2.23 | 2.27 | 2.60 | 2.44 | 2.08 |
| Half. Shap. | 4.75 | 12.75 | 7.75 | 2.55 | 2.41 | 4.01 | 2.47 | 2.08 | 2.27 | 3.07 | 2.57 | 2.05 | 2.22 | 3.02 | 2.32 | 2.04 |
| Delay R crossing $S_I; S_{II}; S_{III}$ | | | | | | | | | | | | | | | | |
| LUB | 7.50 | 19.50 | 13.50 | 3.90 | 3.81 | 6.21 | 4.67 | 3.16 | 3.45 | 4.65 | 4.05 | 3.09 | 3.42 | 4.62 | 3.90 | 3.08 |
| Loc. Shap. | 8.81 | 18.50 | 15.87 | 4.58 | 3.66 | 4.07 | 3.83 | 3.14 | 4.05 | 5.19 | 4.76 | 3.63 | 3.47 | 4.20 | 3.72 | 3.17 |
| Half. Shap. | 6.75 | 18.75 | 9.75 | 3.75 | 3.61 | 6.01 | 3.67 | 3.12 | 3.37 | 4.57 | 3.67 | 3.07 | 3.32 | 4.52 | 3.42 | 3.06 |
| Gain on the new method for R crossing $S_I; S_{II}$ | | | | | | | | | | | | | | | | |
| vs. LUB | 13.63% | 5.55% | 32.60% | 5.55% | 7.61% | 4.72% | 28.65% | 1.87% | 3.19% | 2.38% | 12.71% | 0.72% | 4.13% | 3.07% | 17.14% | 0.93% |
| vs. Loc. Shap. | 12.30% | -21.42% | 20.51% | 9.46% | 0.78% | -52.81% | 2.83% | 0.36% | 8.69% | 1.60% | 11.96% | 7.91% | 2.34% | -16.30% | 4.91% | 1.79% |
| Gain on the new method for R crossing $S_I; S_{II}; S_{III}$ | | | | | | | | | | | | | | | | |
| vs. LUB | 10.00% | 3.84% | 27.77% | 3.84% | 5.21% | 3.20% | 21.29% | 1.25% | 2.17% | 1.61% | 9.25% | 0.48% | 2.80% | 2.07% | 12.30% | 0.62% |
| vs. Loc. Shap. | 23.46% | -1.35% | 38.58% | 18.23% | 1.22% | -47.64% | 4.06% | 0.64% | 16.80% | 11.94% | 22.83% | 15.37% | 4.29% | -7.54% | 8.09% | 3.48% |

lay, and methods difference are small, but the local shaping method is always better when the burst of the interfering flow is big.

6 Conclusion

Looking for worst case performance is an hard task, especially in distributed real-time system where the worst case is hard to determine⁵. Therefore, we are dealing with methods making pessimistic approximations, and trying to reduce this pessimism.

Going from basics works of [12, 13] handling only token-bucket $(\gamma_{r,b})$ traffic, two extensions have been made to reduce this pessimism: [15] handles on each server the shaping introduced by physical links, and [18] models an end-to-end FIFO service to get the benefit of the PBOO result. This works tried to join the two works and have the benefit of both. But for technical reasons, it only handles shaping on the considered flow, not on the interfering one. Then, the new method is always better than the one of [18] (it is a direct generalisation), but performances w.r.t. the local shaping depends on the burst size of the interfering flows.

This of course incites us to study the general case, with shaping on considered and interfering flow. We are currently working on the subject, but it seems really harder.

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⁵It could be NP-hard [2, § III.D].

This appendix is the proof of Theorem 1 of paper “M. Boyer, Half-modeling of shaping in FIFO net with network calculus. In *Proceedings of the of the 18th International Conference on Real-Time and Network Systems*, Toulouse, France, November 2010.”

A Proof of Theorem 1

To prove equality (23), as presented in the sketch of proof, we first have to restrict the value domain of the θ_i variables (eq. (24)).

The second step consists in having an expression of the delay to minimise, *i.e.* that the second term of (24)) can be reduced into (25).

