Information Fusion Based on New Proportional Conflict Redistribution Rules

Florentin Smarandache
Dept. of Mathematics
Univ. of New Mexico
Gallup, NM 8730
U.S.A.
smarand@unm.edu

Jean Dezert
ONERA/DTIM/IED
29 Av. de la Division Leclerc
92320 Châtillon
France
Jean.Dezert@onera.fr

Abstract – *In this paper we propose a new family of fusion rules for the combination of uncertainty and conflicting information. This family of rules is based on new Proportional Conflict Redistributions (PCR) allowing us to deal with highly conflicting sources for static and dynamic fusion applications. Here five PCR rules (PCR1-PCR5) are presented, analyzed and compared through several numerical examples. From PCR1 up to PCR5 one increases in one hand the complexity of the rules, but in other hand one improves the exactitude of the redistribution of conflicting masses. The basic common principle of PCR rules is to redistribute the conflicting mass, after the conjunctive rule has been applied, proportionally with some functions depending on the masses assigned to their corresponding columns in the mass matrix. Alongside of these new five PCR rules, there are infinitely many ways these redistributions (through the choice of the set of weighting factors) can be chosen. PCR1 is equivalent to the Weighted Average Operator (WAO) on Shafer's model only for static fusion problems but these two operators do not preserve the neutral impact of the vacuous belief assignment (VBA). The PCR2-PCR5 rules presented here, preserve the neutral impact of VBA and turn out to be what we consider as reasonable and can serve as alternative to the hybrid DSm rule. PCR4 is an improvement of minC and Dempster's rules of combination and PCR5 is what we feel as the most exact PCR fusion rule developed up to now. The hybrid DSm rule manages the transfer of the belief committed to the conflict through a simple and direct way while the transfer used within PCR rules is more subtle and complex. The PCR rules can be used also and naturally as new efficient alternatives to the Dempster's rule and its other alternatives already proposed in the Dempster-Shafer Theory (DST) over the last twenty years.*

Keywords: Information fusion, Dezert-Smarandache Theory (DSmT), PCR rules, conflict management, belief functions, WAO, minC, Dempster's rule.

1 Introduction

This paper presents a new set of alternative combination rules based on different proportional conflict redistributions (PCR) which can be applied in the framework of the two principal theories dealing with the combination of belief functions: The Dempster-Shafer Theory (DST) [8, 7] and the recent Dezert-Smarandache Theory (DSmT) [9] which overcomes limitations of DST for combining uncertain, imprecise and possibly highly conflicting sources of information for static or dynamic fusion applications. The major differences between these two theories is the nature and hypotheses on the frame Θ on which are defined the basic belief assignments (bba) $m(.)$, i.e. either on the power set 2^Θ for DST or on the hyper-power set (Dedekind's lattice) D^Θ for DSmT. The difference between DST and DSmT lies also in the rules of combination to apply (Dempster's [8], Yager's [13], Dubois and Prade's [3], minC [1], disjunctive rules [3], etc in DST versus general hybrid DSm rule of combination in the DSmT framework). This paper is not devoted specifically to the presentation of all different rules available in literature like in [7, 9] which will be not reported here but only on a new family of rules which have not yet proposed and can be useful for the information fusion community for some fusion applications. These new rules are based on various proportional conflict redistribution methods were stimulated to us by Dr. Wu Li at NASA Research Center, Langley, VA and by the recent minC combination rule developped by Milan Daniel in [2]. Due to space limitations, we assume the reader is familiar with basics of DST [8] and DSmT [9]. This paper is a shorten version of [11] which contains in details all derivations of examples presented here and more. Let's consider a frame $\Theta = \{\theta_1, \dots, \theta_n\}$ of finite number of hypotheses assumed for simplicity to be exhaustive. Let's denote G the classical power set of Θ (i.e. 2^Θ if we assume the Shafer's model with all exclusivity constraints between elements of the frame - if we adopt DST) or denote G the hyper-power set D^Θ (if we adopt DSmT and know that some elements can't be refined because of their intrinsic fuzzy and continuous nature). A basic belief assignment $m(.)$ is then defined as $m : G \to [0, 1]$ with:

$$m(\emptyset) = 0 \qquad \text{and} \qquad \sum_{X \in G} m(X) = 1 \tag{1}$$

Among all possible bbas, the vacuous belief assignment (VBA), defined by $m_v(\Theta) = 1$ which characterizes a full ignorant source, plays a particular and important role for the construction of a satisfying combination rule. Indeed, the major properties that a good rule of combination must satisfy, upon to authors' opinion, are :

