Multi-Criteria Decision-Making Support with Belief Functions

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Outline

Classical decision-making methods with belief functions

Decision-making problem (DMP) FoD $\Theta = \{\theta_1, \dots, \theta_n\}$ = set of possible solutions

Knowing a BBA $m(\cdot)$ over 2^Θ, how should I make my decision δ based on $m(\cdot)$?

In the classical DMP, we restrict $\delta \in \Theta$, i.e. the best decision $\hat{\theta}$ is a singleton of 2^Θ.

Classical DM methods

- Pessimistic Decision-Making attitude: Maximum of belief strategy

$$m(\cdot) \to Bel(\cdot) \quad \text{and} \quad \delta = \hat{\theta} = \arg\max_{\theta_i \in \Theta} Bel(\theta_i)$$

- Optimistic Decision-Making attitude: Maximum of plausibility strategy

$$m(\cdot) \to Pl(\cdot) \quad \text{and} \quad \delta = \hat{\theta} = \arg\max_{\theta_i \in \Theta} Pl(\theta_i)$$

- Compromise Decision-Making attitude: Maximum of probability strategy

$$m(\cdot) \to P(\cdot) \quad \text{and} \quad \delta = \hat{\theta} = \arg\max_{\theta_i \in \Theta} P(\theta_i)$$

where $P(\cdot) \in [Bel(\cdot), Pl(\cdot)]$ is a (subjective) proba measure approximated from the BBA $m(\cdot)$, typically obtained with a lossy transformation, typically BetP, or DSmP

Many methods exist, we only present the most popular – see [DSmT books] (Vol. 3)

Simplest method
Take the mass of each element of Θ and normalize, but it does not take into account partial ignorances

$$P_m(A) = \frac{m(A)}{\sum_{B\in\Theta} m(B)}$$

Method based on plausibility [Cobb Shenoy 2006]
Take the plausibility of each element of Θ and normalize, but it is inconsistent with belief interval

$$P_{Pl}(A) = \frac{Pl(A)}{\sum_{B\in\Theta} Pl(B)}$$

Pignistic probability [Smets 1990]
Redistribute the mass of partial ignorances **equally** to singletons included in them
$\Rightarrow$ higher entropy obtained with $BetP(\cdot)$

$$BetP(A) = \sum_{X\in 2^\Theta} \frac{|X\cap A|}{|A|} m(X)$$

DSmP probability [Dezert Smarandache 2008]
Redistribute mass of partial ignorances **proportionally to masses of singletons** included in them. $\epsilon > 0$ is a small parameter to prevent division by zero in some cases.
$\Rightarrow$ smaller entropy obtained with $DSmP(\cdot)$

$$DSmP_\epsilon(A) = \sum_{Y\in 2^\Theta} \frac{\sum_{\substack{Z\subseteq A\cap Y\\ |Z|=1}} m(Z) + \epsilon|A\cap Y|}{\sum_{\substack{Z\subseteq Y\\ |Z|=1}} m(Z) + \epsilon|Y|} m(Y)$$

Consider $\Theta = \{A, B, C\}$, and the BBA

$$\begin{cases} m(A) = 0.2 \\ m(B \cup C) = 0.8 \end{cases} \Rightarrow \begin{cases} [Bel(A), Pl(A)] = [0.2, 0.2] \\ [Bel(B), Pl(B)] = [0, 0.8] \\ [Bel(C), Pl(C)] = [0, 0.8] \end{cases}$$

- With simplest transformation → inconsistency with Belief Interval

$$P_m(A) = \frac{m(A)}{m(A) + m(B) + m(C)} = \frac{0.2}{0.2 + 0 + 0} = 1 > Pl(A) \quad \text{and} \quad P_m(B) = P_m(C) = 0$$

- With plausibility transformation → inconsistency with Belief Interval

$$P_{Pl}(A) = \frac{0.2}{0.2 + 0.8 + 0.8} \approx 0.112 < Bel(A) \quad \text{and} \quad P_{Pl}(B) = P_{Pl}(C) \approx 0.444$$

- With BetP transformation

$$BetP(A) = m(A) = 0.2 \qquad BetP(B) = BetP(C) = \frac{1}{2} m(B \cup C) = 0.4$$

- With DSmP transformation - same as BetP for this example for any $\epsilon > 0$

$$DSmP(A) = m(A) = 0.2 \qquad DSmP(B) = DSmP(C) = \frac{1}{2} m(B \cup C) = 0.4$$

Example 2 of probabilistic transformations

Consider $\Theta = \{A, B\}$, and $m(A) = 0.3$, $m(B) = 0.1$, $m(A \cup B) = 0.6$

$$\begin{cases} m(A) = 0.3 \\ m(B) = 0.1 \\ m(A \cup B) = 0.6 \end{cases} \Rightarrow \begin{cases} [Bel(A), Pl(A)] = [0.3, 0.9] \\ [Bel(B), Pl(B)] = [0.1, 0.7] \end{cases}$$

- With simplest transformation $P_m(A) = \frac{m(A)}{m(A)+m(B)} = \frac{0.3}{0.3+0.1} = 0.75$ and $P_m(B) = 0.25$
- With plausibility transformation $P_{Pl}(A) = \frac{0.9}{0.9+0.7} = 0.5625$ and $P_{Pl}(B) = 0.4375$
- With BetP transformation $\begin{cases} BetP(A) = m(A) + \frac{1}{2}m(A \cup B) = 0.3 + (0.6/2) = 0.6 \\ BetP(B) = m(B) + \frac{1}{2}m(A \cup B) = 0.1 + (0.6/2) = 0.4 \end{cases}$
- With DSmP transformation $\begin{cases} DSmP_{\epsilon=0}(A) = m(A) + \frac{m(A)}{m(A)+m(B)} \cdot m(A \cup B) = 0.75 \\ DSmP_{\epsilon=0}(B) = m(B) + \frac{m(B)}{m(A)+m(B)} \cdot m(A \cup B) = 0.25 \end{cases}$

Shannon entropy (measure of randomness): $H(P) = -\sum_i p_i \log p_i$

$$H(DSmP) = H(P_m) = 0.8113 \text{ bits} < H(BetP) = 0.9710 \text{ bits} < H(P_{Pl}) = 0.9887 \text{ bits}$$

Decision-making is made easier with DSmP (and P_m here) because the randomness is reduced

Decision-making methods from a BBA (2)

Decision-making based on distances [Han Dezert Yang 2014, Dezert et al. 2016]

A better theoretical approach for decision-making is to use a strict distance metric $d(\cdot,\cdot)$ between two BBAs and to make the decision by

$$\delta = \hat{X} = \arg\min_{X \in \mathcal{X}} d(m, m_X)$$

$\mathcal{X} = \{\text{admissible} X, X \in 2^\Theta\}$ is the set of possible admissible decisions we consider. For instance, if δ must be a singleton, then $\mathcal{X} = \Theta = \{\theta_1, \ldots, \theta_n\}$.

m_X is the BBA focused on X defined by $m_X(Y) = 0$ if $Y \neq X$, and $m_X(Y) = 1$ if $Y = X$

Few strict distance metrics are possible

- Jousselme distance: $d_J(m_1, m_2) \triangleq \sqrt{0.5 \cdot (m_1 - m_2)^T \mathbf{Jac}\, (m_1 - m_2)}$
- Euclidean d_{BI} distance: $d^E_{BI}(m_1, m_2) \triangleq \sqrt{\frac{1}{2^{|\Theta|-1}} \cdot \sum_{A \in 2^\Theta} d^I(BI_1(A), BI_2(A))^2}$
- Chebyshev d_{BI} distance: $d^C_{BI}(m_1, m_2) \triangleq \max_{A \in 2^\Theta} \left\{ d^I\left(BI_1(A), BI_2(A)\right)\right\}$

d^I is Wasserstein distance of interval numbers. In practice, we recommend to use $d^E_{BI}(m_1, m_2)$ [Han Dezert Yang 2017]

Quality of the decision

$$q(\hat{X}) = 1 - \frac{d_{BI}(m, m_{\hat{X}})}{\sum_{X \in \mathcal{X}} d_{BI}(m, m_X)} \in [0, 1]$$

Higher is $q(\hat{X})$ more trustable is the decision $\delta = \hat{X}$

General mono-criterion decision-making problem

How to make a decision among several possible choices, based on some contexts ?

Problem modeling

$q \geqslant 2$ alternatives (choices) $\mathcal{A} = \{A_1, \dots, A_q\}$

$n \geqslant 1$ states of nature (contexts) $\mathcal{S} = \{S_1, \dots, S_n\}$

$$\mathbf{C} \triangleq \begin{array}{c} \\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_q \end{array} \begin{array}{c} \begin{array}{ccccc} S_1 & \dots & S_j & \dots & S_n \end{array} \\ \begin{bmatrix} C_{11} & \dots & C_{1j} & \dots & C_{1n} \\ & & \vdots & & \\ C_{i1} & \dots & C_{ij} & \dots & S_{in} \\ & & \vdots & & \\ C_{q1} & \dots & C_{qj} & \dots & C_{qn} \end{bmatrix} \end{array}$$

$\mathbf{C}$ is the benefit (payoff) matrix of the problem under consideration

Investment company example

An investment company wants to invest a given amount of money in the best option $A^* \in \mathcal{A} = \{A_1, A_2, A_3\}$, where A_1 = car company, A_2 = food company, and A_3 = computer company. Several scenarios (states of nature) S_i are identified depending on national economical situation predictions, which provide the elements of the payoff matrix $\mathbf{C}$ according to a given criteria (growth analysis criterion by example).

Several decision-making frameworks are possible

- Decision under certainty
  If we know the **true** state of nature is S_j, take as decision $\delta = A^*$ with

$$A^* = A_{i*} \quad \text{with} \quad i^* = \arg\max_i \{C_{ij}\}$$

- Decision under risk
  If we know **all** probabilities $p_j = P(S_j)$ of the states of nature, compute the expected benefit $E[C_i] = \sum_j p_j C_{ij}$ of each A_i and take as decision $\delta = A^*$ with

$$A^* = A_{i*} \quad \text{with} \quad i^* = \arg\max_i \{E[C_i]\}$$

- Decision under ignorance
  If **we don't know the probabilities** $p_j = P(S_j)$ of the states of nature, use OWA (**O**rdered **W**eighted **A**veraging) approach [Yager 1988], or Cautious-OWA [Tacnet Dezert 2011], or Fuzzy-Cautious-OWA [Han Dezert Tacnet Han 2012]
- Decision under uncertainty
  If we have **only a BBA** over the states of the nature $\mathcal{S} = \{S_1, \ldots, S_n\}$ defined on the power set $2^{\mathcal{S}}$, we can use Yager extended OWA approach.

$$\mathbf{C} \triangleq \begin{array}{c} \\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_q \end{array} \begin{array}{c} \begin{array}{ccccc} S_1, p_1 & \ldots & S_j, p_j & \ldots & S_n, p_n \end{array} \\ \begin{bmatrix} C_{11} & \ldots & C_{1j} & \ldots & C_{1n} \\ & & \vdots & & \\ C_{i1} & \ldots & C_{ij} & \ldots & C_{in} \\ & & \vdots & & \\ C_{q1} & \ldots & C_{qj} & \ldots & C_{qn} \end{bmatrix} \end{array} \Rightarrow E[\mathbf{C}] = \begin{bmatrix} E[C_1] = \sum_j p_j C_{1j} \\ \vdots \\ E[C_i] = \sum_j p_j C_{ij} \\ \vdots \\ E[C_q] = \sum_j p_j C_{qj} \end{bmatrix}$$

Decision: A^* is the chosen alternative corresponding to highest expected benefit.

