On the coherence of JPDAF formulation based on diffuse prior

Jean Dezert

ONERA/DTIM, 29 Av. Division Leclerc 92320 Châtillon, France

ABSTRACT

From a very simple multitarget tracking example, we point out in this paper a theoretical weakness of joint probabilistic data association filter formulation whenever a non parametric model is used for the probability mass function of the number of false measurements ocurring in validations gates. For such case, very frequently adopted in tracking simulations, we propose a modification of JPDA derivations to provide a better coherence of JPDAF.

1. INTRODUCTION

The well known PDAF (Probabilistic Data Association Filter [2,3]) and JPDAF (Joint Probabilistic Data Association Filter [1,4,5]) developped by Y. Bar-Shalom in eighties are the most popular Bayesian algorithms for target tracking in clutter. JPDAF is a direct extension of PDAF for multitarget tracking. Both algorithms share the same theoretical background based on Bayesian inference combined with total probability theorem [12]. We assume the reader familar with PDAF and JPDAF formulation and we will not present in details the development of PDAF and JPDAF algorithms here. A complete and detailed presentation of these algorithms is available in [6–8]. Improvements of PDAF called PDAF-BD and PDAF-BDAI have been recently presented in [13]. Our intention in this paper is rather to point out a theoretical weakness of the JPDAF formulation with respect to the PDAF one. The basic idea which has motived this paper is the following: consider a multitarget tracking system based on separate PDAFs and another one based only on JPDAF. If the targets are close enough, then JPDAF usually handles pretty well multitarget tracking (regardless of the known track coalescence problem reported in [11,9] and solved in [10]). JPDAF outperfoms target tracking based on separate PDAFs. When targets are faraway from each other, (case of non-overlapping gates or overlapping gates with no measurement in their intersection) then tracking is usually done by separate PDAFs (one PDAF per target) even if theoretically JPDAF could be used too. In such simple case, JPDAF and PDAF should provide same results if the formulation of both algorithms is fully coherent. As it will be shown in the sequel, this is not necessary the case depending on which model for probability mass function (pmf) of the number of false measurements μ_F is used within algorithms. Up to now, the Poisson model or the diffuse model [6] are the only two commonly models usually adopted. We will show on very simple tracking example that actually only Poisson model provides a perfect theoretical coherence between PDAF and JPDAF. Using the diffuse model, JPDAF equations become incompatible with PDAF equations. This is the theoretical weakness of the JPDAF formulation when diffuse model is adopted. To eliminate this problem, we propose a better way to derive joint association probabilities in JPDAF which provides the full compatibility of JPDAF with PDAF in case of diffuse model for μ_F. This modification however breaks down in return the coherence of JPDAF with PDAF if the Poisson model for μ_F is used.

Author can be contacted at following E-mail address: Jean.Dezert@onera.fr

2. JPDAF VERSUS PDAF FORMULATION

Consider the following two targets example with four measurements corresponding to the following data association matrix at time k

$$\Omega = \begin{array}{c|ccc} t_j & 0 & 1 & 2 \\ \mathbf{y}_i & & & \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \\ 4 & 1 & 0 & 1 \end{array} \quad (1)$$

In a such case, the two targets are actually considered as independent since they don't share validated measurements. We have two independent clusters of size one. Column t_0 corresponds to false alarm hypothesis and columns t_1 and t_2 correspond to the origin of measurement associated respectively with target 1 and target 2.

2.1. Data Association Probabilities computed by PDAF

If the tracking of targets t_1 and t_2 is done using two classical PDAFs running in parallel, we will get the following posterior data association probabilities [6]

- **With parametric version of PDAF:** $\mu_F(\Phi) = \frac{(\lambda V)^\Phi}{\Phi!} e^{-\lambda V}, \ \Phi = 0, 1, 2, \ldots$
  - **for target** t_1

$$\beta_0^{t_1} = \frac{1}{c^{t_1}} \times \lambda \times \frac{1 - P_d^{t_1} P_g}{P_d^{t_1} P_g} \quad (2)$$

$$\beta_1^{t_1} = \frac{1}{c^{t_1}} \times e_1^{t_1} \quad (3)$$

$$\beta_2^{t_1} = \frac{1}{c^{t_1}} \times e_2^{t_1} \quad (4)$$

$$\beta_3^{t_1} \equiv 0 \quad (5)$$

$$\beta_4^{t_1} \equiv 0 \quad (6)$$

  - **for target** t_2

$$\beta_0^{t_2} = \frac{1}{c^{t_2}} \times \lambda \times \frac{1 - P_d^{t_2} P_g}{P_d^{t_2} P_g} \quad (7)$$