1. the coherence of the combination result in all possible cases (i.e. for any number of sources, any values of bbas and for any types of frames and models which

can change or stay invariant over time).

2. the commutativity of the rule of combination
3. the neutral impact of the VBA into the fusion.

The requirement for conditions 1 and 2 is legitimate since we are obviously looking for best performances (we don't want a rule yielding to counter-intuitive or wrong solutions) and we don't want that the result depends on the arbitrary order the sources are combined. The neutral impact of VBA to be satisfied by a fusion rule (condition 3), denoted by the generic $\oplus$ operator is very important too. This condition states that the combination of a full ignorant source with a set of $s \geq 1$ non-totally ignorant sources doesn't change the result of the combination of the s sources because the full ignorant source doesn't bring any new specific evidence on any problems under consideration. This condition is thus perfectly reasonable and legitimate. The condition 3 is mathematically represented as follows: for all possible $s \geq 1$ non-totally ignorant sources and for any $X \in G$, the fusion operator $\oplus$ must satisfy

$$[m_1 \oplus \ldots \oplus m_s \oplus m_v](X) = [m_1 \oplus \ldots \oplus m_s](X) \quad (2)$$

The associativity property, while very attractive and generally useful for sequential implementation (which is actually an engineering advantage for computer programming) is not actually a crucial property that a combination rule must satisfy if one looks for the best coherence of the result (and that's only we are looking for here).

2 The general principle of the PCR rules

Let's $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ be the frame of the fusion problem under consideration and two belief assignments $m_1, m_2 : G \rightarrow [0,1]$ such that $\sum_{X \in G} m_i(X) = 1$, $i = 1, 2$. The general principle of the Proportional Conflict Redistribution Rules (PCR for short) is:

- **Step 1**: compute the conjunctive rule, $\forall X \in G$

$$m_{\cap}(X) \triangleq m_{1\ldots s}(X) = \sum_{\substack{X_1,\ldots,X_s \in G \\ X_1 \cap \ldots \cap X_s = X}} \prod_{i=1}^{s} m_i(X_i) \quad (3)$$

- **Step 2**: compute the conflicting masses (partial and/or total), The *total conflicting mass* drawn from two sources, denoted k_{12}, is defined as follows:

$$k_{12} = \sum_{\substack{X_1, X_2 \in G \\ X_1 \cap X_2 = \emptyset}} m_1(X_1) m_2(X_2) \quad (4)$$

  The total conflicting mass is nothing but the sum of *partial conflicting masses*, i.e.

$$k_{12} = \sum_{\substack{X_1, X_2 \in G \\ X_1 \cap X_2 = \emptyset}} m(X_1 \cap X_2) \quad (5)$$

  Here, $m(X_1 \cap X_2)$, where $X_1 \cap X_2 = \emptyset$, represents a partial conflict, i.e. the conflict between the sets X_1 and X_2. Formulas (4) and (5) can be directly generalized for $s \geq 2$ sources [11].

- **Step 3**: then proportionally redistribute the conflicting mass (total or partial) to non-empty sets involved in the model according to all integrity constraints.