Example

$$\mathbf{C} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{ccc} S_1, p_1 = 1/4 & S_2, p_2 = 1/4 & S_3, p_3 = 1/2 \end{array} \\ \begin{bmatrix} 16 & 12 & 20 \\ 32 & 4 & 6 \\ 12 & 20 & 4 \\ 40 & 4 & 8 \end{bmatrix} \end{array} \Rightarrow \begin{bmatrix} E[C_1] = (1/4)16 + (1/4)12 + (1/2)20 = 17 \\ E[C_2] = (1/4)32 + (1/4)4 + (1/2)6 = 12 \\ E[C_3] = (1/4)12 + (1/4)20 + (1/2)4 = 10 \\ E[C_4] = (1/4)40 + (1/4)4 + (1/2)8 = 15 \end{bmatrix}$$

Sorting the expected benefits by their decreasing values gives the ranking

$$A_1 \succ A_4 \succ A_2 \succ A_3$$

The decision to take is $A^* = A_1$

Example of decision under ignorance with OWA

The probabilities $p_j = P(S_j)$ of the states of the nature are unknown

$$\mathbf{C} = \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} \begin{bmatrix} S_1, p_1 = ? & S_2, p_2 = ? & S_3, p_3 = ? & S_4, p_4 = ? \\ 10 & 0 & 20 & 30 \\ 1 & 10 & 20 & 30 \\ 30 & 10 & 2 & 5 \end{bmatrix}$$

- OWA result with **optimistic** attitude $\mathbf{w} = [1\ 0\ 0\ 0] \rightarrow$ we take the max by row

$$\begin{cases} V_1 = OWA(10,0,20,30) = \mathbf{w} \cdot [30\ 20\ 10\ 0]' = 30 \\ V_2 = OWA(1,10,20,30) = \mathbf{w} \cdot [30\ 20\ 10\ 1]' = 30 \\ V_3 = OWA(30,10,2,5) = \mathbf{w} \cdot [30\ 10\ 5\ 2]' = 30 \end{cases} \Rightarrow \text{No best choice exists}$$

- OWA result with **Hurwicz** attitude with $\alpha = 0.5 \Rightarrow \mathbf{w} = [(1/2)\ 0\ 0\ (1/2)]$

$$\begin{cases} V_1 = OWA(10,0,20,30) = \mathbf{w} \cdot [30\ 20\ 10\ 0]' = (30/2) + (0/2) = 15 \\ V_2 = OWA(1,10,20,30) = \mathbf{w} \cdot [30\ 20\ 10\ 1]' = (30/2) + (1/2) = 15.5 \\ V_3 = OWA(30,10,2,5) = \mathbf{w} \cdot [30\ 10\ 5\ 2]' = (30/2) + (2/2) = 16 \end{cases} \Rightarrow A_3 \text{ is the best choice}$$

- OWA result with **normative** attitude $\mathbf{w} = [(1/4)\ (1/4)\ (1/4)\ (1/4)]$

$$\begin{cases} V_1 = OWA(10,0,20,30) = \mathbf{w} \cdot [30\ 20\ 10\ 0]' = 60/4 = 15 \\ V_2 = OWA(1,10,20,30) = \mathbf{w} \cdot [30\ 20\ 10\ 1]' = 61/4 \\ V_3 = OWA(30,10,2,5) = \mathbf{w} \cdot [30\ 10\ 5\ 2]' = 47/4 \end{cases} \Rightarrow A_2 \text{ is the best choice}$$

- OWA result with **pessimistic** attitude $\mathbf{w} = [0\ 0\ 0\ 1] \rightarrow$ we take the min by row

$$\begin{cases} V_1 = OWA(10,0,20,30) = \mathbf{w} \cdot [30\ 20\ 10\ 0]' = 0 \\ V_2 = OWA(1,10,20,30) = \mathbf{w} \cdot [30\ 20\ 10\ 1]' = 1 \\ V_3 = OWA(30,10,2,5) = \mathbf{w} \cdot [30\ 10\ 5\ 2]' = 2 \end{cases} \Rightarrow A_3 \text{ is the best choice}$$

Probas $p_j = P(S_j)$ of the states S_j **are unknown**, but we know a BBA $m(\cdot) : 2^{\mathcal{S}} \mapsto [0, 1]$

$$
\begin{array}{c} \\ \mathbf{C} = [\mathbf{c}_1 \ldots \mathbf{c}_j \ldots \mathbf{c}_n] \triangleq \begin{array}{c} A_1 \\ \vdots \\ A_i \\ \vdots \\ A_q \end{array} \end{array}
\begin{array}{c} \begin{array}{ccccc} S_1, p_1 =? & \ldots & S_j, p_j =? & \ldots & S_n, p_n =? \end{array} \\
\begin{bmatrix} C_{11} & \ldots & C_{1j} & \ldots & C_{1n} \\ & & \vdots & & \\ C_{i1} & \ldots & C_{ij} & \ldots & C_{in} \\ & & \vdots & & \\ C_{q1} & \ldots & C_{qj} & \ldots & C_{qn} \end{bmatrix} \end{array}
$$

Method 1: Approximate $m(\cdot)$ by a proba measure $\Rightarrow$ decison-making under risk

Method 2: Extended OWA method [Yager 1988]

1. Decisional attitude: choose the decisional attitude (optimistic,pessimistic, etc)
2. Apply OWA on each sub-matrix $\mathbf{C}(X_k)$ of benefits associated with the focal element X_k, $k = 1, \ldots, r$ of $m(\cdot)$ to get valuations $V_i(X_k)$, $i = 1, \ldots, q$

$$\mathbf{C}(X_k) = [\mathbf{c}_j | S_j \subseteq X_k]$$

3. Compute the generalized expected benefits for $i = 1, \ldots, q$

$$E[C_i] = \sum_{k=1}^{r} m(X_k) V_i(X_k)$$

4. Decision: take the decision $\delta = A^* = A_{i*}$ with $i^* = \arg\max_i \{E[C_i]\}$

Probas $p_j = P(S_j)$ of the states S_j **are unknown**, but we know a BBA $m(\cdot) : 2^{\mathcal{S}} \mapsto [0, 1]$

$$\mathbf{C} = \begin{matrix} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} \begin{matrix} S_1, p_1 =? & S_2, p_2 =? & S_3, p_3 =? & S_4, p_4 =? & S_5, p_5 =? \\ \left[\begin{matrix} 7 \\ 12 \\ 9 \\ 6 \end{matrix} \right. & \begin{matrix} 5 \\ 10 \\ 13 \\ 9 \end{matrix} & \begin{matrix} 12 \\ 5 \\ 3 \\ 11 \end{matrix} & \begin{matrix} 13 \\ 11 \\ 10 \\ 15 \end{matrix} & \left. \begin{matrix} 6 \\ 2 \\ 9 \\ 4 \end{matrix} \right] \end{matrix}$$

The uncertainty is modeled by a BBA with 3 focal elements as follows

BBA\FE	$X_1 \triangleq S_1 \cup S_3 \cup S_4$	$X_2 \triangleq S_2 \cup S_5$	$X_3 \triangleq S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$
$m(\cdot)$	0.6	0.3	0.1

Construction of benefit sub-matrices for each focal element of $m(\cdot)$

$$\mathbf{C}(X_1) = \begin{matrix} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} \begin{matrix} S_1 & S_3 & S_4 \\ \left[\begin{matrix} 7 \\ 12 \\ 9 \\ 6 \end{matrix} \right. & \begin{matrix} 12 \\ 5 \\ 3 \\ 11 \end{matrix} & \left. \begin{matrix} 13 \\ 11 \\ 10 \\ 15 \end{matrix} \right] \end{matrix} \quad \mathbf{C}(X_2) = \begin{matrix} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} \begin{matrix} S_2 & S_5 \\ \left[\begin{matrix} 5 \\ 10 \\ 13 \\ 9 \end{matrix} \right. & \left. \begin{matrix} 6 \\ 2 \\ 9 \\ 4 \end{matrix} \right] \end{matrix} \quad \mathbf{C}(X_3) = \begin{matrix} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} \begin{matrix} S_1 & S_2 & S_3 & S_4 & S_5 \\ \left[\begin{matrix} 7 \\ 12 \\ 9 \\ 6 \end{matrix} \right. & \begin{matrix} 5 \\ 10 \\ 13 \\ 9 \end{matrix} & \begin{matrix} 12 \\ 5 \\ 3 \\ 11 \end{matrix} & \begin{matrix} 13 \\ 11 \\ 10 \\ 15 \end{matrix} & \left. \begin{matrix} 6 \\ 2 \\ 9 \\ 4 \end{matrix} \right] \end{matrix}$$