$$\beta_1^{t_2} \equiv 0 \quad (8)$$

$$\beta_2^{t_2} \equiv 0 \quad (9)$$

$$\beta_3^{t_2} = \frac{1}{c^{t_2}} \times e_3^{t_2} \quad (10)$$

$$\beta_4^{t_2} = \frac{1}{c^{t_2}} \times e_4^{t_2} \quad (11)$$

where c^{t_j}, for $j = 1, 2$, is a normalization constant given by

$$c^{t_j} = \lambda \frac{1 - P_d^{t_j} P_g}{P_d^{t_j} P_g} + \sum_{i=1}^{m_k^{t_j}} e_i^{t_j} \quad (12)$$

where λ is the spatial density of clutter; $m_k^{t_j}$ is the number of validated measurements in validation gate V^{t_j}; P_g is the gating probability; $P_d^{t_j}$ is the detection probability of target t_j ($j = 1, 2$) and $e_i^{t_j}(\mathbf{y}_i(k)) \triangleq P_g^{-1}\mathcal{N}[\mathbf{y}_i(k); \hat{\mathbf{y}}^{t_j}(k|k-1), \mathbf{S}^{t_j}(k)]$. $\mathbf{y}_i$ is the ith validated measurement; $\hat{\mathbf{y}}^{t_j}$ is the predicted measurement for target t_j and $\mathbf{S}^{t_j}(k)$ the predicted covariance of innovation (see [6–8] for details).

- **With non parametric (diffuse) version of PDAF:** $\mu_F(\Phi) = 1/N, \ \Phi = 0, 1, \dots, N-1$
  - **for target** t_1

$$\beta_0^{t_1} = \frac{1}{c^{t_1}} \times \frac{m_k^{t_1}}{V^{t_1}} \times \frac{1 - P_d^{t_1} P_g}{P_d^{t_1} P_g} \tag{13}$$

$$\beta_1^{t_1} = \frac{1}{c^{t_1}} \times e_1^{t_1} \tag{14}$$

$$\beta_2^{t_1} = \frac{1}{c^{t_1}} \times e_2^{t_1} \tag{15}$$

$$\beta_3^{t_1} \equiv 0 \tag{16}$$

$$\beta_4^{t_1} \equiv 0 \tag{17}$$

  - **for target** t_2

$$\beta_0^{t_2} = \frac{1}{c^{t_2}} \times \frac{m_k^{t_2}}{V^{t_2}} \times \frac{1 - P_d^{t_2} P_g}{P_d^{t_2} P_g} \tag{18}$$

$$\beta_1^{t_2} \equiv 0 \tag{19}$$

$$\beta_2^{t_2} \equiv 0 \tag{20}$$

$$\beta_3^{t_2} = \frac{1}{c^{t_2}} \times e_3^{t_2} \tag{21}$$

$$\beta_4^{t_2} = \frac{1}{c^{t_2}} \times e_4^{t_2} \tag{22}$$

where V^{t_j} is the validation gate volume and c^{t_j} for $j = 1, 2$ is given by

$$c^{t_j} = \frac{m_k^{t_j}}{V^{t_j}} \frac{1 - P_d^{t_j} P_g}{P_d^{t_j} P_g} + \sum_{i=1}^{m_k^{t_j}} e_i^{t_j} \tag{23}$$

2.2. Data Association Probabilities computed by JPDAF

Consider now the same example where both targets are tracked by JPDAF instead of two separate PDAFs. Because of independence of elementary clusters (i.e. each "cluster" corresponds to only one target), we expect in theory that marginal association probabilities $\beta_i^{t_j}$ computed by JPDAF will correspond exactly to those given by the PDAF for both models μ_F (diffuse and Poisson). Unfortunately, as it will be shown, this is not the case when diffuse model for μ_F is chosen.

In our example, the following nine feasible event matrices $\hat{\Omega}(\Theta) = [\hat{\omega}_{it}(\Theta)]$ must be taken into account in the JPDA corresponding to feasible joint association events $\Theta_1, \dots, \Theta_9$

$$\Theta_1 \longrightarrow \hat{\Omega}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \Theta_2 \longrightarrow \hat{\Omega}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\Theta_3 \longrightarrow \hat{\Omega}_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \Theta_4 \longrightarrow \hat{\Omega}_4 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Theta_5 \longrightarrow \hat{\Omega}_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \Theta_6 \longrightarrow \hat{\Omega}_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\Theta_7 \longrightarrow \hat{\Omega}_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Theta_8 \longrightarrow \hat{\Omega}_8 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\Theta_9 \longrightarrow \hat{\Omega}_9 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Following [8], the posterior probability of any feasible joint association event Θ is given by the general form