The way the conflicting mass is redistributed yields to five versions of PCR, denoted PCR1, PCR2, ... PCR5 as it will be shown in the sequel. The PCR combination rules work for any degree of conflict $k_{12} \in [0,1]$, for any models (Shafer's model, free DSm model or any hybrid DSm model). PCR rules work both in DST and DSmT frameworks and for static or dynamical fusion applications. The sophistication/complexity (but correctness) of proportional conflict redistribution increases from the first PCR1 rule up to the last rule PCR5. The development of different PCR rules presented here comes from the fact that PCR1 does not preserve the neutral impact of VBA. All other improved rules PCR2-PCR5 preserve the commutativity, the neutral impact of VBA and present, upon to our opinion, a more and more exact solution for the conflict management that any satisfactory combination rule must tend to.

3 The PCR1 fusion rule

3.1 Definition

PCR1 is the simplest and the easiest version of PCR for combination described in details in [10]. The basic idea for PCR1 is to compute only the total conflicting mass k_{12} (not worrying about the partial conflicting masses) and to redistribute it to *all non-empty sets* proportionally with respect to their corresponding non-empty column sum of the associated mass matrix. The PCR1 fusion for 2 sources[1] is defined $\forall (X \neq \emptyset) \in G$ by :

$$m_{PCR1}(X) = m_{12}(X) + \frac{c_{12}(X)}{d_{12}} \cdot k_{12} \quad (6)$$

where $m_{12}(X)$ is the conjunctive consensus on X given by (3), $c_{12}(X)$ is the non-zero sum of the column of X in the mass matrix $\mathbf{M} = [\mathbf{m}_1\ \mathbf{m}_2]'$ (where $\mathbf{m}_i$ for $i = 1, 2$ is the row vector of belief assignments committed by the source i to elements of G), i.e. $c_{12}(X) = m_1(X) + m_2(X) \neq 0$, k_{12} is the total conflicting mass, and d_{12} is the sum of all non-zero column sums of all non-empty sets (in many cases $d_{12} = 2$, but in some degenerate cases it can be less) (see [10]).

PCR1 is an alternative combination rule to WAO (Weighted Average Operator) proposed by Jøsang and al. in [5]. Both are particular cases of WO (The Weighted Operator) of Inagaki [4] and Lefèvre and al. [6] because the conflicting mass is redistributed with respect to some weighting factors. In the PCR1, the proportionalization is done for each non-empty set with respect to the non-zero sum of its corresponding mass matrix - instead of its mass column average as in WAO. But, PCR1 extends WAO, since PCR1 works *also* for the degenerate cases (like within some dynamical fusion applications) when all column sums of all non-empty sets are zero. In such cases, the conflicting mass

[1] PCR1 fusion has been extended for $s \geq 2$ sources in [11]

is transferred to the non-empty disjunctive form of all non-empty sets together; when this disjunctive form happens to be empty, then either the problem degenerates truly to a void problem and thus all conflicting mass is transferred onto the empty set, or we can assume (if one has enough reason to justify such assumption) that the frame of discernment might contain new unknown hypotheses all summarized by θ_0 and under this assumption all conflicting mass is transferred onto the unknown possible θ_0. A nice feature of PCR1 rule, is that it works in all cases (degenerate and non degenerate). PCR1 corresponds to a specific choice of proportionality coefficients in the infinite continuum family of possible rules of combination involving conjunctive consensus operator. As pointed out judiciously to the authors by Prof. Smets in a private communication, PCR1 applied on the power set and for non-degenerate cases gives the same results as WAO. However PCR1 works also for degenerate cases contrariwise to WAO as we prove in next example.

3.2 Example for PCR1 (degenerate case)

For non degenerate cases, PCR1 and WAO provide the same results. So it is interesting to focus the reader's attention on the difference between PCR1 and WAO in a simple degenerate case. As example, let's consider three different suspects A, B and C in a criminal investigation (i.e. $\Theta = \{A, B, C\}$) and the two following simple Bayesian witnesses reports