Using **pessimistic** decisional attitude

- Apply OWA for each sub-matrix $\mathbf{C}(X_k)$, $k = 1, 2, 3$

$$\mathbf{C}(X_1) = \begin{matrix} & s_1 & s_3 & s_4 \\ A_1 & 7 & 12 & 13 \\ A_2 & 12 & 5 & 11 \\ A_3 & 9 & 3 & 10 \\ A_4 & 6 & 11 & 15 \end{matrix} \Rightarrow \begin{cases} V_1(X_1) = OWA(7,12,13) = [0\ 0\ 1]\cdot[13\ 12\ 7]' = 7 \\ V_2(X_1) = OWA(12,5,11) = [0\ 0\ 1]\cdot[12\ 11\ 5]' = 5 \\ V_3(X_1) = OWA(9,3,10) = [0\ 0\ 1]\cdot[10\ 9\ 3]' = 3 \\ V_4(X_1) = OWA(6,11,15) = [0\ 0\ 1]\cdot[15\ 11\ 6]' = 6 \end{cases}$$

$$\mathbf{C}(X_2) = \begin{matrix} & s_2 & s_5 \\ A_1 & 5 & 6 \\ A_2 & 10 & 2 \\ A_3 & 13 & 9 \\ A_4 & 9 & 4 \end{matrix} \Rightarrow \begin{cases} V_1(X_2) = OWA(5,6) = [0\ 1]\cdot[6\ 5]' = 5 \\ V_2(X_2) = OWA(10,2) = [0\ 1]\cdot[10\ 2]' = 2 \\ V_3(X_2) = OWA(13,9) = [0\ 1]\cdot[13\ 9]' = 9 \\ V_4(X_2) = OWA(9,4) = [0\ 1]\cdot[9\ 4]' = 4 \end{cases}$$

$$\mathbf{C}(X_3) = \begin{matrix} & s_1 & s_2 & s_3 & s_4 & s_5 \\ A_1 & 7 & 5 & 12 & 13 & 6 \\ A_2 & 12 & 10 & 5 & 11 & 2 \\ A_3 & 9 & 13 & 3 & 10 & 9 \\ A_4 & 6 & 9 & 11 & 15 & 4 \end{matrix} \Rightarrow \begin{cases} V_1(X_3) = OWA(7,5,12,13,6) = [0\ 0\ 0\ 0\ 1]\cdot[13\ 12\ 7\ 6\ 5]' = 5 \\ V_2(X_3) = OWA(12,10,5,11,2) = [0\ 0\ 0\ 0\ 1]\cdot[12\ 11\ 10\ 5\ 2]' = 2 \\ V_3(X_3) = OWA(9,13,3,10,9) = [0\ 0\ 0\ 0\ 1]\cdot[13\ 10\ 9\ 9\ 3]' = 3 \\ V_4(X_3) = OWA(6,9,11,15,4) = [0\ 0\ 0\ 0\ 1]\cdot[15\ 11\ 9\ 6\ 4]' = 4 \end{cases}$$

- Compute generalized expected benefits $E[C_i] = \sum_k m(X_k)V_i(X_k)$ with $m(X_1) = 0.6$, $m(X_2) = 0.3$ and $m(X_3) = 0.1$

$$E[C_1] = 0.6\cdot 7 + 0.3\cdot 5 + 0.1\cdot 5 = 6.2$$
$$E[C_2] = 0.6\cdot 5 + 0.3\cdot 2 + 0.1\cdot 2 = 3.8$$
$$E[C_3] = 0.6\cdot 3 + 0.3\cdot 9 + 0.1\cdot 3 = 4.8$$
$$E[C_4] = 0.6\cdot 6 + 0.3\cdot 4 + 0.1\cdot 4 = 5.2$$

- Take final decision with alternative having highest expected benefit $\rightarrow A^* = A_1$

Using **optimistic** decisional attitude

- Apply OWA for each sub-matrix $\mathbf{C}(X_3)$, $k = 1, 2, 3$

$$\mathbf{C}(X_1) = \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} \overset{\begin{matrix} s_1 & s_3 & s_4 \end{matrix}}{\begin{bmatrix} 7 & 12 & 13 \\ 12 & 5 & 11 \\ 9 & 3 & 10 \\ 6 & 11 & 15 \end{bmatrix}} \Rightarrow \begin{cases} V_1(X_1) = OWA(7,12,13) = [1\ 0\ 0] \cdot [13\ 12\ 7]' = 13 \\ V_2(X_1) = OWA(12,5,11) = [1\ 0\ 0] \cdot [12\ 11\ 5]' = 12 \\ V_3(X_1) = OWA(9,3,10) = [1\ 0\ 0] \cdot [10\ 9\ 3]' = 10 \\ V_4(X_1) = OWA(6,11,15) = [1\ 0\ 0] \cdot [15\ 11\ 6]' = 15 \end{cases}$$

$$\mathbf{C}(X_2) = \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} \overset{\begin{matrix} s_2 & s_5 \end{matrix}}{\begin{bmatrix} 5 & 6 \\ 10 & 2 \\ 13 & 9 \\ 9 & 4 \end{bmatrix}} \Rightarrow \begin{cases} V_1(X_2) = OWA(5,6) = [1\ 0] \cdot [6\ 5]' = 6 \\ V_2(X_2) = OWA(10,2) = [1\ 0] \cdot [10\ 2]' = 10 \\ V_3(X_2) = OWA(13,9) = [1\ 0] \cdot [13\ 9]' = 13 \\ V_4(X_2) = OWA(9,4) = [1\ 0] \cdot [9\ 4]' = 9 \end{cases}$$

$$\mathbf{C}(X_3) = \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} \overset{\begin{matrix} s_1 & s_2 & s_3 & s_4 & s_5 \end{matrix}}{\begin{bmatrix} 7 & 5 & 12 & 13 & 6 \\ 12 & 10 & 5 & 11 & 2 \\ 9 & 13 & 3 & 10 & 9 \\ 6 & 9 & 11 & 15 & 4 \end{bmatrix}} \Rightarrow \begin{cases} V_1(X_3) = OWA(7,5,12,13,6) = [1\ 0\ 0\ 0\ 0] \cdot [13\ 12\ 7\ 6\ 5]' = 13 \\ V_2(X_3) = OWA(12,10,5,11,2) = [1\ 0\ 0\ 0\ 0] \cdot [12\ 11\ 10\ 5\ 2]' = 12 \\ V_3(X_3) = OWA(9,13,3,10,9) = [1\ 0\ 0\ 0\ 0] \cdot [13\ 10\ 9\ 9\ 3]' = 13 \\ V_4(X_3) = OWA(6,9,11,15,4) = [1\ 0\ 0\ 0\ 0] \cdot [15\ 11\ 9\ 6\ 4]' = 15 \end{cases}$$

- Compute generalized expected benefits $E[C_i] = \sum_k m(X_k)V_i(X_k)$ with $m(X_1) = 0.6$, $m(X_2) = 0.3$ and $m(X_3) = 0.1$

$$E[C_1] = 0.6 \cdot 13 + 0.3 \cdot 6 + 0.1 \cdot 13 = 10.9$$
$$E[C_2] = 0.6 \cdot 12 + 0.3 \cdot 10 + 0.1 \cdot 12 = 11.4$$
$$E[C_3] = 0.6 \cdot 10 + 0.3 \cdot 13 + 0.1 \cdot 13 = 11.2$$
$$E[C_4] = 0.6 \cdot 15 + 0.3 \cdot 9 + 0.1 \cdot 15 = 13.2$$

- Take final decision with alternative having highest expected benefit $\rightarrow A^* = A_4$

Advantage, limitation and improvement of OWA

Advantage of OWA

Very simple to apply

Limitation of OWA

The result strongly depends on the decisional attitude chosen when applying OWA
How to avoid this? → complicate methods exist to select weights (using entropy)

Improvements of OWA

Use jointly the two most extreme decisional attitudes (pessimistic and optimistic) to be more cautious, which can be done as follows

1. Applying OWA using Hurwicz attitude by taking $\alpha = 1/2$
   → a balance only between min and max benefit values
2. Applying modified OWA based on belief functions
   → we use all benefit values between min and max

   - Cautious OWA (COWA) [Tacnet Dezert 2011]

     Pessimistic and optimistic generalized expected benefits allow to build belief intervals, and to get BBAs that are combined with PCR6 to get combined BBA from which the final decision is taken.

   - Fuzzy Cautious OWA (FCOWA) [Han Dezert Tacnet Han 2012]

     A version of COWA more efficient and more simple to implement

Fuzzy Cautious OWA method

At first, apply OWA with pessimistic and optimistic attitudes to get bounds $[E^{\min}[C_i], E^{\max}[C_i]]$ of expected benefits of each alternative A_i

Main steps of Fuzzy Cautious OWA (FCOWA) [Han Dezert Tacnet Han 2012]

1. Normalize each column $E^{\min}[C]$ and $E^{\max}[C]$ **separately** to obtain $E^{\text{Fuzzy}}(C)$
2. Conversion of the two normalized columns, i.e. two Fuzzy Membership Functions (FMF), into **two pessimistic and optimistic BBAs** $m_{\text{Pess}}(\cdot)$ and $m_{\text{Opti}}(\cdot)$
3. Combination of $m_{\text{Pess}}(\cdot)$ and $m_{\text{Opti}}(\cdot)$ to get a fused BBA $m(\cdot)$
4. Final decision drawn from $m(\cdot)$ by a chosen decision rule, for example by max of BetP, DSmP, or by min of d_{BI}

Advantages of FCOWA

- only 2 BBAs are involved in the combination $\Rightarrow$ only one fusion step is needed
- the BBAs in FCOWA (built by using alpha-cuts) are consonant support (FE are nested), which brings less computational complexity than with COWA
- good performances of FCOWA w.r.t. COWA
- good robustness of FCOWA to scoring errors w.r.t. COWA

Physical meaning

In FCOWA, the 2 SoE are pessimistic OWA and optimistic OWA. The combination result can be regarded as a tradeoff between these two attitudes.

The uncertainty of the states is modeled by the following BBA (previous example)

BBA\FE	$X_1 \triangleq S_1 \cup S_3 \cup S_4$	$X_2 \triangleq S_2 \cup S_5$	$X_3 \triangleq S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$
$m(\cdot)$	0.6	0.3	0.1

From the benefit matrix, we get the expected pessimistic and optimistic benefits (previous example)

$$\mathbf{C} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{ccccc} S_1 & S_2 & S_3 & S_4 & S_5 \end{array} \\ \begin{bmatrix} 7 & 5 & 12 & 13 & 6 \\ 12 & 10 & 5 & 11 & 2 \\ 9 & 13 & 3 & 10 & 9 \\ 6 & 9 & 11 & 15 & 4 \end{bmatrix} \end{array} \Rightarrow \mathbf{E}[C] = \begin{bmatrix} E^{min}[C_1] = 6.2 & E^{max}[C_1] = 10.9 \\ E^{min}[C_2] = 3.8 & E^{max}[C_2] = 11.4 \\ E^{min}[C_3] = 4.8 & E^{max}[C_3] = 11.2 \\ E^{min}[C_4] = 5.2 & E^{max}[C_4] = 13.2 \end{bmatrix}$$

Step 1 of FCOWA: Normalization of each column of expected benefit matrix $\mathbf{E}[C]$

$$\mathbf{E}^{Fuzzy}(C) = \begin{bmatrix} 6.2/6.2 & 10.9/13.2 \\ 3.8/6.2 & 11.4/13.2 \\ 4.8/6..2 & 11.2/13.2 \\ 5.2/6.2 & 13.2/13.2 \end{bmatrix} \approx \begin{array}{c} \begin{array}{cc} FMF1\mu_1(\cdot) & FMF2\mu_2(\cdot) \end{array} \\ \begin{bmatrix} 1 & 0.8258 \\ 0.6129 & 0.8636 \\ 0.7742 & 0.8485 \\ 0.8387 & 1 \end{bmatrix} \end{array}$$

Step 2 of FCOWA: Construction of m_{Pess} from μ_1, and m_{Opti} from μ_2

based on α-cut method [Orlov 1978, Goodman 1982, Florea et al. 2003, Yi et al. 2016]
We sort μ values in increasing order $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_M \leqslant 1$
From the FMF μ we compute mass $m(B_j) = \frac{\alpha_j - \alpha_{j-1}}{\alpha_M}$ where focal element B_j is defined by $B_j = \{A_i \in \Theta | \mu(A_i) \geqslant \alpha_j\}$.