The last step is the application of Lemma 5, which reduces expression $\inf_{x_i \geq 0} \left\{ \sum_{i=1}^n x_i + \bigvee_{i=1}^n [b_i - a_i x_i]^+ \right\}$.

A.1 First step: reducing the variable range

1. We are first going to restrict the θ_1 domain, *i.e.* proving eq.(24).

Let us first show two intermediate results.

$$(a) \forall \theta_1 \in \left[0, T_1 + \frac{b_1}{R_1}\right], \dot{\theta}_1 = T_1 + \frac{b_1}{R_1} : \beta_{R'_1, T'_{\theta_1}} \mathbb{1}_{\{\>\theta_1\}} \leq \beta_{R'_1, T_{\theta_1}} \mathbb{1}_{\{\>\dot{\theta}_1\}}$$

PROOF First, notice that $\theta_1 \leq T_{\theta_1}^1 \iff \theta_1 \leq T_1 + \frac{b_1}{R_1}$. And if $\theta_1 \leq T_{\theta_1}^1$, then $\beta_{R'_1, T'_{\theta_1}} \mathbb{1}_{\{\>\theta_1\}} = \beta_{R'_1, T_{\theta_1}} \mathbb{1}_{\{\>\theta_1\}}$, the test term is useless, we are handling simple rate-latency $\beta_{R,T}$ functions. Furthermore, $T \leq T' \implies \beta_{R,T} \geq \beta_{R,T'}$. Since $T_{\theta_1}^1$ is a decreasing function of θ_1 , we have $\forall \theta_1 \in [0, T_1 + \frac{b_1}{R_1}]$, $T_{\theta_1}^1 \leq T_{\dot{\theta}_1}^1$. To sum up $\forall \theta_1 \in [0, T_1 + \frac{b_1}{R_1}]$:

$$\beta_{R'_1, T'_{\theta_1}} \mathbb{1}_{\{\>\theta_1\}} = \beta_{R'_1, T_{\theta_1}} \leq \beta_{R'_1, T_{\dot{\theta}_1}} = \beta_{R'_1, T'_{\dot{\theta}_1}} \quad \blacksquare$$

$$(b) g \geq g' \implies h(f, g * h) \leq h(f, g' * h).$$

PROOF We know that $f \geq g \geq g' \implies h(f, g) \leq h(f, g')$. We also know that convolution is isotone on wide-sense increasing functions, that is to say, $g \geq g' \implies g * h \geq g' * h$. \blacksquare

With previous results, we know that, for all sets of value $\theta_1, \dots, \theta_n$, if $\theta_1 \leq \dot{\theta}_1 = T_1 + \frac{b_1}{R_1}$, then $h(\bigwedge_i \gamma_{r_i, b_i}, \bigast_{i=1}^n \beta_{R'_i, T_{\theta_i}} \mathbb{1}_{\{\>\theta_i\}}) \geq h(\bigwedge_i \gamma_{r_i, b_i}, \beta_{R'_1, T_{\theta_1}} \mathbb{1}_{\{\>\dot{\theta}_1\}} \bigast_{i=2}^n \beta_{R'_i, T_{\theta_i}} \mathbb{1}_{\{\>\theta_i\}})$.

Then, we can restrict the research of the minimum to values of $\theta \geq T_1 + \frac{b_1}{R_1}$, *i.e.*

$$\inf_{\forall i \in [1, n]: \theta_i \geq 0} h\left(\bigwedge_i \gamma_{r_i, b_i}, \bigast_i \beta_{R'_i, T_{\theta_i}} \mathbb{1}_{\{\>\theta_i\}}\right) = \inf_{\theta_1 \geq T_1 + \frac{b_1}{R_1}, \forall i \in [2, n]: \theta_i \geq 0} h\left(\bigwedge_i \gamma_{r_i, b_i}, \bigast_i \beta_{R'_i, T_{\theta_i}} \mathbb{1}_{\{\>\theta_i\}}\right) \quad (26)$$

2. We can iterate the previous step to restrict the same way domains of θ_2 to θ_n .

A.2 Expression of delay

This proof uses a different way than [18]: they are using the pseudo-inverse of the function $\bigast_{i=1}^n \beta_{R'_i, T_{\theta_i}} \mathbb{1}_{\{\>\theta_i\}}$, and we are using the distributivity of the distributivity of min on delay.