$$\begin{aligned} P\{\Theta|\mathbf{Y}^k\} &= \frac{1}{c} p[\mathbf{Y}(k)|\Theta, m_k, \mathbf{Y}^{k-1}] \times P\{\Theta|\delta(\Theta), \Phi(\Theta), m_k\} \times P\{\delta(\Theta), \Phi(\Theta)|m_k\} \\ &= \frac{1}{c}\Big[V^{-\Phi(\Theta)} \prod_{i=1}^{m_k} [f_{t_i}(\mathbf{y}_i(k))]^{\tau_i(\Theta)}\Big] \times \Big[\frac{\Phi(\Theta)!}{m_k!}\Big] \times \Big[\mu_F(\Phi(\Theta)) \prod_{t=1}^{T} [P_d^t]^{\delta_t(\Theta)} [1 - P_d^t]^{1-\delta_t(\Theta)}\Big] \end{aligned} \tag{24}$$

where $f_{t_i}(\mathbf{y}_i(k)) \triangleq \mathcal{N}[\mathbf{y}_i(k); \hat{\mathbf{y}}^{t_i}(k|k-1), \mathbf{S}^{t_i}(k)] \equiv P_g e_i^{t_i}(\mathbf{y}_i(k))$ and c is a normalization constant. The false measurement indicator $\Phi(\Theta)$, the target detection indicator $\delta_t(\Theta)$ and the measurement association indicator $\tau_i(\Theta)$ are all functions of the event Θ under consideration and are given by

$$\delta_t(\Theta) \triangleq \sum_{i=1}^{m_k} \hat{\omega}_{it}(\Theta) \leq 1 \qquad t = 1, \dots, T \tag{25}$$

$$\tau_i(\Theta) \triangleq \sum_{t=1}^{T} \hat{\omega}_{it}(\Theta) \tag{26}$$

$$\Phi(\Theta) \triangleq \sum_{i=1}^{m_k}[1-\tau_i(\Theta)] \tag{27}$$

The marginal data association probabilities β_i^t that measurement i belongs to target t is obtained by summing over all feasible events Θ for which this condition is true, that is

$$\beta_i^t(k) = \sum_{\Theta(k)} P\{\Theta|\mathbf{Y}^k\}\hat{\omega}_{it}(\Theta) \qquad i=1,\dots,m_k \tag{28}$$

$$\beta_0^t(k) = 1 - \sum_{i=1}^{m_k}\beta_i^t(k) \tag{29}$$

In our example, one gets

- **for target** $t_1 = 1$

$$\beta_0^{t_1} = P\{\Theta_1|\mathbf{Y}^k\} + P\{\Theta_8|\mathbf{Y}^k\} + P\{\Theta_9|\mathbf{Y}^k\} \tag{30}$$

$$\beta_1^{t_1} = P\{\Theta_2|\mathbf{Y}^k\} + P\{\Theta_3|\mathbf{Y}^k\} + P\{\Theta_4|\mathbf{Y}^k\} \tag{31}$$

$$\beta_2^{t_1} = P\{\Theta_5|\mathbf{Y}^k\} + P\{\Theta_6|\mathbf{Y}^k\} + P\{\Theta_7|\mathbf{Y}^k\} \tag{32}$$

$$\beta_3^{t_1} \equiv 0 \tag{33}$$

$$\beta_4^{t_1} \equiv 0 \tag{34}$$

- **for target** $t_2 = 2$

$$\beta_0^{t_2} = P\{\Theta_1|\mathbf{Y}^k\} + P\{\Theta_2|\mathbf{Y}^k\} + P\{\Theta_5|\mathbf{Y}^k\} \tag{35}$$

$$\beta_1^{t_2} \equiv 0 \tag{36}$$

$$\beta_2^{t_2} \equiv 0 \tag{37}$$

$$\beta_3^{t_2} = P\{\Theta_3|\mathbf{Y}^k\} + P\{\Theta_6|\mathbf{Y}^k\} + P\{\Theta_8|\mathbf{Y}^k\} \tag{38}$$

$$\beta_4^{t_2} = P\{\Theta_4|\mathbf{Y}^k\} + P\{\Theta_7|\mathbf{Y}^k\} + P\{\Theta_9|\mathbf{Y}^k\} \tag{39}$$

If now we replace each $P\{\Theta_j|\mathbf{Y}^k\}$ by its expression (24), one gets for target t_1

$$\begin{aligned}\beta_0^{t_1} = \;& \frac{1}{c^{t_1}} \times \frac{4!}{4!}\mu_F(4)V^{-4}(1-P_d^{t_1})(1-P_d^{t_2}) \\ &+ \frac{1}{c^{t_1}} \times \frac{3!}{4!}\mu_F(3)V^{-3}(1-P_d^{t_1})P_d^{t_2}f_3^{t_2} \\ &+ \frac{1}{c^{t_1}} \times \frac{3!}{4!}\mu_F(3)V^{-3}(1-P_d^{t_1})P_d^{t_2}f_4^{t_2}\end{aligned}$$