Example: From the FMF μ_1, one has

$$\alpha_1 = \mu_1(A_2) = 0.6129 < \alpha_2 = \mu_1(A_3) = 0.7742 < \alpha_3 = \mu_1(A_4) = 0.8387 < \alpha_4 = \mu_1(A_1) = 1$$

Focal element $B_3 = \{A_i \in \Theta | \mu(A_i) \geqslant \alpha_3\} = \{A_1, A_4\}$ because $\mu_1(A_1) > \alpha_3$ and $\mu_1(A_4) > \alpha_3$. Hence

$$m_{Pess}(B_3) = m_{Pess}(A_1 \cup A_4) = \frac{\alpha_3 - \alpha_2}{\alpha_4} = \frac{0.8387 - 0.7742}{1} = 0.0645$$

Finally, we get

Focal Element	$m_{Pess}(.)$	Focal Element	$m_{Opti}(.)$
$A_1 \cup A_2 \cup A_3 \cup A_4$	0.6129	$A_1 \cup A_2 \cup A_3 \cup A_4$	0.8257
$A_1 \cup A_3 \cup A_4$	0.1613	$A_2 \cup A_3 \cup A_4$	0.0227
$A_1 \cup A_4$	0.0645	$A_2 \cup A_4$	0.0152
A_1	0.1613	A_4	0.1364

Step 3 of FCOWA: Combination of BBAs m_{Pess} and m_{Opti} to get the fused BBA $m(\cdot)$
Step 4 of FCOWA: Decision-making from $m(\cdot)$

Methods for Multi-Criteria Decision-Making support

Classical Multi-Criteria Decision-Making (MCDM) problem

How to make a choice among several alternatives based on different criteria?

Problem modeling 1 $\Rightarrow$ using pairwise comparison matrices $\rightarrow$ **AHP methods**
We consider a set of criteria $C_1, \ldots, C_N$ with preferences of importance established from a pairwise comparison matrix (PCM) $\mathbf{M}$. For each criteria C_j, a set of preferences of the alternatives is established from a given **pairwise comparison matrix** $\mathbf{M}_j$.

Problem modeling 2 $\Rightarrow$ using directly the score matrix $\rightarrow$ **TOPSIS methods**

- A set of $M \geqslant 2$ alternatives $\mathcal{A} \triangleq \{A_1, \ldots, A_M\}$
- A set of $N > 1$ Criteria $\mathbf{C} \triangleq \{C_1, \ldots, C_N\}$
- A set of $N > 1$ criteria **importance weights** $W = \{w_1, \ldots, w_N\}$, with $w_j \in [0, 1]$ and $\sum_j w_j = 1$

$$\mathbf{S} \triangleq \begin{array}{c} \\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_M \end{array} \begin{array}{c} \begin{array}{ccccc} C_1, w_1 & \ldots & C_j, w_j & \ldots & C_N, w_N \end{array} \\ \begin{bmatrix} S_{11} & \ldots & S_{1j} & \ldots & S_{1N} \\ & & \vdots & & \\ S_{i1} & \ldots & S_{ij} & \ldots & S_{iN} \\ & & \vdots & & \\ S_{M1} & \ldots & S_{Mj} & \ldots & S_{MN} \end{bmatrix} \end{array}$$

$\mathbf{S}$ is the score matrix of the MCDM problem under consideration

Car example: How to buy a car based on some criteria (i.e. cost, safety, etc.)?

Important remarks

- All methods developed so far suffer from rank reversal problem [Wang Luo 2009], which means that the rank is changed by adding (or deleting) an alternative in the problem. We consider rank reversal as very serious drawback.
- Most of existing methods require score normalization at first, except for ERV (Estimator Ranking Vector) method [Yin et al. 2013]. Normalization has been identified as one of the origins of rank reversal problem.
- There is no MCDM method which makes consensus among users, . . . but some are very popular
  - AHP (Analytic Hierarchy Process) method is very popular in operational research community but not exempt of problems
  - TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) method is very popular **but the choice of normalization is disputed**

What is presented in this course

- Belief-Function-based TOPSIS methods called BF-TOPSIS to solve classical and non-classical MCDM problems [Dezert Han Yin 2016, Carladous et al. 2016]

What is not presented

- AHP method and its extension DSm-AHP using belief functions [Saaty 1980, Dezert et al. 2010, Dezert Tacnet 2011]

TOPSIS = **T**echnique for **O**rder **P**reference by **S**imilarity to **I**deal **S**olution

Classical TOPSIS method [Hwang Yoon 1981]

1. Build the normalized score matrix $\mathbf{R} = [R_{ij}] = [S_{ij}/\sqrt{\sum_i S_{ij}^2}]$
2. Calculate the weighted normalized decision matrix $\mathbf{D} = [w_j \cdot R_{ij}]$
3. Determine the positive (best) ideal solution A^{best} by **taking the best/max value** in each column of $\mathbf{D}$
4. Determine the negative (worst) ideal solution A^{worst} by **taking the worst/min value** in each column of $\mathbf{D}$
5. Compute L2-distances $d(A_i, A^{best})$ of A_i, (i=1,...,M) to A^{best}, and $d(A_i, A^{worst})$ of A_i to A^{worst}
6. Calculate the relative closeness of A_i to best ideal solution A^{best} by

$$C(A_i, A^{best}) \triangleq \frac{d(A_i, A^{worst})}{d(A_i, A^{worst}) + d(A_i, A^{best})}$$

   When $C(A_i, A^{best}) = 1$, its means that $A_i = A^{best}$ because $d(A_i, A^{best}) = 0$

   When $C(A_i, A^{best}) = 0$, its means that $A_i = A^{worst}$ because $d(A_i, A^{worst}) = 0$

7. Rank alternatives A_i according to $C(A_i, A^{best})$ in descending order, and select the highest preferred solution

A very simple example for TOPSIS

$$\mathbf{S} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} C_1, w_1 = 1/2 \quad C_2, w_2 = 1/2 \\ \begin{bmatrix} 6 & 2 \\ 3 & 5 \\ 4 & 4 \end{bmatrix} \end{array}$$

1. Step 1 & 2 (normalization & columns weighting):

$$\mathbf{R} = [S_{ij}/\sqrt{\sum_i S_{ij}^2}] \Rightarrow \mathbf{R} = \begin{array}{c} C_1, 1/2 \quad C_2, 1/2 \\ \begin{bmatrix} 0.7682 & 0.2981 \\ 0.3841 & 0.7454 \\ 0.5121 & 0.5963 \end{bmatrix} \end{array} \Rightarrow \mathbf{D} = \begin{bmatrix} 0.3841 & 0.1491 \\ 0.1921 & 0.3727 \\ 0.2561 & 0.2981 \end{bmatrix}$$

2. Step 3 & 4 (best and worst solutions) $A^{best} = [0.3841\ 0.3727]$, $A^{worst} = [0.1921\ 0.1491]$
3. Step 5 (L_2-distance of A_i to A^{best} and to A^{worst}):

$$A^{best} = [0.3841\ 0.3727] \quad A^{worst} = [0.1921\ 0.1491]$$

$$\begin{array}{l} A_1 = [0.3841\ 0.1491] \\ A_2 = [0.1921\ 0.3727] \\ A_3 = [0.2561\ 0.2981] \end{array} \begin{bmatrix} d(A_1, A^{best}) = 0.2236 & d(A_1, A^{worst}) = 0.1921 \\ d(A_2, A^{best}) = 0.1921 & d(A_2, A^{worst}) = 0.2236 \\ d(A_3, A^{best}) = 0.1482 & d(A_3, A^{worst}) = 0.1622 \end{bmatrix}$$

4. Step 6 (relative closeness of A_i to A^{best}): $C(A_i, A^{best}) \triangleq \frac{d(A_i, A^{worst})}{d(A_i, A^{worst}) + d(A_i, A^{best})}$

$$C(A_1, A^{best}) = 0.4620 \qquad C(A_2, A^{best}) = 0.5380 \qquad C(A_3, A^{best}) = 0.5227$$

5. Step 7 (ranking by decreasing order of $C(A_i, A^{best})$): $A_2 \succ A_3 \succ A_1$

Based on TOPSIS, the decision δ to make is $\delta = A_2$

BF-TOPSIS is a TOPSIS-alike method based on belief functions [Dezert Han Yin 2016]

Advantages of BF-TOPSIS

- no need for ad-hoc choice of scores normalization
- relatively simple to implement
- more robust to rank reversal phenomena (although not exempt)

Main problem to overcome

Working with belief functions requires the construction of BBAs.
How to build efficiently BBAs from the score values?

Solution → see next slides

Four BF-TOPSIS methods available with different complexities

1. BF-TOPSIS1: smallest complexity
2. BF-TOPSIS2: medium complexity
3. BF-TOPSIS3: high complexity (because of PCR6 fusion rule)
4. BF-TOPSIS4: high complexity (because of ZPCR6 fusion rule)

BF-TOPSIS for working with imprecise scores is presented in [Dezert Han Tacnet 2017], with implementation improvement in [Mahato et al. 2018].

- Positive support of A_i based on all scores values of a criteria C_j

$$\mathrm{Sup}_j(A_i) \triangleq \sum_{k\in\{1,\dots M\}|S_{kj}\leqslant S_{ij}} |S_{ij} - S_{kj}|$$

$\mathrm{Sup}_j(A_i)$ measures how much A_i is better (higher) than other alternatives

- Negative support of A_i based on all scores values of a criteria C_j

$$\mathrm{Inf}_j(A_i) \triangleq - \sum_{k\in\{1,\dots M\}|S_{kj}\geqslant S_{ij}} |S_{ij} - S_{kj}|$$

$\mathrm{Inf}_j(A_i)$ measures how much A_i is worse (lower) than other alternatives

Important inequality see proof in [Dezert Han Yin 2016]

$$\frac{\mathrm{Sup}_j(A_i)}{A^j_{\text{max}}} \leqslant 1 - \frac{\mathrm{Inf}_j(A_i)}{A^j_{\text{min}}}$$

iff $A^j_{\text{max}} \triangleq \max_i \mathrm{Sup}_j(A_i)$ and $A^j_{\text{min}} \triangleq \min_i \mathrm{Inf}_j(A_i)$ are different from zero.