We first apply (4) on each server S_i . Then, each S_i offers to flow R a residual service $\beta_i^{\theta_i} = \beta_{R'_i, T_{\theta_i}} \mathbb{1}_{\{\>\theta_i\}}$ with $R'_i = R_i - r'_i$ and $T_{\theta_i}^i = \frac{R_i T_i + b'_i - r'_i \theta_i}{R'_i}$.

Then, the delay of R is the same as if it was traversing an unique server of service curve $\beta^{\theta_1, \dots, \theta_n} = \bigast_{i=1}^n \beta_{R'_i, T_{\theta_i}} \mathbb{1}_{\{\>\theta_i\}}$ and from Lemma 3 $\beta^{\theta_1, \dots, \theta_n} = \delta_{\sum_{i=1}^n (\theta_i \vee T_{\theta_i}^i)} \wedge \bigwedge_{i=1}^n \beta_{R'_i, (\sum_{j=1}^n \theta_j \vee T_{\theta_j}^j) - [\theta_i - T_{\theta_i}^i]^+}$

The horizontal delay can be decomposed using Lemma 4. $h(\bigwedge_i \gamma_{r_i, b_i}, \beta^{\theta_1, \dots, \theta_n}) = h(\bigwedge_i \gamma_{r_i, b_i}, \delta_{\sum_{i=1}^n (\theta_i \vee T_{\theta_i}^i)}) \vee \bigvee_i h(\bigwedge_i \gamma_{r_i, b_i}, \beta_{R'_i, (\sum_{j=1}^n \theta_j \vee T_{\theta_j}^j) - [\theta_i - T_{\theta_i}^i]^+})$.

From Lemma 1, we have an expression for the delay. Let $k'_i = \min \{j \mid r_j \leq R_i - r'_i\}$.

$$\text{Then, } h(\bigwedge_i \gamma_{r_i, b_i}, \beta^{\theta_1, \dots, \theta_n}) = \sum_{i=1}^n [\theta_i \vee T_{\theta_i}^i]^+ \vee \bigvee_i \left\{ \sum_{j=1}^n [\theta_j \vee T_{\theta_j}^j]^+ - [\theta_i - T_{\theta_i}^i]^+ + \frac{y_{k'_i}}{R'_i} - x_{k'_i} \right\}$$

We are only intersted in values of $\theta_i \geq T_i + \frac{b'_i}{R_i}$. In this cases, $\theta_i \geq T_{\theta_i}^i$, $[\theta_i - T_{\theta_i}^i]^+ = \frac{R_i \theta_i - R_i T_i - b'_i}{R_i - r'_i}$ and the expression be simplified.

Then, under assumption $\theta_i \geq T_i + \frac{b'_i}{R_i}$ we can rewrite the delay:

$$\begin{aligned} & h\left(\bigwedge_i \gamma_{r_i, b_i}, \bigast_{i=1}^n \beta_{R'_i, T_{\theta_i}} \mathbb{1}_{\{\>\theta_i\}}\right) \\ &= \sum_{i=1}^n \theta_i \vee \bigvee_{i=1}^n \left\{ \left(\sum_{i=1}^n \theta_i \right) - \theta_i + T_{\theta_i}^i + \frac{y_{k'_i}}{R'_i} - x_{k'_i} \right\} \\ &= \sum_{j=1}^n \theta_j \vee \bigvee_{i=1}^n \left\{ \left(\sum_{i=j}^n \theta_j \right) + \frac{R_i T_i + b'_i + y_{k'_i} - R_i \theta_i}{R_i - r'_i} - x_{k'_i} \right\} \\ &= \bigvee_{i=1}^n \left\{ \left(\sum_{j=1}^n \theta_j \right) + \left[\frac{R_i T_i + b'_i + y_{k'_i} - R_i \theta_i}{R_i - r'_i} - x_{k'_i} \right]^+ \right\} \end{aligned}$$

with variable substitution $\theta'_i = \theta_i - (T_i + \frac{b'_i}{R_i})$ it becomes