$$\begin{aligned}\beta_1^{t_1} &= \frac{1}{c^{t_1}} \times \frac{3!}{4!}\mu_F(3)V^{-3}P_d^{t_1}f_1^{t_1}(1-P_d^{t_2}) + \frac{1}{c^{t_1}} \times \frac{2!}{4!}\mu_F(2)V^{-2}P_d^{t_1}f_1^{t_1}P_d^{t_2}f_3^{t_2} + \frac{1}{c^{t_1}} \times \frac{2!}{4!}\mu_F(2)V^{-2}P_d^{t_1}f_1^{t_1}P_d^{t_2}f_4^{t_2} \\ &= \frac{1}{c^{t_1}} \times P_d^{t_1}f_1^{t_1}[\frac{1}{4}\mu_F(3)V^{-3}(1-P_d^{t_2}) + \frac{1}{12}\mu_F(2)V^{-2}P_d^{t_2}(f_3^{t_2}+f_4^{t_2})]\end{aligned}$$

$$\beta_2^{t_1} = \frac{1}{c^{t_1}} \times \frac{3!}{4!}\mu_F(3)V^{-3}P_d^{t_1}f_2^{t_1}(1-P_d^{t_2}) + \frac{1}{c^{t_1}} \times \frac{2!}{4!}\mu_F(2)V^{-2}P_d^{t_1}f_2^{t_1}P_d^{t_2}f_3^{t_2} + \frac{1}{c^{t_1}} \times \frac{2!}{4!}\mu_F(2)V^{-2}P_d^{t_1}f_2^{t_1}P_d^{t_2}f_4^{t_2}$$
$$= \frac{1}{c^{t_1}} \times P_d^{t_1}f_2^{t_1}[\frac{1}{4}\mu_F(3)V^{-3}(1-P_d^{t_2}) + \frac{1}{12}\mu_F(2)V^{-2}P_d^{t_2}(f_3^{t_2}+f_4^{t_2})]$$

with the normalization constant c^{t_1} given by

$$c^{t_1} = \frac{1}{4}\mu_F(3)V^{-3}(f_1^{t_1}+f_2^{t_1})P_d^{t_1}(1-P_d^{t_2}) + \frac{1}{4}\mu_F(3)V^{-3}(1-P_d^{t_1})P_d^{t_2}(f_3^{t_2}+f_4^{t_2})$$
$$+ \frac{1}{12}\mu_F(2)V^{-2}P_d^{t_1}(f_1^{t_1}+f_2^{t_1})P_d^{t_2}(f_3^{t_2}+f_4^{t_2}) + \mu_F(4)V^{-4}(1-P_d^{t_1})(1-P_d^{t_2})$$

which can be more conveniently expressed as

$$c^{t_1} = (1-P_d^{t_2})[\frac{1}{4}\mu_F(3)V^{-3}P_d^{t_1}(f_1^{t_1}+f_2^{t_1}) + \mu_F(4)V^{-4}(1-P_d^{t_1})]$$
$$+ P_d^{t_2}(f_3^{t_2}+f_4^{t_2})[\frac{1}{4}\mu_F(3)V^{-3}(1-P_d^{t_1}) + \frac{1}{12}\mu_F(2)V^{-2}P_d^{t_1}(f_1^{t_1}+f_2^{t_1})]$$

Same kind of derivations can be done for target t_2 as well and will not presented here due to space limitation.

3. COHERENCE OF JPDAF WITH PDAF FOR POISSON MODEL

If we assume a Poisson pmf for μ_F with spatial density λ of false measurements,

$$\mu_F(\Phi) = \frac{(\lambda V)^\Phi}{\Phi!}e^{-\lambda V} \tag{40}$$

after some elementary algebra, the previous normalization constant c^{t_1} can be factorized as follows

$$c^{t_1} = \frac{e^{-\lambda V}}{12}[\lambda(1-P_d^{t_1}) + P_d^{t_1}(f_1^{t_1}+f_2^{t_1})][\frac{\lambda^3}{2}(1-P_d^{t_2}) + \frac{\lambda^2}{2}P_d^{t_2}(f_3^{t_2}+f_4^{t_2})]$$

and marginal association probabilities

$$\beta_0^{t_1} = \frac{1}{c^{t_1}} \times \lambda(1-P_d^{t_1}) \times \frac{e^{-\lambda V}}{12}[\frac{\lambda^3}{2}(1-P_d^{t_2}) + \frac{\lambda^2}{2}P_d^{t_2}(f_3^{t_2}+f_4^{t_2})]$$
$$\beta_1^{t_1} = \frac{1}{c^{t_1}} \times P_d^{t_1}f_1^{t_1} \times \frac{e^{-\lambda V}}{12}[\frac{\lambda^3}{2}(1-P_d^{t_2}) + \frac{\lambda^2}{2}P_d^{t_2}(f_3^{t_2}+f_4^{t_2})]$$
$$\beta_2^{t_1} = \frac{1}{c^{t_1}} \times P_d^{t_1}f_2^{t_1} \times \frac{e^{-\lambda V}}{12}[\frac{\lambda^3}{2}(1-P_d^{t_2}) + \frac{\lambda^2}{2}P_d^{t_2}(f_3^{t_2}+f_4^{t_2})]$$