	A	B	C
$m_1(.)$	0.3	0.4	0.3
$m_2(.)$	0.5	0.1	0.4
$m_{12}(.)$	0.15	0.04	0.12

The conjunctive consensus $m_{12}(.)$ corresponds to the last row of the previous table. The conflicting mass is $k_{12} = 0.69$. Now let's assume that a little bit later, one learns that $B = \emptyset$ because the second suspect brings a strong alibi, then the initial consensus on B (i.e. $m_{12}(B) = 0.04$) must enter now in the new conflicting mass $k'_{12} = 0.69 + 0.04 = 0.73$ since $B = \emptyset$. Applying the PCR1 formula, one gets now:

$$\begin{cases} m_{PCR1}(B) = 0 \\ m_{PCR1}(A) = 0.15 + \frac{0.8}{0.8+0.7} \cdot 0.73 = 0.5393 \\ m_{PCR1}(C) = 0.12 + \frac{0.7}{0.8+0.7} \cdot 0.73 = 0.4607 \end{cases}$$

while the WAO provides

$$\begin{cases} m_{WAO}(B) = 0 \\ m_{WAO}(A) = 0.15 + (1/2)(0.3 + 0.5)(0.73) = 0.4420 \\ m_{WAO}(C) = 0.12 + (1/2)(0.3 + 0.4)(0.73) = 0.3755 \end{cases}$$

We can verify easily that $m_{PCR1}(A) + m_{PCR1}(B) + m_{PCR1}(C) = 1$ while $m_{WAO}(A) + m_{WAO}(B) + m_{WAO}(C) = 0.8175 < 1$. This example shows clearly the difference between PCR1 and WAO and the ability of PCR1 to deal with degenerate cases contrariwise to WAO.

3.3 Main drawback of PCR1 and WAO

A severe drawback of PCR1 and of WAO is the non-preservation of the neutral impact of the VBA as shown in [10]. In other words, for $s \geq 1$, one gets for $m_1(.) \neq m_v(.), \ldots, m_s(.) \neq m_v(.)$:

$$m_{PCR1}(.) = [m_1 \oplus \ldots m_s \oplus m_v](.) \neq [m_1 \oplus \ldots m_s](.)$$

For the cases of the combination of only one non-vacuous belief assignment $m_1(.)$ with the vacuous belief assignment $m_v(.)$ where $m_1(.)$ has mass assigned to an empty element, say $m_1(\emptyset) > 0$ as in Smets' TBM [12], or as in DSmT dynamic fusion where one finds out that a previous non-empty element A, whose mass $m_1(A) > 0$, becomes empty after a certain time, then this mass of an empty set has to be transferred to other elements using PCR1, but for such case $[m_1 \oplus m_v](.)$ is different from $m_1(.)$. This severe drawback of WAO and PCR1 forces us to develop the next PCR rules satisfying the neutral impact of VBA with better redistributions of the conflicting information.

4 The PCR2 fusion rule

4.1 Definition

In PCR2, the total conflicting mass k_{12} is distributed only to *the non-empty sets involved in the conflict* (not to all non-empty sets) and taken the canonical form of the conflict proportionally with respect to their corresponding non-empty column sum. The redistribution is then more exact (accurate) than in PCR1 and WAO. A nice feature of PCR2 is the preservation of the neutral impact of the VBA and of course its ability to deal with all cases/models. A non-empty set $X_1 \in G$ is considered *involved in the conflict* if there exists another set $X_2 \in G$ such that $X_1 \cap X_2 = \emptyset$ and $m_{12}(X_1 \cap X_2) > 0$. The general formula for the PCR2 fusion of 2 sources is given by :