$$\begin{aligned} &= \bigvee_{i=1}^n \left\{ \left(\sum_{j=1}^n \theta'_j + T_j + \frac{b'_j}{R_j} \right) + \left[\frac{y_{k'_i} - R_i \theta'_i}{R_i - r'_i} - x_{k'_i} \right]^+ \right\} \\ &= \left(\sum_{j=1}^n T_j + \frac{b'_j}{R_j} \right) + \bigvee_{i=1}^n \left\{ \left(\sum_{j=1}^n \theta'_j \right) + \left[\frac{y_{k'_i} - R_i \theta'_i}{R_i - r'_i} - x_{k'_i} \right]^+ \right\} \end{aligned}$$

Then, looking for the inf of such expression leads to:

$$\begin{aligned} &\inf_{\forall i \in [1, n]: \theta_i \geq 0} h \left(\bigwedge_i \gamma_{r_i, b_i}, \bigstar_{i=1}^n \beta_{R_i, T_{\theta_i}} \mathbb{1}_{\{\theta_i\}} \right) = \\ &\quad \left(\sum_{i=1}^n T_i + \frac{b'_i}{R_i} \right) \\ &+ \inf_{\forall i \in [1, n]: \theta'_i \geq 0} \bigvee_{i=1}^n \left\{ \left(\sum_{j=1}^n \theta'_j \right) + \left[\frac{y_{k'_i} - R_i \theta'_i}{R_i - r'_i} - x_{k'_i} \right]^+ \right\} \end{aligned}$$

A.3 Last step

Here comes the last proof: giving an analytical solution to the inf term in (25). Abstracting from the specific network calculus problem, we are solving

$$\inf_{x_i \geq 0} \left\{ \sum_{i=1}^n x_i + \bigvee_{i=1}^n [b_i - a_i x_i]^+ \right\} \quad (27)$$

with $a_i = \frac{R_i}{R_i - r'_i} \geq 1$ and $b_i = \frac{y_{k'_i} - (R_i - r'_i)x_{k'_i}}{R_i - r'_i} \geq 0$ (6).

Then, using Lemma 5, we can compute the infimum of this expression. This is a generalisation of what is done in [18], where all b_i have the same value.

So, using Lemma 5, if $\sum_{i=1}^n \frac{R_i - r'_i}{R_i} \geq 1$, then, the delay value is

$$\sum_{i=1}^n \frac{y_{k'_i} - (R_i - r'_i)x_{k'_i}}{R_i} \quad (28)$$

Otherwise, the index must be re-ordered. Let $p : [1, n] \mapsto [1, n]$ be a permutation such that $(b_{p(i)})$ is a wide-sense increasing sequence. Let $q \stackrel{\text{def}}{=} \min \left\{ p(i) \mid 1 \geq \sum_{j=1}^{n-p(i)+1} \frac{R_i - r'_i}{R_i} \right\}$. In this case, the delay

⁶The fact that these b_i are non negative come from the definitions of k'_i : it is the smaller index such that the slopes before $x_{k'_i}$ is greater than $(R_i - r'_i)$. It implies that $y_{k'_i} \geq (R_i - r'_i)x_{k'_i}$

value is

$$\begin{aligned} &\sum_{p(i) \leq k} \frac{y_{k'_i} - (R_i - r'_i)x_{k'_i}}{R_i} + \\ &\quad \frac{y_{k'_q} - (R_q - r'_q)x_{k'_q}}{R_q - r'_q} \left(1 - \sum_{p(i) \leq k} \frac{R_i - r'_i}{R_i} \right) \end{aligned}$$