Using a simplification by $\frac{e^{-\lambda V}}{12}[\frac{\lambda^3}{2}(1-P_d^{t_2}) + \frac{\lambda^2}{2}P_d^{t_2}(f_3^{t_2}+f_4^{t_2})]/P_d^{t_1}$ yields

$$\beta_0^{t_1} = \frac{\lambda(1-P_d^{t_1})/P_d^{t_1}}{\lambda(1-P_d^{t_1})/P_d^{t_1} + f_1^{t_1} + f_2^{t_1}} \equiv \frac{1}{c^{t_1}} \times \lambda \times \frac{1-P_d^{t_1}P_g}{P_d^{t_1}P_g} \tag{41}$$
$$\beta_1^{t_1} = \frac{f_1^{t_1}}{\lambda(1-P_d^{t_1})/P_d^{t_1} + f_1^{t_1} + f_2^{t_1}} \equiv \frac{1}{c^{t_1}} \times e_1^{t_1} \tag{42}$$
$$\beta_2^{t_1} = \frac{f_2^{t_1}}{\lambda(1-P_d^{t_1})/P_d^{t_1} + f_1^{t_1} + f_2^{t_1}} \equiv \frac{1}{c^{t_1}} \times e_2^{t_1} \tag{43}$$

It can be easily checked that these expressions for marginal association probabilities are *fully coherent* with (2)-(4) as soon as gating probability $P_g = 1$. Same conclusion can be drawn about marginal probabilities $\beta_0^{t_2}, \beta_3^{t_2}$ and $\beta_4^{t_2}$ for target t_2.

4. INCOHERENCE OF JPDAF WITH PDAF FOR DIFFUSE MODEL

If now we consider the diffuse model [6] for μ_F, i.e. $\mu_F(\Phi) \equiv \epsilon$, $\forall \Phi \geq 0$, we get for constant c^{t_1}

$$
\begin{aligned}
c^{t_1} = \quad & \epsilon(1-P_d^{t_2})/V \times [V^{-3}(1-P_d^{t_1}) + \frac{1}{4}V^{-2}P_d^{t_1}(f_1^{t_1}+f_2^{t_1})] \\
& + \frac{\epsilon}{4}P_d^{t_2}(f_3^{t_2}+f_4^{t_2})[V^{-3}(1-P_d^{t_1}) + \frac{1}{3}V^{-2}P_d^{t_1}(f_1^{t_1}+f_2^{t_1})]
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
c^{t_1} = \quad & \epsilon(1-P_d^{t_1})/V \times [V^{-3}(1-P_d^{t_2}) + \frac{1}{4}V^{-2}P_d^{t_2}(f_3^{t_2}+f_4^{t_2})] \\
& + \frac{\epsilon}{4}P_d^{t_1}(f_1^{t_1}+f_2^{t_1})[V^{-3}(1-P_d^{t_2}) + \frac{1}{3}V^{-2}P_d^{t_2}(f_3^{t_2}+f_4^{t_2})]
\end{aligned}
$$

However in this case, the normalization constant c^{t_1} cannot be factorized in two separable factors depending on targets t_1 and t_2 like in the Poisson case. The marginal association probabilities can only be expressed as

$$
\beta_0^{t_1} = \frac{1}{c^{t_1}} \times \epsilon(1-P_d^{t_1})/V \times [V^{-3}(1-P_d^{t_2}) + \frac{1}{4}V^{-2}P_d^{t_2}(f_3^{t_2}+f_4^{t_2})] \quad (44)
$$

$$
\beta_1^{t_1} = \frac{1}{c^{t_1}} \times \frac{\epsilon}{4}P_d^{t_1}f_1^{t_1} \times [V^{-3}(1-P_d^{t_2}) + \frac{1}{3}V^{-2}P_d^{t_2}(f_3^{t_2}+f_4^{t_2})] \quad (45)
$$

$$
\beta_2^{t_1} = \frac{1}{c^{t_1}} \times \frac{\epsilon}{4}P_d^{t_1}f_2^{t_1} \times [V^{-3}(1-P_d^{t_2}) + \frac{1}{3}V^{-2}P_d^{t_2}(f_3^{t_2}+f_4^{t_2})] \quad (46)
$$

Because of the non separability of normalization constant c^{t_1}, the marginal association probabilities $\beta_0^{t_1}, \beta_1^{t_1}$ and $\beta_2^{t_1}$ cannot be reduced to expressions (13)-(15) obtained by standard PDAF based on same diffuse model for μ_F. This proves the theoretical weakness of JPDAF formulation when diffuse model is chosen for μ_F.