Reminder $$\frac{Sup_j(A_i)}{A^j_{max}} \leqslant 1 - \frac{Inf_j(A_i)}{A^j_{min}}$$

Belief function modeling

$$Bel_{ij}(A_i) \triangleq \frac{Sup_j(A_i)}{A^j_{max}} \quad \text{and} \quad Bel_{ij}(\bar{A}_i) \triangleq \frac{Inf_j(A_i)}{A^j_{min}}$$

If $A^j_{max} = 0$, we set $Bel_{ij}(X_i) = 0$
If $A^j_{min} = 0$, we set $Pl_{ij}(A_i) = 1$ so that $Bel_{ij}(\bar{A}_i) = 0$

By construction, $$0 \leqslant Bel_{ij}(A_i) \leqslant (Pl_{ij}(A_i) = 1 - Bel_{ij}(\bar{A}_i)) \leqslant 1$$

BBA construction from Belief Interval

From $[Bel_{ij}(A_i), Pl_{ij}(A_i)]$, one gets the $M \times N$ BBAs matrix $\mathbf{M} = [m_{ij}(\cdot)]$ by taking

$$\begin{aligned} m_{ij}(A_i) &= Bel_{ij}(A_i) \\ m_{ij}(\bar{A}_i) &= Bel_{ij}(\bar{A}_i) = 1 - Pl_{ij}(A_i) \\ m_{ij}(A_i \cup \bar{A}_i) &= Pl_{ij}(A_i) - Bel_{ij}(A_i) \end{aligned}$$

Advantages of this BBA construction

1. if all S_{ij} are the same for a given column, we get $\forall A_i$, $Sup_j(A_i) = Inf_j(A_i) = 0$ and therefore $m_{ij}(A_i \cup \bar{A}_i) = 1$ which is the vacuous BBA, which makes sense.
2. it is invariant to the bias and scaling effects of score values. Indeed, if S_{ij} are replaced by $S'_{ij} = a \cdot S_{ij} + b$, with a scale factor $a > 0$ and a bias $b \in \mathbb{R}$, then $m_{ij}(\cdot)$ and $m'_{ij}(\cdot)$ remain equal.
3. if a numerical value S_{ij} is missing or indeterminate, then we use the vacuous belief assignment $m_{ij}(A_i \cup \bar{A}_i) = 1$.
4. We can also discount the BBA $m_{ij}(\cdot)$ by a reliability factor using the classical Shafer's discounting method if one wants to express some doubts on the reliability of $m_{ij}(\cdot)$.

In summary

From $[S_{ij}]$, we know how to build the matrix $\mathbf{M} = [(m_{ij}(A_i), m_{ij}(\bar{A}_i), m_{ij}(A_i \cup \bar{A}_i))]$

How to use these BBAs to rank A_i to make a decision? $\rightarrow$ BF-TOPSIS methods

Steps of BF-TOPSIS1 [Dezert Han Yin 2016]

1. From $\mathbf{S}$, compute BBAs $m_{ij}(A_i)$ $m_{ij}(\bar{A}_i)$, and $m_{ij}(A_i \cup \bar{A}_i)$
2. Set $m_{ij}^{\text{best}}(A_i) \triangleq 1$, and $m_{ij}^{\text{worst}}(\bar{A}_i) \triangleq 1$ and compute distances $d_{BI}^{E}(m_{ij}, m_{ij}^{\text{best}})$ and $d_{BI}^{E}(m_{ij}, m_{ij}^{\text{worst}})$ to ideal solutions.
3. Compute the weighted average distances of A_i to ideal solutions

$$d^{\text{best}}(A_i) \triangleq \sum_{j=1}^{N} w_j \cdot d_{BI}^{E}(m_{ij}, m_{ij}^{\text{best}})$$

$$d^{\text{worst}}(A_i) \triangleq \sum_{j=1}^{N} w_j \cdot d_{BI}^{E}(m_{ij}, m_{ij}^{\text{worst}})$$

4. Compute the relative closeness of A_i with respect to ideal best solution A^{best}

$$C(A_i, A^{\text{best}}) \triangleq \frac{d^{\text{worst}}(A_i)}{d^{\text{worst}}(A_i) + d^{\text{best}}(A_i)}$$

5. Rank A_i by $C(A_i, A^{\text{best}})$ in descending order.

Steps of BF-TOPSIS2 [Dezert Han Yin 2016]

1. From $\mathbb{S}$, compute BBAs $m_{ij}(A_i)$ $m_{ij}(\bar{A}_i)$, and $m_{ij}(A_i \cup \bar{A}_i)$
2. Set $m_{ij}^{\text{best}}(A_i) \triangleq 1$, and $m_{ij}^{\text{worst}}(\bar{A}_i) \triangleq 1$ and compute distances $d_{BI}^{E}(m_{ij}, m_{ij}^{\text{best}})$ and $d_{BI}^{E}(m_{ij}, m_{ij}^{\text{worst}})$ to ideal solutions.
3. For each criteria C_j, compute the relative closeness of A_i to best ideal solution A^{best} by

$$C_j(A_i, A^{\text{best}}) \triangleq \frac{d_{BI}^{E}(m_{ij}, m_{ij}^{\text{worst}})}{d_{BI}^{E}(m_{ij}, m_{ij}^{\text{worst}}) + d_{BI}^{E}(m_{ij}, m_{ij}^{\text{best}})}$$

4. Compute the weighted average of $C_j(A_i, A^{\text{best}})$ by

$$C(A_i, A^{\text{best}}) \triangleq \sum_{j=1}^{N} w_j \cdot C_j(A_i, A^{\text{best}})$$

5. Rank A_i by $C(A_i, A^{\text{best}})$ in descending order.

Steps of BF-TOPSIS3 [Dezert Han Yin 2016]

1. Compute BBAs $m_{ij}(A_i)$, $m_{ij}(\bar{A}_i)$ and $m_{ij}(A_i \cup \bar{A}_i)$ and apply importance discounting of each BBA with weight w_j, see [Smarandache Dezert Tacnet 2010]
2. For each A_i, fuse the discounted BBAs with PCR6 to get BBAs $m_i^{\text{PCR6}}(\cdot)$
3. Set $m_i^{\text{best}}(A_i) \triangleq 1$, and $m_i^{\text{worst}}(\bar{A}_i) \triangleq 1$. Compute distances

$$d^{\text{best}}(A_i) \triangleq d_{BI}^{E}(m_i^{\text{PCR6}}, m_i^{\text{best}})$$

$$d^{\text{worst}}(A_i) \triangleq d_{BI}^{E}(m_i^{\text{PCR6}}, m_i^{\text{worst}})$$

4. Compute the relative closeness of A_i, $i = 1, \ldots, M$, with respect to ideal best solution A^{best}

$$C(A_i, A^{\text{best}}) \triangleq \frac{d^{\text{worst}}(A_i)}{d^{\text{worst}}(A_i) + d^{\text{best}}(A_i)}$$

5. Rank A_i by $C(A_i, A^{\text{best}})$ in descending order.

BF-TOPSIS4 method

Same as BF-TOPSIS3, but PCR6 rule is replaced by ZPCR6 rule (i.e. PCR6 rule including Zhang's degree of intersection) [Smarandache Dezert 2015]

BF-TOPSIS methods **are consistent** with direct ranking in mono-criteria case

Example (Mono-criteria) Preference order → greater value is better

$$\mathbf{S} \triangleq \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \end{array} \overset{C_1}{\begin{bmatrix} 10 \\ 20 \\ -5 \\ 0 \\ 100 \\ -11 \\ 0 \end{bmatrix}} \Rightarrow \mathbf{M} \triangleq \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \end{array} \begin{bmatrix} m_{i1}(A_i) & m_{i1}(\bar{A}_i) & m_{i1}(A_i \cup \bar{A}_i) \\ 0.0955 & 0.5236 & 0.3809 \\ 0.1809 & 0.4188 & 0.4003 \\ 0.0102 & 0.8115 & 0.1783 \\ 0.0273 & 0.6806 & 0.2921 \\ 1.0000 & 0 & 0 \\ 0 & 1.0000 & 0 \\ 0.0273 & 0.6806 & 0.2921 \end{bmatrix} \Rightarrow \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \end{array} \overset{C(A_i, A^{\text{best}})}{\begin{bmatrix} 0.1130 \\ 0.1948 \\ 0.0257 \\ 0.0485 \\ 1.0000 \\ 0 \\ 0.0485 \end{bmatrix}}$$

Results

Ranking methods	**Preferences order**
By direct ranking	$A_5 \succ A_2 \succ A_1 \succ (A_4 \sim A_7) \succ A_3 \succ A_6$
By BF-TOPSIS	$A_5 \succ A_2 \succ A_1 \succ (A_4 \sim A_7) \succ A_3 \succ A_6$
By DS fusion	$A_5 \succ (A_1 \sim A_2 \sim A_3 \sim A_4 \sim A_6 \sim A_7)$
By PCR6 fusion	$A_5 \succ A_2 \succ A_1 \succ A_4 \succ (A_3 \sim A_6 \sim A_7)$

Ranking results of DS (Dempster-Shafer) fusion and PCR6 fusion of the BBAs do not match with direct ranking even in mono criteria case because of strong dependencies between BBAs in their construction.

In this example, we have $Score(A_5) >> Score(A_2)$

$$\mathbf{S} \triangleq \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \end{matrix} \overset{C_1}{\begin{bmatrix} 10 \\ 20 \\ -5 \\ 0 \\ 100 \\ -11 \\ 0 \end{bmatrix}} \Rightarrow \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \end{matrix} \overset{C(A_i, A^{\text{best}})}{\begin{bmatrix} 0.1130 \\ 0.1948 \\ 0.0257 \\ 0.0485 \\ 1.0000 \\ 0 \\ 0.0485 \end{bmatrix}} \Rightarrow A_5 \succ A_2 \succ A_1 \succ (A_4 \sim A_7) \succ A_3 \succ A_6$$

Let's modify the example with $Score(A_5) \sim Score(A_2)$

$$\mathbf{S} \triangleq \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \end{matrix} \overset{C_1}{\begin{bmatrix} 10 \\ 20 \\ -5 \\ 0 \\ 21 \\ -11 \\ 0 \end{bmatrix}} \Rightarrow \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \end{matrix} \overset{C(A_i, A^{\text{best}})}{\begin{bmatrix} 0.5072 \\ 0.9472 \\ 0.0675 \\ 0.1584 \\ 1.0000 \\ 0 \\ 0.1584 \end{bmatrix}} \Rightarrow A_5 \succ A_2 \succ A_1 \succ (A_4 \sim A_7) \succ A_3 \succ A_6$$

We see that A_2 is very close to ideal best solution, even if final result is unchanged.