A.4 A useful Lemma on specific inf

Lemma 5 ($\inf_{x_i \geq 0} \left\{ \sum_{i=1}^n x_i + \bigvee_{i=1}^n [b_i - a_i x_i]^+ \right\}$)
Let be $\forall i \in [1, n] : b_i \geq 0, a_i \geq 1$. We can assume, without loss of generality, that the b_i are sorted in non decreasing order ($b_i \leq b_{i+1}$). Then

$$\begin{aligned} &\inf_{x_i \geq 0} \left\{ \sum_{i=1}^n x_i + \bigvee_{i=1}^n [b_i - a_i x_i]^+ \right\} \\ &= \begin{cases} \sum_{i=1}^n \frac{b_i}{a_i} & \text{if } \sum_{i=1}^n \frac{1}{a_i} \leq 1 \\ \sum_{j=k}^n \frac{b_j}{a_j} + b_k \left(1 - \sum_{j=k}^n \frac{1}{a_j} \right) & \text{otherwise} \end{cases} \quad (29) \end{aligned}$$

with $k \stackrel{\text{def}}{=} \max \left\{ i \mid 1 \geq \sum_{j=i}^n \frac{1}{a_j} \right\}$. \square

This lemma have been solved in [18] in a more specific way:

$$\inf_{\tau'_i \geq 0} \left\{ \sum_i \tau'_i + \bigvee_i \left[\frac{\sigma - \rho^i \tau'_i}{\rho^i - \rho_i} \right]^+ \right\} \quad (30)$$

with $\sigma > 0, \rho^i \geq \rho_i \geq 0$. This problem can be also written, in a more general way $\inf_{\tau'_i \geq 0} \left\{ \sum_i \tau'_i + \bigvee_i \left[\frac{u - v_i \tau'_i}{w_i} \right]^+ \right\}$ with $u > 0, v_i, w_i \geq 0$. This problem is equivalent to (29), with $u = 1, b_i = \frac{1}{w_i}, a_i = \frac{v_i}{w_i}$, except the case $b_i = 0$. So, we redo the proof to see if the null case could makes a difference. It does not. Here is the proof.

In the following, \bar{x} will denotes a vector of n values (x_1, \dots, x_n) , and $f(\bar{x}) = f(x_1, \dots, x_n) = \sum_{i=1}^n x_i + \bigvee_{i=1}^n [b_i - a_i x_i]^+$. We are looking for the minimum for f on $\mathbb{R}_{\geq 0}^n$.

Except for step 1, this proof is a clone of the one of [18], generalised to handle different values of b_i .

$$\begin{aligned} 1. \quad &\inf_{0 \leq x_i} \left\{ \sum_{i=1}^n x_i + \bigvee_{i=1}^n [b_i - a_i x_i]^+ \right\} = \\ &\inf_{0 \leq x_i \leq \frac{b_i}{a_i}} \left\{ \sum_{i=1}^n x_i + \bigvee_{i=1}^n (b_i - a_i x_i) \right\} \end{aligned}$$

PROOF They are two contradictory terms in this expression: $\sum_{i=1}^n x_i$ increases with any x_i , and terms $b_i - a_i x_i$ decreases. More formally, for any vector of value $\bar{x} = (x_1, \dots, x_n)$ with some $x_i \geq \frac{b_i}{a_i}$, then, it exist $\bar{x}' = (x_1, \dots, x_{i-1}, \frac{b_i}{a_i}, x_{i+1}, \dots, x_n)$ such that $f(\bar{x}') \leq f(\bar{x})$. ■

We have done exactly the same kind of proof for (24).

In [18], this is done in a different way: they first study the case $0 \leq x_i \leq \frac{b_i}{a_i}$ (which is ‘‘Case 1: $0 \leq \rho^i \tau_i' \leq \sigma$ for any i .’’)

$$2. \quad \inf_{0 \leq x_i \leq \frac{b_i}{a_i}} f(\bar{x}) = \bigwedge_{j=1}^n \inf_{\bar{x} \in X_j} \sum_{i=1}^n x_i + b_j - a_j x_j$$

with $X_j \stackrel{\text{def}}{=} \left\{ \bar{x} \in \mathbb{R}_{\geq 0}^n \mid \forall i : x_i \leq \frac{b_i}{a_i} \text{ and } b_i - a_i x_i \leq b_j - a_j x_j \right\}$