5. A NEW COHERENT JPDAF FORMULATION FOR DIFFUSE MODEL

Actually, the expression of $P\{\Theta|\delta(\Theta),\Phi(\Theta),m_k\}$ entering in (24) for derivation of $P\{\Theta|\mathbf{Y}^k\}$ is questionable. First note that in our example, from the set of *feasible* event matrices, only the following 4 pairs (δ,Φ) of indicators are only feasible

$(\delta = [0\ 0], \Phi = 4)$ which is associated only with Θ_1
$(\delta = [1\ 0], \Phi = 3)$ which can be associated with Θ_2 or Θ_5
$(\delta = [0\ 1], \Phi = 3)$ which can be associated with Θ_8 or Θ_9
$(\delta = [1\ 1], \Phi = 2)$ which can be associated with Θ_3, Θ_4, Θ_6 or Θ_7

From this remark, it makes more sense to take instead of $\Phi(\Theta)!/m_k!$ for $P\{\Theta|\delta(\Theta),\Phi(\Theta),m_k\}$ as done in the standard JPDAF formulation, the new following probabilities

$$
\begin{aligned}
& P\{\Theta_1|\delta = [0\ 0], \Phi = 4, m_k\} = 1 \\
& P\{\Theta_2|\delta = [1\ 0], \Phi = 3, m_k\} = P\{\Theta_5|\delta = [1\ 0], \Phi = 3, m_k\} = 1/2 \\
& P\{\Theta_8|\delta = [0\ 1], \Phi = 3, m_k\} = P\{\Theta_9|\delta = [0\ 1], \Phi = 3, m_k\} = 1/2 \\
& P\{\Theta_3|\delta = [1\ 1], \Phi = 2, m_k\} = P\{\Theta_4|\delta = [1\ 1], \Phi = 2, m_k\} = 1/4 \\
& P\{\Theta_6|\delta = [1\ 1], \Phi = 2, m_k\} = P\{\Theta_7|\delta = [1\ 1], \Phi = 2, m_k\} = 1/4
\end{aligned}
$$

with such new choice for $P\{\Theta|\delta(\Theta), \Phi(\Theta), m_k\}$ derivation we now get for target t_1

$$\begin{aligned}\beta_0^{t_1} = \ & \frac{1}{c^{t_1}} \times \mu_F(4)V^{-4}(1-P_d^{t_1})(1-P_d^{t_2}) \\ & + \frac{1}{c^{t_1}} \times \frac{1}{2}\mu_F(3)V^{-3}(1-P_d^{t_1})P_d^{t_2}f_3^{t_2} \\ & + \frac{1}{c^{t_1}} \times \frac{1}{2}\mu_F(3)V^{-3}(1-P_d^{t_1})P_d^{t_2}f_4^{t_2}\end{aligned}$$

$$\begin{aligned}\beta_1^{t_1} = \ & \frac{1}{c^{t_1}} \times \frac{1}{2}\mu_F(3)V^{-3}P_d^{t_1}f_1^{t_1}(1-P_d^{t_2}) \\ & + \frac{1}{c^{t_1}} \times \frac{1}{4}\mu_F(2)V^{-2}P_d^{t_1}f_1^{t_1}P_d^{t_2}f_3^{t_2} \\ & + \frac{1}{c^{t_1}} \times \frac{1}{4}\mu_F(2)V^{-2}P_d^{t_1}f_1^{t_1}P_d^{t_2}f_4^{t_2} \\ = \ & \frac{1}{c^{t_1}} \times P_d^{t_1}f_1^{t_1}[\frac{1}{2}\mu_F(3)V^{-3}(1-P_d^{t_2}) + \frac{1}{4}\mu_F(2)V^{-2}P_d^{t_2}(f_3^{t_2}+f_4^{t_2})]\end{aligned}$$

$$\begin{aligned}\beta_2^{t_1} = \ & \frac{1}{c^{t_1}} \times \frac{1}{2}\mu_F(3)V^{-3}P_d^{t_1}f_2^{t_1}(1-P_d^{t_2}) \\ & + \frac{1}{c^{t_1}} \times \frac{1}{4}\mu_F(2)V^{-2}P_d^{t_1}f_2^{t_1}P_d^{t_2}f_3^{t_2} \\ & + \frac{1}{c^{t_1}} \times \frac{1}{4}\mu_F(2)V^{-2}P_d^{t_1}f_2^{t_1}P_d^{t_2}f_4^{t_2} \\ = \ & \frac{1}{c^{t_1}} \times P_d^{t_1}f_2^{t_1}[\frac{1}{2}\mu_F(3)V^{-3}(1-P_d^{t_2}) + \frac{1}{4}\mu_F(2)V^{-2}P_d^{t_2}(f_3^{t_2}+f_4^{t_2})]\end{aligned}$$