When all scores are the same

- $\Rightarrow$ all BBAs are the same and equal to the vacuous BBA
- $\Rightarrow$ all closeness measures to best ideal solution are equal

$$\mathbf{S} \triangleq \begin{matrix} \\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_M \end{matrix} \begin{matrix} C_1 \\ \begin{bmatrix} s \\ \vdots \\ s \\ \vdots \\ s \end{bmatrix} \end{matrix} \Rightarrow \mathbf{M} \triangleq \begin{matrix} \\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_M \end{matrix} \begin{matrix} m_{i1}(A_i \cup \bar{A}_i) \\ \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix} \end{matrix} \Rightarrow \begin{matrix} \\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_M \end{matrix} \begin{matrix} C(A_i, A^{\text{best}}) \\ \begin{bmatrix} c \\ \vdots \\ c \\ \vdots \\ c \end{bmatrix} \end{matrix}$$

Conclusion: No specific choice can be drawn, which is perfectly normal.

Multi-Criteria example [Wang Luo 2009]

We consider 5 alternatives, and 4 criteria

$$\mathbf{S} \triangleq \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{cccc} C_1, \frac{1}{6} & C_2, \frac{1}{3} & C_3, \frac{1}{3} & C_4, \frac{1}{6} \end{array} \\ \begin{bmatrix} 36 & 42 & 43 & 70 \\ 25 & 50 & 45 & 80 \\ 28 & 45 & 50 & 75 \\ 24 & 40 & 47 & 100 \\ 30 & 30 & 45 & 80 \end{bmatrix} \end{array}$$

Rank reversal with TOPSIS

Set of alternatives	TOPSIS
$\{A_1, A_2, A_3\}$	$A_3 \succ A_2 \succ A_1$
$\{A_1, A_2, A_3, A_4\}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$\{A_1, A_2, A_3, A_4, A_5\}$	$A_3 \succ A_2 \succ A_4 \succ A_1 \succ A_5$
	Rank reversal

Rank reversal with BF-TOPSIS

Set of alternatives	BF-TOPSIS1 & BF-TOPSIS2	BF-TOPSIS3 & BF-TOPSIS4
$\{A_1, A_2, A_3\}$	$A_2 \succ A_3 \succ A_1$	$A_3 \succ A_2 \succ A_1$
$\{A_1, A_2, A_3, A_4\}$	$A_3 \succ A_2 \succ A_4 \succ A_1$	$A_3 \succ A_2 \succ A_4 \succ A_1$
$\{A_1, A_2, A_3, A_4, A_5\}$	$A_3 \succ A_2 \succ A_4 \succ A_1 \succ A_5$	$A_3 \succ A_2 \succ A_4 \succ A_1 \succ A_5$
	Rank reversal	No rank reversal

Car selection example

How to buy a car among 4 possible choices, and based on 5 different criteria with weights $w_1 = 5/17$, $w_2 = 4/17$, $w_3 = 4/17$, $w_4 = 1/17$, and $w_5 = 3/17$

- C_1 = price (in €); the least is the best
- C_2 = fuel consumption (in L/km); the least is the best
- C_3 = CO_2 emission (in g/km); the least is the best
- C_4 = fuel tank volume (in L); the biggest is the best
- C_5 = trunk volume (in L); the biggest is the best

Building the score matrix from http://www.choisir-sa-voiture.com

$$\mathbf{S} \triangleq \begin{array}{l} A_1 = \text{TOYOTA YARIS 69 VVT-i Tendance} \\ A_2 = \text{SUZUKI SWIFT MY15 1.2 VVT So'City} \\ A_3 = \text{VOLKSWAGEN POLO 1.0 60 Confortline} \\ A_4 = \text{OPEL CORSA 1.4 Turbo 100 ch Start/Stop Edition} \end{array} \overset{\begin{array}{ccccc} C_1, \frac{5}{17} & C_2, \frac{4}{17} & C_3, \frac{4}{17} & C_4, \frac{1}{17} & C_5, \frac{3}{17} \end{array}}{\begin{bmatrix} 15000 & 4.3 & 99 & 42 & 73 \\ 15290 & 5.0 & 116 & 42 & 892 \\ 15350 & 5.0 & 114 & 45 & 952 \\ 15490 & 5.3 & 123 & 45 & 1120 \end{bmatrix}}$$

A_1 **is the expected best choice** because the 3 most important criteria meet their best values for car A_1.

With classical TOPSIS $A_4 \succ A_1 \succ A_3 \succ A_2$ (counter-intuitive)

With all BF-TOPSIS methods $A_1 \succ A_3 \succ A_2 \succ A_4$ (which fits with what we expect)

Non classical MCDM problem

How to make a choice in $\mathcal{A}$ from multi-criteria scores expressed on power-set of $\mathcal{A}$?

$$
\begin{array}{ll}
 & \begin{array}{l} X_i \in 2^{\mathcal{A}} \\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_M \\ \vdots \\ A_1 \cup A_2 \\ \vdots \\ A_1 \cup \ldots \cup A_i \cup \ldots \cup A_M \end{array}
\end{array}
\quad
\mathbf{S} \triangleq
\begin{array}{c}
\begin{array}{ccccc} C_1, w_1 & \ldots & C_j, w_j & \ldots & C_N, w_N \end{array} \\
\begin{bmatrix}
S_{11} & \ldots & S_{1j} & \ldots & S_{1N} \\
 & & \vdots & & \\
S_{i1} & \ldots & S_{ij} & \ldots & S_{iN} \\
 & & \vdots & & \\
S_{M1} & \ldots & S_{Mj} & \ldots & S_{MN} \\
 & & \vdots & & \\
S_{(M+1)1} & \ldots & S_{(M+1)j} & \ldots & S_{(M+1)N} \\
 & & \vdots & & \\
S_{(2^M-1)1} & \ldots & S_{(2^M-1)j} & \ldots & S_{(2^M-1)N}
\end{bmatrix}
\end{array}
$$

See [Dezert Han Tacnet Carladous Yin 2016, Carladous 2017] for details

How to build $m(.) : 2^{\mathcal{A} \triangleq \{A_1, A_2, \ldots, A_M\}} \mapsto [0,1]$ from scores $\mathbf{S} \triangleq [S_{ij}]$?

Direct extension of BBA construction [Dezert Han Tacnet Carladous Yin 2016]

- Positive support of $X_i \in 2^{\mathcal{A}}$ based on all scores values of a criteria C_j

$$Sup_j(X_i) \triangleq \sum_{Y \in 2^{\mathcal{A}} | S_j(Y) \leqslant S_j(X_i)} |S_j(X_i) - S_j(Y)|$$

$Sup_j(X_i)$ measures how much X_i is better (higher) than other Y of $2^{\mathcal{A}}$

- Negative support of $X_i \in 2^{\mathcal{A}}$ based on all scores values of a criteria C_j

$$Inf_j(X_i) \triangleq - \sum_{Y \in 2^{\mathcal{A}} | S_j(Y) \geqslant S_j(X_i)} |S_j(X_i) - S_j(Y)|$$

$Inf_j(X_i)$ measures how much X_i is worse (lower) than other Y of $2^{\mathcal{A}}$

Belief function modeling

$$0 \leqslant \frac{Sup_j(X_i)}{X^j_{max}} \leqslant 1 - \frac{Inf_j(X_i)}{X^j_{min}} \leqslant 1 \Rightarrow \begin{cases} Bel_{ij}(X_i) \triangleq \frac{Sup_j(X_i)}{X^j_{max}}, & \text{with } X^j_{max} = \max_i Sup_j(X_i) \\ Bel_{ij}(\bar{X}_i) \triangleq \frac{Inf_j(X_i)}{X^j_{min}}, & \text{with } X^j_{min} = \min_i Inf_j(X_i) \end{cases}$$

Concrete (complicate) examples of non classical MCDM for Protecting housing areas against torrential floods has been studied in Carladous thesis [Carladous 2017]

Simple example

Five students $A_1, \ldots, A_5$ have to be ranked based on two criteria

- C_1 = long jump performance
- C_2 = collected funds for an animal protection project

The scores are given as follows

$$\mathbf{S} = \begin{array}{c} X_i \in 2^{\mathbf{A}} \\ A_1 \\ A_3 \\ A_4 \\ A_5 \\ A_1 \cup A_2 \\ A_3 \cup A_4 \end{array} \begin{array}{c} \begin{array}{cc} C_1, w_1 & C_2, w_2 \end{array} \\ \begin{bmatrix} 3.7\,\text{m} & ? \\ 3.6\,\text{m} & ? \\ 3.8\,\text{m} & ? \\ 3.7\,\text{m} & 640\,€ \\ ? & 600\,€ \\ ? & 650\,€ \end{bmatrix} \end{array}$$

Difficulties:

- Scores are given in different units and different scales
- Some scores values can be missing
- Criteria C_j do not have same weights of importance w_j (in general)

Step 1: BBA matrix construction

$$\mathbf{S} = \begin{array}{c} \text{FE} \in 2^{\mathcal{A}} \\ A_1 \\ A_3 \\ A_4 \\ A_5 \\ A_1 \cup A_2 \\ A_3 \cup A_4 \end{array} \begin{array}{c} \begin{array}{cc} C_1, w_1 & C_2, w_2 \end{array} \\ \begin{bmatrix} 3.7\,\text{m} & ? \\ 3.6\,\text{m} & ? \\ 3.8\,\text{m} & ? \\ 3.7\,\text{m} & 640\,€ \\ ? & 600\,€ \\ ? & 650\,€ \end{bmatrix} \end{array} \Rightarrow \mathbf{M} = \begin{array}{c} \begin{array}{cc} C_1, w_1 & C_2, w_2 \end{array} \\ \begin{bmatrix} (0.25, 0.25, 0.50) & (0, 0, 1) \\ (0, 1, 0) & (0, 0, 1) \\ (1, 0, 0) & (0, 0, 1) \\ (0.25, 0.25, 0.50) & (0.6667, 0.1111, 0.2222) \\ (0, 0, 1) & (0, 1, 0) \\ (0, 0, 1) & (1, 0, 0) \end{bmatrix} \end{array}$$

Step 2: distances to ideal best and worst solutions

Focal elem.	$d_{BI}(m_{i1}, m^{best})$	$d_{BI}(m_{i1}, m^{worst})$	$d_{BI}(m_{i2}, m^{best})$	$d_{BI}(m_{i2}, m^{worst})$
A_1	0.6016	0.2652	0.7906	0.2041
A_3	0.8416	0	0.7906	0.2041
A_4	0	0.8416	0.7906	0.2041
A_5	0.6016	0.2652	0.2674	0.5791
$A_1 \cup A_2$	0.5401	0.3536	0.6770	0
$A_3 \cup A_4$	0.5401	0.3536	0	0.6770

Steps 3-5: weighted distances with $w_1 = 1/3$ and $w_2 = 2/3$, closeness and ranking

Focal elem.	$d^{\text{best}}(X_i)$	$d^{\text{worst}}(X_i)$	$C(X_i, X^{\text{best}})$	Ranking
A_1	0.7276	0.2245	0.2358	4
A_3	0.8076	0.1361	0.1442	6
A_4	0.5270	0.4166	0.4415	3
A_5	0.3788	0.4745	0.5561	2
$A_1 \cup A_2$	0.6314	0.1179	0.1573	5
$A_3 \cup A_4$	0.1800	0.5692	0.7597	1

BF-ICrA for MCDM simplification

Purpose: Identify criteria that behave similarly for simplifying MCDM

Atanassov ICrA Method [Atanassov et al. 2014]

From the MCDM score matrix $\mathbf{M}$, build an inter criteria matrix (ICM) $\mathbf{K}$ whose components express the degree of agreement and disagreement between each possible pair of criteria.