- $\forall (X \neq \emptyset) \in G$, for X *not involved in the conflict*

$$m_{PCR2}(X) = m_{12}(X) \tag{7}$$

- $\forall (X \neq \emptyset) \in G$, for X *involved in the conflict*

$$m_{PCR2}(X) = m_{12}(X) + \frac{c_{12}(X)}{e_{12}} \cdot k_{12} \tag{8}$$

where $m_{12}(X)$ is the conjunctive consensus on X given by (3), $c_{12}(X)$ is the non-zero sum of the column of X in the mass matrix, i.e. $c_{12}(X) = m_1(X) + m_2(X) \neq 0$, k_{12} is the total conflicting mass, and e_{12} is the sum of all non-zero column sums of all non-empty sets *only* involved in the conflict (in many cases $e_{12} = 2$, but in some degenerate cases it can be less). In the degenerate case when all column sums of all non-empty sets involved in the conflict are zero, then the conflicting mass is transferred to the non-empty disjunctive form of all sets together which were involved in the conflict together. But if this disjunctive form happens to be empty, then the problem reduces to a degenerate void problem and thus all conflicting mass is transferred to the empty set or we can assume (if one has enough reason to justify such assumption) that the frame of discernment might contain new unknown hypotheses all summarized by θ_0 and under this assumption all conflicting mass is transferred onto the unknown possible θ_0. The PCR2 formula can be easily generalized for $s \geq 2$ sources. The neutral impact of VBA through PCR2 is directly proven because the vacuous belief assignment has no impact on conjunctive consensus [11]. This can be checked easily by the reader on simple numerical examples.

4.2 Example for PCR2 versus PCR1

Lets take the frame of discernment $\Theta = \{A, B\}$, Shafer's model (i.e. all intersections empty), and the following two bbas:

	A	B	$A \cup B$
$m_1(.)$	0.7	0.1	0.2
$m_2(.)$	0.5	0.4	0.1
$c_{12}(.)$	1.2	0.5	0.3
$m_{12}(.)$	0.52	0.13	0.02

The sums of columns of the mass matrix $[\mathbf{m}_1\mathbf{m}_2]'$ corresponds to the row $c_{12}(.)$ of the table. The conjunctive consensus $m_{12}(.)$ corresponds to the last row of the previous table. The conflicting mass is $k_{12} = m_{12}(A \cap B) = 0.33$. Applying PCR1 and PCR2 rules yield the final results[2]

With PCR1	With PCR2
$m_{PCR1}(A) = 0.7180$	$m_{PCR2}(A) = 0.752941$
$m_{PCR1}(B) = 0.2125$	$m_{PCR2}(B) = 0.227059$
$m_{PCR1}(A \cup B) = 0.0695$	$m_{PCR2}(A \cup B) = 0.02$

5 The PCR3 fusion rule

5.1 General principle of PCR3

In PCR3, one transfers *partial conflicting masses*, instead of the total conflicting mass, to non-empty sets involved in partial conflict (taken the canonical form of each partial conflict). If an intersection is empty, say $A \cap B = \emptyset$, then the mass $m(A \cap B)$ of the partial conflict is transferred to the non-empty sets A and B proportionally with respect to the non-zero column sum of masses assigned to A and respectively to B by the bbas $m_1(.)$ and $m_2(.)$. The PCR3 rule works if at least one set between A and B is non-empty and its column sum is non-zero. When both sets A and B are empty, or both corresponding column sums of the mass matrix are zero, or only one set is non-empty and its column sum is zero, then the mass $m(A \cap B)$ is transferred to the non-empty disjunctive form $u(A) \cup u(B)$ defined[3] as [9]:

$$\begin{cases} u(X) = X \text{ if } X \text{ is a singleton} \\ u(X \cup Y) = u(X) \cup u(Y) \\ u(X \cap Y) = u(X) \cup u(Y) \end{cases} \quad (9)$$

If this disjunctive form is empty then $m(A \cap B)$ is transferred to the non-empty total ignorance; but if even the total ignorance is empty then either the problem degenerates truly to a void problem and thus all conflicting mass is transferred onto the empty set, or we can assume (if one has enough reason to justify such assumption) that the frame of discernment might contain new unknown hypotheses all summarized by θ_0 and under this assumption all conflicting mass is transferred onto the unknown possible θ_0. If another intersection, say $A \cap C \cap D = \emptyset$, then again the mass $m(A \cap C \cap D) > 0$ is transferred to the non-empty sets A, C, and D proportionally with respect to the non-zero sum of masses assigned to A, C, and respectively D by the sources; if all three sets A, C, D are empty or the sets which are non-empty have their corresponding column sums equal to zero, then the mass $m(A \cap C \cap D)$ is transferred to the non-empty disjunctive form $u(A) \cup u(C) \cup u(D)$; if this disjunctive form is empty then the mass $m(A \cap C \cap D)$ is transferred to the non-empty total ignorance; but if even the total ignorance is empty (a completely degenerate void case) all conflicting mass is transferred onto the empty set (which means that the problem is truly void), or (if we prefer to adopt an optimistic point of view) all conflicting mass is transferred onto a new unknown extra and closure element θ_0 representing all missing hypotheses of Θ.