with the normalization constant c^{t_1} given by

$$\begin{aligned}c^{t_1} = & \frac{1}{2}\mu_F(3)V^{-3}(f_1^{t_1}+f_2^{t_1})P_d^{t_1}(1-P_d^{t_2}) + \frac{1}{2}\mu_F(3)V^{-3}(1-P_d^{t_1})P_d^{t_2}(f_3^{t_2}+f_4^{t_2}) \\ & + \frac{1}{4}\mu_F(2)V^{-2}P_d^{t_1}(f_1^{t_1}+f_2^{t_1})P_d^{t_2}(f_3^{t_2}+f_4^{t_2}) + \mu_F(4)V^{-4}(1-P_d^{t_1})(1-P_d^{t_2})\end{aligned}$$

which can be more conveniently expressed as

$$\begin{aligned}c^{t_1} = \ & (1-P_d^{t_2})[\frac{1}{2}\mu_F(3)V^{-3}P_d^{t_1}(f_1^{t_1}+f_2^{t_1}) + \mu_F(4)V^{-4}(1-P_d^{t_1})] \\ & + P_d^{t_2}(f_3^{t_2}+f_4^{t_2})[\frac{1}{2}\mu_F(3)V^{-3}(1-P_d^{t_1}) + \frac{1}{4}\mu_F(2)V^{-2}P_d^{t_1}(f_1^{t_1}+f_2^{t_1})]\end{aligned}$$

Same kind of derivations can be done for target t_2 as well.

5.1. Coherence of new formulation for diffuse JPDAF

If we consider the diffuse model for μ_F, we get for constant c^{t_1}

$$\begin{aligned}c^{t_1} = \ & \epsilon(1-P_d^{t_2})[\frac{1}{2}V^{-3}P_d^{t_1}(f_1^{t_1}+f_2^{t_1}) + V^{-4}(1-P_d^{t_1})] \\ & + \epsilon P_d^{t_2}(f_3^{t_2}+f_4^{t_2})[\frac{1}{2}V^{-3}(1-P_d^{t_1}) + \frac{1}{4}V^{-2}P_d^{t_1}(f_1^{t_1}+f_2^{t_1})]\end{aligned}$$

which can now be factorized as

$$c^{t_1} = \epsilon \times [V^{-1}(1-P_d^{t_2}) + \frac{1}{2}P_d^{t_2}(f_3^{t_2}+f_4^{t_2})][V^{-3}(1-P_d^{t_1}) + \frac{1}{2}V^{-2}P_d^{t_1}(f_1^{t_1}+f_2^{t_1})]$$

The marginal association probabilities can then be expressed as

$$\beta_0^{t_1} = \frac{1}{c^{t_1}} \times \epsilon \times V^{-3}(1-P_d^{t_1}) \times [V^{-1}(1-P_d^{t_2}) + \frac{1}{2}P_d^{t_2}(f_3^{t_2}+f_4^{t_2})] \quad (47)$$

$$\beta_1^{t_1} = \frac{1}{c^{t_1}} \times \epsilon \times \frac{1}{2}V^{-2}P_d^{t_1}f_1^{t_1} \times [V^{-2}(1-P_d^{t_2}) + \frac{1}{2}P_d^{t_2}(f_3^{t_2}+f_4^{t_2})] \quad (48)$$

$$\beta_2^{t_1} = \frac{1}{c^{t_1}} \times \epsilon \times \frac{1}{2}V^{-2}P_d^{t_1}f_2^{t_1} \times [V^{-1}(1-P_d^{t_2}) + \frac{1}{2}P_d^{t_2}(f_3^{t_2}+f_4^{t_2})] \quad (49)$$

After a simplification by $\epsilon \times \frac{1}{2}V^{-2}[V^{-1}(1-P_d^{t_2}) + \frac{1}{2}P_d^{t_2}(f_3^{t_2}+f_4^{t_2})]$, one finally gets

$$\beta_0^{t_1} = \frac{\frac{2}{V}(1-P_d^{t_1})/P_d^{t_1}}{\frac{2}{V}(1-P_d^{t_1})/P_d^{t_1} + f_1^{t_1} + f_2^{t_1}} \equiv \frac{\frac{2}{V}(1-P_d^{t_1})/P_d^{t_1}P_g}{\frac{2}{V}(1-P_d^{t_1})/P_d^{t_1}P_g + e_1^{t_1} + e_2^{t_1}} \quad (50)$$