Agreement score between C_j and $C_{j'}$

$$K^{\mu}_{jj'} = \sum_{i=1}^{M-1} \sum_{i'=i+1}^{M} [sgn(S_{ij} - S_{i'j})sgn(S_{ij'} - S_{i'j'}) + sgn(S_{i'j} - S_{ij})sgn(S_{i'j'} - S_{ij'})]$$

$K^{\mu}_{jj'}$ is the number of cases in which $S_{ij} > S_{i'j}$ and $S_{ij'} > S_{i'j'}$ hold simultaneously.

Disagreement score between C_j and $C_{j'}$

$$K^{\nu}_{jj'} = \sum_{i=1}^{M-1} \sum_{i'=i+1}^{M} [sgn(S_{ij} - S_{i'j})sgn(S_{i'j'} - S_{ij'}) + sgn(S_{i'j} - S_{ij})sgn(S_{ij'} - S_{i'j'})]$$

$K^{\nu}_{jj'}$ is the number of cases in which $S_{ij} > S_{i'j}$ and $S_{ij'} < S_{i'j'}$ hold simultaneously.

The signum function is chosen as $$sgn(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \leqslant 0 \end{cases}$$

Atanassov did prove that $0 \leq K^{\mu}_{jj'} + K^{\nu}_{jj'} \leq \frac{M(M-1)}{2}$ (M is the # of alternatives)

Hence $0 \leq \frac{2K^{\mu}_{jj'}}{M(M-1)} + \frac{2K^{\nu}_{jj'}}{M(M-1)} \leq 1$

degree of agreement between C_j and $C_{j'}$ ← $\mu_{jj'} \triangleq \frac{2K^{\mu}_{jj'}}{M(M-1)}$

$\nu_{jj'} \triangleq \frac{2K^{\nu}_{jj'}}{M(M-1)}$ → degree of disagreement between C_j and $C_{j'}$

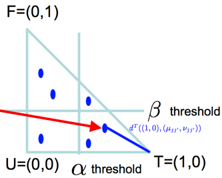
Inter-Criteria Matrix $\mathbf{K} = [K_{jj'}] = [(\mu_{jj'}, \nu_{jj'})]$

Inter-Criteria Analysis (ICrA)

Examine the location of points in the TFU triangle

- (α, β) agreement : If $\mu_{jj'} > \alpha$ and $\nu_{jj'} < \beta$.
- (α, β) disagreement : If $\mu_{jj'} < \beta$ and $\nu_{jj'} > \alpha$.
- (α, β) uncertainty: Otherwise.

$$d^T_{C_j C_{j'}} = d((1,0), (\mu_{jj'}, \nu_{jj'})) = \sqrt{(1-\mu_{jj'})^2 + \nu^2_{jj'}}$$



One can identify easily the criteria that are in strong agreement (i.e. those close to $T = (1,0)$), or in strong disagreement (i.e. those close to $F = (0,1)$).

Advantages of Atanassov's ICrA: Relatively easy to implement and use

Limitations of Atanassov's ICrA

1. Construction of $\mu_{jj'}$ and $\nu_{jj'}$ is very crude because it only counts the ">" or "<" inequalities, but not how bigger or how lower the score values are in making the comparison.
2. The construction of the Inter-Criteria Matrix $\mathbf{K}$ is not unique. It depends on the choice of signum function.
3. Atanassov ICrA method depends on the choice of α and β thresholds

Important remark: $\mu_{jj'}$ and $\nu_{jj'}$ can be interpreted in the BF framework by considering the Frame of Discernment (FoD)

$$\Theta = \{\theta = \text{"}C_j \text{ and } C_{j'} \text{ agree"}, \bar{\theta} = \text{"}C_j \text{ and } C_{j'} \text{ disagree"}\}$$

and the following relationships

$$m_{jj'}(\theta) = \mu_{jj'}$$
$$m_{jj'}(\bar{\theta}) = \nu_{jj'}$$
$$m_{jj'}(\theta \cup \bar{\theta}) = 1 - \mu_{jj'} - \nu_{jj'}$$

→ Development of a new BF-ICrA method

BF-ICrA is presented in [Dezert et al. 2019], with application in [Fidanova et al. 2019].

Step 1 of BF-ICrA: Construction of BBA matrix

We use method developed in BF-TOPSIS.
For each column (criteria) C_j of the score matrix $\mathbf{S}$) we compute the BBAs

$$m_{ij}(A_i) = Bel_{ij}(A_i)$$
$$m_{ij}(\bar{A}_i) = Bel_{ij}(\bar{A}_i)$$
$$m_{ij}(A_i \cup \bar{A}_i) = 1 - m_{ij}(A_i) - m_{ij}(\bar{A}_i)$$

with

$$\begin{cases} Bel_{ij}(A_i) = \frac{Sup_j(A_i)}{\max_i Sup_j(A_i)} \\ Bel_{ij}(\bar{A}_i) = \frac{Inf_j(A_i)}{\min_i Inf_j(A_i)} \end{cases}$$

and

$$\begin{cases} Sup_j(A_i) = \sum_{k\in\{1,\ldots,M\}|S_{kj}\leqslant S_{ij}} |S_{ij} - S_{kj}| \\ Inf_j(A_i) = -\sum_{k\in\{1,\ldots,M\}|S_{kj}\geqslant S_{ij}} |S_{ij} - S_{kj} \end{cases}$$

So finally from score matrix $\mathbf{S}$, we get BBA matrix $\mathbf{M}$

$$\mathbf{S} = [S_{ij}] \rightarrow \mathbf{M} = [m_{ij}(\cdot)] = [(m_{ij}(A_i), m_{ij}(\bar{A}_i), m_{ij}(A_i \cup \bar{A}_i))]$$

Step 2 of BF-ICrA: Construction of Inter-Criteria Matrix (ICM) matrix $\mathbf{K} = [K_{jj'}]$

We want to compute $\mathbf{K} = [K_{jj'}] = [(m_{jj'}(\theta), m_{jj'}(\bar{\theta}), m_{jj'}(\theta \cup \bar{\theta}))]$

Step 2-a: For each alternative A_i we compute

$$m^i_{jj'}(\theta) = m_{ij}(A_i)m_{ij'}(A_i) + m_{ij}(\bar{A}_i)m_{ij'}(\bar{A}_i) \qquad \text{Mass of agreement}$$

$$m^i_{jj'}(\bar{\theta}) = m_{ij}(A_i)m_{ij'}(\bar{A}_i) + m_{ij}(\bar{A}_i)m_{ij'}(A_i) \qquad \text{Mass of disagreement}$$

$$m^i_{jj'}(\theta \cup \bar{\theta}) = 1 - m^i_{jj'}(\theta) - m^i_{jj'}(\bar{\theta}) \qquad \text{Mass of uncertainty}$$

Step 2-b: We fuse the M BBAs $m^i_{jj'}(\cdot)$ to obtain the BBA $mjj'(\cdot)$

- If M is not too large, we recommend PCR6 fusion rule
- If M is too large for PCR6 working in computer memory, we use the averaging rule

Step 3 of BF-ICrA: Simplification of MCDM problem from ICM matrix $\mathbf{K}$

Compute the $d_{BI}(m_{jj'}, m_T)$ distance between $m_{jj'}(\cdot)$ and the full agreement BBA $m_T(\theta) = 1$ where the d_{BI} distance is defined by [Han Dezert Yang 2014]

$$d_{BI}(m_1, m_2) = \sqrt{\frac{1}{2^{|\Theta|-1}} \sum_{X \in 2^\Theta} d^I([Bel_1(X), Pl_1(X)], [Bel_2(X), Pl_2(X)])^2}$$

d^I is Wasserstein distance of interval numbers defined by

$$d^I([a_1, b_1], [a_2, b_2]) = \sqrt{\left[\frac{a_1 + b_1}{2} - \frac{a_2 + b_2}{2}\right]^2 + \frac{1}{3}\left[\frac{b_1 - a_1}{2} - \frac{b_2 - a_2}{2}\right]^2}$$

Since all criteria in strong agreement behave similarly from decision-making standpoint, we can identify (quasi-)redundant criteria from d_{BI} values and take them out of original MCDM problem and solve (if possible) a simplified MCDM problem.

Step 4: Solve simplified MCDM problem (with criteria weighting adjustments) using an available technique (AHP, BF-TOPSIS, etc)

MCDM Problem: How to choose a car to buy based on multiple-criteria?

Constraint: our budget is limited to 12000 euros.

List of 10 cars

- A_1 = DACIA SANDERO SCe 75;
- A_2 = RENAULT CLIO TCe 75;
- A_3 = SUZUKI CELERIO 1.0 VVT Avantage;
- A_4 = FORD KA+ Ka+ 1.2 70 ch S&S Essential;
- A_5 = MITSUBISHI SPACE STAR 1.0 MIVEC 71;
- A_6 = KIA PICANTO 1.0 essence MPi 67 ch BVM5 Motion;
- A_7 = HYUNDAI I10 1.0 66 BVM5 Initia;
- A_8 = CITROEN C1 VTi 72 S&S Live;
- A_9 = TOYOTA AYGO 1.0 VVT-i x;
- A_{10} = PEUGEOT 108 VTi 72ch S&S BVM5 Like.