5.2 The PCR3 formula

For the combination of two bbas, the PCR3 formula is given by: $\forall (X \neq \emptyset) \in G$,

$$\begin{aligned} m_{PCR3}(X) &= m_{12}(X) \\ &+ c_{12}(X) \cdot \sum_{\substack{Y \in G \\ c(Y \cap X) = \emptyset}} \frac{m_1(Y)m_2(X) + m_1(X)m_2(Y)}{c_{12}(X) + c_{12}(Y)} \\ &+ \sum_{\substack{X_1, X_2 \in G \setminus \{X\} \\ c(X_1 \cap X_2) = \emptyset \\ u(X_1) \cup u(X_2) = X}} [m_1(X_1)m_2(X_2) + m_1(X_2)m_2(X_1)] \\ &+ \phi_\Theta(X) \sum_{\substack{X_1, X_2 \in G \setminus \{X\} \\ c(X_1 \cap X_2) = \emptyset \\ u(X_1) = u(X_2) = \emptyset}} [m_1(X_1)m_2(X_2) \\ &\qquad + m_1(X_2)m_2(X_1)] \end{aligned} \quad (10)$$

where $c(Y \cap X)$ is the canonical form of $Y \cap X$ and $c(X_1 \cap X_2)$ is the canonical form of $X_1 \cap X_2$, $c_{12}(X_i)$ ($X_i \in G$) is the non-zero sum of the mass matrix column corresponding to the set X_i, i.e. $c_{12}(X_i) = m_1(X_i) + m_2(X_i) \neq 0$, and where $\phi_\Theta(.)$ is the characteristic function of the total ignorance defined by

$$\begin{cases} \phi_\Theta(X) = 1 \text{ if } X = \Theta \triangleq \theta_1 \cup \ldots \cup \theta_n \text{ (full ignorance)} \\ \phi_\Theta(X) = 0 \quad \text{otherwise} \end{cases}$$

For the fusion of $s \geq 2$ bbas, one extends the above procedure to formulas (9) and (10). PCR3 preserves the neutral impact of the VBA and works for any cases/models (see [11] for details and more examples).

5.3 Example for PCR3

Let's take $\Theta = \{A, B, C\}$, Shafer's model (i.e. all intersections empty), and the 2 following Bayesian bbas

[2]The verification is left to the reader or can be found in [11].

[3]These relationships can be generalized for any number of sets.

	A	B	C
$m_1(.)$	0.6	0.3	0.1
$m_2(.)$	0.4	0.4	0.2
$c_{12}(.)$	1	0.7	0.3
$m_{12}(.)$	0.24	0.12	0.02

The total conflict is $k_{12} = m_{12}(A \cap B) + m_{12}(A \cap C) + m_{12}(B \cap C) = 0.36 + 0.16 + 0.10 = 0.62$, which is a sum of factors. In this specific example, PCR3 provides a result different from PCR1 and PCR2, but PCR2 and PCR1 provides same results. One gets the final result:

- With PCR1 or PCR2:

$$\boxed{\begin{array}{l} m_{PCR1}(A) = m_{PCR2}(A) = 0.550 \\ m_{PCR1}(B) = m_{PCR2}(B) = 0.337 \\ m_{PCR1}(C) = m_{PCR2}(C) = 0.113 \end{array}}$$

- With PCR3:

$$\boxed{\begin{array}{l} m_{PCR3}(A) = 0.574842 \\ m_{PCR3}(B) = 0.338235 \\ m_{PCR3}(C) = 0.086923 \end{array}}$$