$$\beta_1^{t_1} = \frac{f_1^{t_1}}{\frac{2}{V}(1-P_d^{t_1})/P_d^{t_1} + f_1^{t_1} + f_2^{t_1}} \equiv \frac{e_1^{t_1}}{\frac{2}{V}(1-P_d^{t_1})/P_d^{t_1}P_g + e_1^{t_1} + e_2^{t_1}} \quad (51)$$

$$\beta_2^{t_1} = \frac{f_2^{t_1}}{\frac{2}{V}(1-P_d^{t_1})/P_d^{t_1} + f_1^{t_1} + f_2^{t_1}} \equiv \frac{e_2^{t_1}}{\frac{2}{V}(1-P_d^{t_1})/P_d^{t_1}P_g + e_1^{t_1} + e_2^{t_1}} \quad (52)$$

(53)

As we can easily check, these expressions for marginal association probabilities become now *fully coherent* with expressions (13)-(15). Same conclusion can be drawn about marginal probabilities $\beta_0^{t_2}$, $\beta_3^{t_2}$ and $\beta_4^{t_2}$ for target t_2.

5.2. Incoherence of new formulation for Poisson JPDAF

If we assume now a Poisson pmf for μ_F, the normalization constant c^{t_1} cannot be factorized but can only be expressed as

$$\begin{aligned} c^{t_1} = \quad & e^{-\lambda V}\frac{\lambda}{2}(1-P_d^{t_2})[\frac{\lambda^2}{6}P_d^{t_1}(f_1^{t_1}+f_2^{t_1}) + \frac{\lambda^3}{12}(1-P_d^{t_1})] \\ & + e^{-\lambda V}P_d^{t_2}(f_3^{t_2}+f_4^{t_2})[\frac{\lambda^2}{8}P_d^{t_1}(f_1^{t_1}+f_2^{t_1}) + \frac{\lambda^3}{12}(1-P_d^{t_1})] \end{aligned}$$

or equivalently

$$\begin{aligned} c^{t_1} = \quad & e^{-\lambda V}\frac{\lambda}{2}(1-P_d^{t_1})[\frac{\lambda^2}{6}P_d^{t_2}(f_3^{t_2}+f_4^{t_2}) + \frac{\lambda^3}{12}(1-P_d^{t_2})] \\ & + e^{-\lambda V}P_d^{t_1}(f_1^{t_1}+f_2^{t_1})[\frac{\lambda^2}{8}P_d^{t_2}(f_3^{t_2}+f_4^{t_2}) + \frac{\lambda^3}{12}(1-P_d^{t_2})] \end{aligned}$$

and marginal association probabilities can be written

$$\beta_0^{t_1} = \frac{1}{c^{t_1}} \times e^{-\lambda V}\frac{\lambda}{2}(1-P_d^{t_1}) \times [\frac{\lambda^2}{6}P_d^{t_2}(f_3^{t_2}+f_4^{t_2}) + \frac{\lambda^3}{12}(1-P_d^{t_2})] \quad (54)$$

$$\beta_1^{t_1} = \frac{1}{c^{t_1}} \times e^{-\lambda V}P_d^{t_1}f_1^{t_1} \times [\frac{\lambda^2}{8}P_d^{t_2}(f_3^{t_2}+f_4^{t_2}) + \frac{\lambda^3}{12}(1-P_d^{t_2})] \quad (55)$$

$$\beta_2^{t_1} = \frac{1}{c^{t_1}} \times e^{-\lambda V}P_d^{t_1}f_2^{t_1} \times [\frac{\lambda^2}{8}P_d^{t_2}(f_3^{t_2}+f_4^{t_2}) + \frac{\lambda^3}{12}(1-P_d^{t_2})] \quad (56)$$

Because of the non separability of normalization constant c^{t_1}, these expressions cannot be reduced to (2)-(4). Therefore this new formulation for JPDAF (matched for diffuse model) becomes *not coherent* with PDAF when Poisson model is used for μ_F.

6. CONCLUSION

From a very simple multitarget tracking example, investigations on coherence of the JPDAF formulation have been presented. It has been shown that the standard JPDAF formulation is only coherent with PDAF formulation if only a Poisson model is used for the pmf of number of false measurements μ_F. The standard JPDAF formulation becomes incoherent when a diffuse model for μ_F is taken. Therefore, special caution must be taken before running JPDAF based on diffuse model. Another derivation of joint association probabilities proposed in this paper has shown that the coherence of JPDAF can however been obtained for a diffuse model for μ_F. Unfortunately our new formulation breaks the coherence of JPDAF if Poisson model for μ_F is chosen instead of diffuse model. In concluding remark, it clearly appears that there exists (until now) no unique general formulation of JPDAF equations which provides the full coherence of JPDAF with PDAF for any pmf of the number of false measurements μ_F.

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