PROOF Another definition of X_j could be $\left\{ \bar{x} \mid \forall i : x_i \leq \frac{b_i}{a_i} \text{ and } b_j - a_j x_j = \bigvee_{i=1}^n a_i - b_i x_i \right\}$. It obviously comes that $\bigcup_{j=1}^n X_j = \left\{ \bar{x} \in \mathbb{R}_{\geq 0}^n \mid \forall i : x_i \leq \frac{b_i}{a_i} \right\}$. Then

$$\begin{aligned} & \inf_{0 \leq x_i \leq \frac{b_i}{a_i}} \left\{ \sum_{i=1}^n x_i + \bigvee_{i=1}^n (b_i - a_i x_i) \right\} \\ &= \bigwedge_{j=1}^n \inf_{\bar{x} \in X_j} \left\{ \sum_{i=1}^n x_i + \bigvee_{i=1}^n (b_i - a_i x_i) \right\} \\ &= \bigwedge_{j=1}^n \inf_{\bar{x} \in X_j} \left\{ \sum_{i=1}^n x_i + b_j - a_j x_j \right\} \quad \blacksquare \end{aligned}$$

The X_j are just a renaming of the T_j of [18].

$$3. \quad \inf_{\bar{x} \in X_j} \left\{ \sum_{i=1}^n x_i + b_j - a_j x_j \right\} = \inf_{0 \leq x_j \leq \frac{b_j}{a_j}} \sum_{i=1}^n \left[\frac{b_i - b_j + a_j x_j}{a_i} \right]^+ + b_j - a_j x_j$$

PROOF Consider x_j as a constant. This function increases with any x_i ($i \neq j$). Then, its inf is reached with the smallest values of each x_i in the interval range.

$$\begin{aligned} & 0 \leq b_i - a_i x_i \leq b_j - a_j x_j \\ \iff & -b_i \leq -a_i x_i \leq b_j - b_i - a_j x_j \\ \iff & \frac{b_i}{a_i} \geq x_i \geq \frac{-b_j + b_i + a_j x_j}{a_i} \end{aligned}$$

We also known that $x_i \geq 0$. Then, the minimal value for each x_i is $\left[\frac{b_i - b_j + a_j x_j}{a_i} \right]^+$, and each $x_i \neq x_j$ can

be replaced by this value.

$$\begin{aligned} & \inf_{\bar{x} \in X_j} \left\{ \sum_{i=1}^n x_i + b_j - a_j x_j \right\} = \\ & \inf_{0 \leq x_j \leq \frac{b_j}{a_j}} \sum_{\substack{i=1 \\ i \neq j}}^n \left[\frac{b_i - b_j + a_j x_j}{a_i} \right]^+ + x_j + b_j - a_j x_j \end{aligned}$$

Notice that $x_j = \frac{b_j - b_i + a_j x_j}{a_j}$, then it comes.

$$\inf_{0 \leq x_j \leq \frac{b_j}{a_j}} \sum_{i=1}^n \left[\frac{b_i - b_j + a_j x_j}{a_i} \right]^+ + b_j - a_j x_j \quad \blacksquare$$

$$4. \quad \inf_{0 \leq x_i} \left\{ \sum_{i=1}^n x_i + \bigvee_{i=1}^n [b_i - a_i x_i]^+ \right\} = \inf_{0 \leq x \leq b_n} \sum_{i=1}^n \left[\frac{b_i - x}{a_i} \right]^+ + x$$

PROOF In the previous steps, we have reduced

$$\begin{aligned} & \text{the problem } \inf_{0 \leq x_i} \left\{ \sum_{i=1}^n x_i + \bigvee_{i=1}^n [b_i - a_i x_i]^+ \right\} \text{ into} \\ & \bigwedge_{j=1}^n \inf_{0 \leq x_j \leq \frac{b_j}{a_j}} \sum_{i=1}^n \left[\frac{b_i - b_j + a_j x_j}{a_i} \right]^+ + b_j - a_j x_j \end{aligned}$$

Let us redefine the independent variable x_j into $x'_j = b_j - a_j x_j$. We have $0 \leq x_j \leq \frac{b_j}{a_j} \iff 0 \geq -a_j x_j \geq -b_j \iff b_j \geq x'_j \geq 0$. Then

$$\begin{aligned} & \bigwedge_{j=1}^n \inf_{0 \leq x_j \leq \frac{b_j}{a_j}} \sum_{i=1}^n \left[\frac{b_i - b_j + a_j x_j}{a_i} \right]^+ + b_j - a_j x_j = \\ & \bigwedge_{j=1}^n \inf_{0 \leq x'_j \leq b_j} \sum_{i=1}^n \left[\frac{b_i - x'_j}{a_i} \right]^+ + x'_j \end{aligned}$$