We give here the full derivation of $m_{PCR3}(A)$ for better understanding ($m_{PCR3}(B)$ and $m_{PCR3}(C)$ are derived similarly).

$$\begin{aligned} m_{PCR3|12}(A) &= m_{12}(A) \\ &+ c_{12}(A) \cdot \frac{m_1(B)m_2(A) + m_1(A)m_2(B)}{c_{12}(A) + c_{12}(B)} \\ &+ c_{12}(A) \cdot \frac{m_1(C)m_2(A) + m_1(A)m_2(C)}{c_{12}(A) + c_{12}(C)} \\ &= 0.24 + 1 \cdot \frac{0.3 \cdot 0.4 + 0.6 \cdot 0.4}{1 + 0.7} \\ &\quad + 1 \cdot \frac{0.1 \cdot 0.4 + 0.6 \cdot 0.2}{1 + 0.3} = 0.574842 \end{aligned}$$

Note that in this simple case, the two last sums involved in formula (10) do not count because here there doesn't exist positive mass products $m_1(X_1)m_2(X_2)$ to compute for any $X \in G$, $X_1, X_2 \in G \setminus \{X\}$ such that $X_1 \cap X_2 = \emptyset$ and $u(X_1) \cup u(X_2) = X$, neither for $X_1 \cap X_2 = \emptyset$ and $u(X_1) = u(X_2) = \emptyset$.

6 The PCR4 fusion rule

6.1 General principle of PCR4

PCR4 redistributes the partial conflicting mass to the elements involved in the partial conflict, considering the canonical form of the partial conflict. PCR4 is an improvement of previous PCR rules but also of Milan Daniel's minC operator [9]. Daniel uses the proportionalization with respect to the results of the conjunctive rule, but not with respect to the masses assigned to each set by the sources of information as done in PCR1-3 and also as in the most effective PCR5 rule explicated in the next section. Actually, PCR4 also uses the proportionalization with respect to the results of the conjunctive rule, but with PCR4 the conflicting mass $m_{12}(A \cap B)$ when $A \cap B = \emptyset$ is distributed to A and B only because only A and B were involved in the conflict ($A \cup B$ was not involved in the conflict since $m_{12}(A \cap B) = m_1(A)m_2(B) + m_2(A)m_1(B)$), while minC redistributes $m_{12}(A \cap B)$ to A, B, *and* $A \cup B$ in both of its versions a) and b) (see [9] , Chap. 10 for details). Also, for the mixed elements such as $C \cap (A \cup B) = \emptyset$, the mass $m(C \cap (A \cup B))$ is redistributed to C, $A \cup B$, $A \cup B \cup C$ in minC version a), and worse in minC version b) to A, B, C, $A \cup B$, $A \cup C$, $B \cup C$ and $A \cup B \cup C$. PCR4 rule improves this and redistributes the mass $m(C \cap (A \cup B))$ to C and $A \cup B$ *only*, since only them were involved in the conflict: i.e. $m_{12}(C \cap (A \cup B)) = m_1(C)m_2(A \cup B) + m_2(C)m_1(A \cup B)$, clearly the other elements A, B, $A \cup B \cup C$ that get some mass in minC were not involved in the conflict $C \cap (A \cup B)$. Thus PCR4 does a more exact redistribution than both minC versions.

6.2 The PCR4 formula

PCR4 formula for two sources is given by $\forall X \in G \setminus \{\emptyset\}$,

$$\begin{aligned} m_{PCR4}(X) &= m_{12}(X) \\ &+ \sum_{\substack{Y \in G \\ c(Y \cap X) = \emptyset}} m_{12}(X) \cdot \frac{m_{12}(X \cap Y)}{m_{12}(X) + m_{12}(Y)} \end{aligned} \quad (11)$$

where $m_{12}(X)$ corresponds to the conjunctive consensus on X, and where all denominators $[m_{12}(X) + m_{12}(Y)]$ are different from zero. If a denominator corresponding to some Y is zero, the fraction it belongs to is discarded and the mass $m_{12}(X \cap Y)$ is transferred to X and Y proportionally with respect to their non-zero column sum of masses; if both their column sums of masses are zero, then one transfers to the partial ignorance $X \cup Y$; if even this partial ignorance is empty then one transfers to the total ignorance. In [11], we show the neutral impact of VBA for PCR4 and provide the general PCR4 formula for $s \geq 2$ sources with more detailed examples and comparisons with minC operator.