List of 17 criteria of original MCDM problem

- C_1 is the price (€); smaller is better
- C_2 is the length (mm); larger is better
- C_3 is the height (mm); larger is better
- C_4 is the width without mirror (mm); smaller is better
- C_5 is the wheelbase (mm);larger is better
- C_6 is the max loading volume (L);larger is better
- C_7 is the tank capacity (L);larger is better
- C_8 is the unloaded weight (Kg); smaller is better
- C_9 is the cylinder volume(cm^3);larger is better
- C_{10} is the acceleration 0-100 Km/h (s);larger is better
- C_{11} is the max speed (Km/h);larger is better
- C_{12} is the power (Kw);larger is better
- C_{13} is the horse power (hp);larger is better
- C_{14} is the mixed consumption (L/100Km); smaller is better
- C_{15} is the extra-urban consumption (L/100Km); smaller is better
- C_{16} is the urban consumption (L/100Km); smaller is better
- C_{17} is the CO2 emission level (g/Km) smaller is better

MCDM Score matrix

obtained from `https://automobile.choisir.com/comparateur/voitures-neuves`

$S \triangleq$

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	C_{16}	C_{17}
A_1	7990	4069	1523	1733	2589	1200	50	969	998	14.2	158	55	75	5.2	4.5	6.5	117
A_2	10990	4063	1448	1732	2589	1146	45	1138	898	12.3	178	56	75	5	4.2	6.3	113
A_3	9790	3600	1530	1600	2425	1053	35	815	998	13.9	155	50	68	3.9	3.6	4.5	89
A_4	10350	3941	1524	1774	2490	1029	42	1063	1198	14.6	164	51	70	5.1	4.4	6.3	117
A_5	10990	3795	1505	1665	2450	910	35	865	999	16.7	172	52	71	4.6	4.1	5.3	105
A_6	11000	3595	1485	1595	2400	1010	35	860	998	14.3	161	49	67	4.4	3.7	5.6	106
A_7	11050	3665	1500	1660	2385	1046	40	1008	998	14.7	156	49	66	5.1	4.3	6.5	117
A_8	11550	3466	1465	1615	2340	780	35	840	998	14	160	53	72	3.7	3.4	4.3	85
A_9	11590	3465	1460	1615	2340	812	35	915	998	13.8	160	51	69	4.1	3.6	4.9	93
A_{10}	11950	3475	1460	1615	2340	780	35	840	998	12.6	160	53	72	3.7	3.4	4.3	85

To make the preference order homogeneous, we multiply values of columns C_1, C_4, C_8, and C_{14} to C_{17} by -1 so that our MCDM problem is described by a modified score matrix with homogeneous preference order ("larger is better") for each column **before applying** the BF-ICrA method.

Example of BF-ICrA - Cont'd

Computation of distance matrix with BF-ICrA

$\mathbf{D}(\theta) =$

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}	c_{16}	c_{17}
c_1	0.1401	0.2225	0.2434	0.7318	0.2054	0.2114	0.1901	0.6506	0.4113	0.3907	0.5493	0.4320	0.4128	0.7489	0.7766	0.7383	0.7369
c_2	0.2225	0.0709	0.3946	0.9034	0.0827	0.1471	0.0977	0.8414	0.5985	0.5349	0.3081	0.2659	0.2675	0.8848	0.8945	0.8726	0.8750
c_3	0.2434	0.3946	0.1014	0.5689	0.4016	0.3319	0.4383	0.4161	0.2387	0.2821	0.7145	0.6605	0.6078	0.6368	0.6948	0.6039	0.6445
c_4	0.7318	0.9034	0.5689	0.0904	0.8721	0.7634	0.9054	0.1515	0.6548	0.5438	0.7242	0.7382	0.7272	0.1545	0.1370	0.1742	0.1780
c_5	0.2054	0.0827	0.4016	0.8721	0.0805	0.1436	0.0958	0.8146	0.6524	0.5514	0.3145	0.2537	0.2618	0.8372	0.8536	0.8225	0.8214
c_6	0.2114	0.1471	0.3319	0.7634	0.1436	0.1165	0.1673	0.7520	0.6222	0.5261	0.4767	0.4227	0.4001	0.8501	0.8432	0.8589	0.8565
c_7	0.1901	0.0977	0.4383	0.9054	0.0958	0.1673	0.0355	0.8820	0.5295	0.5681	0.3585	0.2302	0.2715	0.8541	0.8632	0.8565	0.8253
c_8	0.6506	0.8414	0.4161	0.1515	0.8146	0.7520	0.8820	0.1171	0.4588	0.4349	0.7325	0.6920	0.6597	0.1689	0.1890	0.1558	0.1746
c_9	0.4113	0.5985	0.2387	0.6548	0.6524	0.6222	0.5295	0.4588	0.0636	0.2331	0.7125	0.7476	0.7367	0.5695	0.6200	0.5405	0.5947
c_{10}	0.3907	0.5349	0.2821	0.5438	0.5514	0.5261	0.5681	0.4349	0.2331	0.1466	0.5893	0.7070	0.6988	0.5852	0.6389	0.5466	0.5845
c_{11}	0.5493	0.3081	0.7145	0.7242	0.3145	0.4767	0.3585	0.7325	0.7125	0.5893	0.1294	0.2887	0.3331	0.5907	0.5922	0.5748	0.5704
c_{12}	0.4320	0.2659	0.6605	0.7382	0.2537	0.4227	0.2302	0.6920	0.7476	0.7070	0.2887	0.1292	0.1403	0.5571	0.5907	0.5278	0.5030
c_{13}	0.4128	0.2675	0.6078	0.7272	0.2618	0.4001	0.2715	0.6597	0.7367	0.6988	0.3331	0.1403	0.1340	0.5819	0.6086	0.5541	0.5411
c_{14}	0.7489	0.8848	0.6368	0.1545	0.8372	0.8501	0.8541	0.1689	0.5695	0.5852	0.5907	0.5571	0.5819	0.0705	0.0842	0.0682	0.0632
c_{15}	0.7766	0.8945	0.6948	0.1370	0.8536	0.8432	0.8632	0.1890	0.6200	0.6389	0.5922	0.5907	0.6086	0.0842	0.0849	0.0902	0.0842
c_{16}	0.7383	0.8726	0.6039	0.1742	0.8225	0.8589	0.8565	0.1558	0.5405	0.5466	0.5748	0.5278	0.5541	0.0682	0.0902	0.0584	0.0575
c_{17}	0.7369	0.8750	0.6445	0.1780	0.8214	0.8565	0.8253	0.1746	0.5947	0.5845	0.5704	0.5030	0.5411	0.0632	0.0842	0.0575	0.0509

- C_2, C_5 and C_7 are in very strong agreement and somehow redundant for MCDM. We keep C_7 (tank capacity) criteria.
- C_{12} and C_{13} are not too far either and we can simplify the MCDM by keeping only criterion C_{12} (the power) instead of C_{12} and C_{13}
- C_{14}, C_{15}, C_{16} and C_{17} are in very strong agreement. We keep C_{16} (urban consumption) in simplified MCDM

Criteria of simplified MCDM problem to solve

C_1, C_3, C_4, C_6, C_7, C_8, C_9, C_{10}, C_{11}, C_{12} and C_{16}

Example of BF-ICrA - Cont'd

The simplified MCDM car problem after BF-ICrA

Here we choose weights directly from simplified MCDM, but we could choose them by adjustment of original MCDM weights (if available).

$S_{\text{simplified}} \triangleq$

	C_1	C_3	C_4	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{16}
A_1	7990	1523	1733	1200	50	969	998	14.2	158	55	6.5
A_2	10990	1448	1732	1146	45	1138	898	12.3	178	56	6.3
A_3	9790	1530	1600	1053	35	815	998	13.9	155	50	4.5
A_4	10350	1524	1774	1029	42	1063	1198	14.6	164	51	6.3
A_5	10990	1505	1665	910	35	865	999	16.7	172	52	5.3
A_6	11000	1485	1595	1010	35	860	998	14.3	161	49	5.6
A_7	11050	1500	1660	1046	40	1008	998	14.7	156	49	6.5
A_8	11550	1465	1615	780	35	840	998	14	160	53	4.3
A_9	11590	1460	1615	812	35	915	998	13.8	160	51	4.9
A_{10}	11950	1460	1615	780	35	840	998	12.6	160	53	4.3

Choice of importance scores $\text{imp}(C_j) \in \{1 = \text{least important}, 2, 3, 4, 5 = \text{most important}\}$

$\text{imp}(C_1) = \text{imp}(C_{16}) = 5$ — C_1 is price & C_{16} is urban consumption

$\text{imp}(C_6) = \text{imp}(C_7) = 4$ — C_6 is max loading vol. & C_7 is tank vol.

$\text{imp}(C_{10}) = \text{imp}(C_{11}) = \text{imp}(C_{12}) = 3$ — C_{10} is accel. & C_{11} is max speed & C_{12} is power

$\text{imp}(C_8) = \text{imp}(C_9) = 2.$ — C_8 is unloaded weight & C_9 is cylinder vol.

$\text{imp}(C_3) = \text{imp}(C_4) = 1$ — C_3 is height & C_4 is width

After normalization, the importance weights are

$$\mathbf{w} = [\frac{5}{33} \quad \frac{1}{33} \quad \frac{1}{33} \quad \frac{4}{33} \quad \frac{4}{33} \quad \frac{2}{33} \quad \frac{2}{33} \quad \frac{3}{33} \quad \frac{3}{33} \quad \frac{3}{33} \quad \frac{5}{33}]$$

Solution of the simplified MCDM car problem

- with BF-TOPSIS1 & BF-TOPSIS2 methods:

$$A_2 \succ A_1 \succ A_4 \succ A_7 \succ A_5 \succ A_6 \succ A_{10} \succ A_9 \succ A_8 \succ A_3$$

- with BF-TOPSIS3 & BF-TOPSIS4 methods:

$$A_2 \succ A_1 \succ A_4 \succ A_7 \succ A_5 \succ A_{10} \succ A_9 \succ A_6 \succ A_8 \succ A_3$$

- with classical AHP method (with double normalization of score matrix):

$$A_2 \succ A_1 \succ A_4 \succ A_7 \succ A_5 \succ A_6 \succ A_9 \succ A_8 \succ A_3 \succ A_{10}$$

Best choice for buying the car (for the chosen criteria and importance weights)

- The car A_2 (RENAULT CLIO TCe 75) is the first best choice
- The car A_1 (DACIA SANDERO SCe 75) is the second best choice

We can observe the stability of the order of first best solutions with the different MCDM methods.

- To start working with BF, we recommend Smets TBM MatLab codes that include many useful efficient functions based on Fast Möbius Transforms

  http://iridia.ulb.ac.be/~psmets/

- Some toolboxes for working with BF can be found from Belief Functions and Applications Society (BFAS) web site

  http://www.bfasociety.org/

- Explanations for implementation of generalized belief functions can be found in

  A. Martin, Implementing general belief function framework with a practical codification for low complexity, in [DSmT books], Vol. 3, Chap 7, 2009.

- Implementation of fusion rules by sampling techniques (java package) http://refereefunction.fredericdambreville.com

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