But there is no need to keep n different variables x'_j , since they is no dependency between them: they appear into different subterms. Let us denote simply $x = x'_j$. The term then becomes

$$\bigwedge_{j=1}^n \inf_{0 \leq x \leq b_j} \sum_{i=1}^n \left[\frac{b_i - x}{a_i} \right]^+ + x$$

It can be re-written as $\bigwedge_{j=1}^n \inf_{0 \leq x \leq b_j} f(x)$ with $f(x) \stackrel{\text{def}}{=} \sum_{i=1}^n \left[\frac{b_i - x}{a_i} \right]^+ + x$ function. This is always the same expression which is studied, on intervals $[0, b_i]$. But, the infimum is of course meet on the larger interval, i.e. $\bigwedge_{j=1}^n \inf_{0 \leq x \leq b_j} f(x) = \inf_{0 \leq x \leq \sqrt[n]{b_1} \dots b_n} f(x) = \inf_{0 \leq x \leq b_n} f(x)$ ■

We now have a piecewise linear continuous function, whose minimum must be found.

$$5. \quad \inf_{0 \leq x \leq b_n} \sum_{i=1}^n \left[\frac{b_i - x}{a_i} \right]^+ + x =$$

$$\begin{cases} \sum_{i=1}^n \frac{b_i}{a_i} & \text{if } \sum_{i=1}^n \frac{1}{a_i} \leq 1 \\ \sum_{j=k}^n \frac{b_j}{a_j} + b_k \left(1 - \sum_{j=k}^n \frac{1}{a_j} \right) & \text{otherwise} \end{cases}$$

with $k \stackrel{\text{def}}{=} \max \left\{ i \mid 1 \geq \sum_{j=i}^n \frac{1}{a_j} \right\}$.

PROOF We have to find the minimum on $[0, b_n]$ of the piecewise linear function $f(x) \stackrel{\text{def}}{=} \sum_{i=1}^n \left[\frac{b_i - x}{a_i} \right]^+ + x$.

The x term is increasing, and each $\left[\frac{b_i - x}{a_i} \right]^+$ is decreasing up to b_i . Let us define $b_0 = 0$.

On each interval $[b_{i-1}, b_i]$, the function value is $\sum_{j=i}^n \frac{b_j - x}{a_j} + x$, and its slope is $1 - \sum_{j=i}^n \frac{1}{a_j}$.

If $\sum_{j=1}^n \frac{1}{a_j} \leq 1$, then, the function is wide sense increasing, and the minimum is reached at $x = 0$ (and the value is $\sum_{i=1}^n \frac{b_i}{a_i}$).

Otherwise, the functions is decreasing, up to an inflexion point b_k where is began to increase (we know that is can not be always decreasing: on last interval, $[b_{n-1}, b_n]$ the slope value is $1 - \frac{1}{a_n}$, and all $a_i \geq 1$).

Let be $k \stackrel{\text{def}}{=} \max \left\{ i \mid 1 \geq \sum_{j=i}^n \frac{1}{a_j} \right\}$. Then, the function f reaches its minimum at b_k , and its value is

$$\begin{aligned} \sum_{j=k}^n \frac{b_j - b_k}{a_j} + b_k &= \sum_{j=k}^n \frac{b_j}{a_j} + b_k \left(1 - \sum_{j=k}^n \frac{1}{a_j} \right) \\ &= \sum_{j=k}^n \frac{b_j}{a_j} + b_k \left(1 - \sum_{j=k}^n \frac{1}{a_j} \right) \blacksquare \end{aligned}$$

It should be noticed that, with convention $a_0 = \infty$, $b_0 = 0$, both formulas for cases $\sum_{j=1}^n \frac{1}{a_j} \leq 1$ and $\sum_{j=1}^n \frac{1}{a_j} > 1$ are equivalent.