6.3 An example for PCR4

Let's consider a more complex example involving some null masses (i.e. $m_{12}(A) = m_{12}(B) = 0$) in the conjunctive consensus between sources. So, let's take $\Theta = \{A, B, C, D\}$, the Shafer's model and the two following belief assignments:

	A	B	C	D
$m_1(.)$	0	0.4	0.5	0.1
$m_2(.)$	0.6	0	0.1	0.3
$m_{12}(.)$	0	0	0.05	0.03

The conjunctive consensus $m_{12}(.)$ corresponds to the last row of the table with the total conflicting mass

$$\begin{aligned} k_{12} &= m_{12}(A \cap B) + m_{12}(A \cap C) + m_{12}(A \cap D) \\ &\quad + m_{12}(B \cap C) + m_{12}(B \cap D) + m_{12}(C \cap D) \\ &= 0.24 + 0.30 + 0.06 + 0.04 + 0.12 + 0.16 = 0.92 \end{aligned}$$

Because $m_{12}(A) = m_{12}(B) = 0$, the denominator $m_{12}(A) + m_{12}(B) = 0$ and the transfer onto A and B should be done proportionally to $m_2(A)$ and $m_1(B)$, thus $x/0.6 = y/0.4 = 0.24/(0.6 + 0.4) = 0.24$ whence $x = 0.144$, $y = 0.096$ thus $m_{PCR4}(A) = m_{12}(A) + x = 0 + 0.144 = 0.144$. From PCR4 Formula, the partial conflict $m_{12}(A \cap C) = 0.30$ goes to C since $m_{12}(A) = 0$, $m_{12}(A \cap D) = 0.06$ goes to D since $m_{12}(A) = 0$, $m_{12}(B \cap C) = 0.04$ goes to C since $m_{12}(B) = 0$, $m_{12}(B \cap D) = 0.12$ goes to D since $m_{12}(B) = 0$ and the partial conflict $m_{12}(C \cap D) = 0.16$ is proportionally redistributed to C and D only according to $z/0.05 = w/0.03 = 0.16/(0.05 + 0.03) = 2$ whence $z = 0.10$ and $w = 0.06$. Summing all redistibuted partial conflicts, one finally gets for PCR4 versus minC (version a)) the following results

With PCR4	With minC a)
$m_{PCR4}(A) = 0.144$	$m_{minC}(A) = 0.120$
$m_{PCR4}(B) = 0.096$	$m_{minC}(B) = 0.120$
$m_{PCR4}(C) = 0.490$	$m_{minC}(C) = 0.490$
$m_{PCR4}(D) = 0.270$	$m_{minC}(D) = 0.270$

The distinction between PCR4 and minC here is that minC (version a)) halves the conflicting mass $m_{12}(A \cap B) = 0.24$ between A and B, while PCR4 redistributes it to A and B proportionally to their masses $m_2(A)$ and $m_1(B)$. The minC version b) will redistribute partial conflict $m_{12}(A \cap B) = 0.24$ to A, B and also to $A \cup B$ which is less exact than PCR4 and minC version a) since $A \cup B$ is not involved into the partial conflict $A \cap B$.

7 The PCR5 fusion rule

7.1 General principle of PCR5

Similarly to PCR2-4, PCR5 redistributes the partial conflicting mass to the elements involved in the partial conflict, considering the canonical form of the partial conflict. PCR5 is the most mathematically exact redistribution of conflicting mass to non-empty sets following the logic of the conjunctive rule. But this is harder to implement. PCR2 to PCR5 fusion rules preserve the neutral impact of the VBA because in any partial conflict, as well in the total conflict which is a sum of all partial conflicts, the canonical form of each partial conflict does not include Θ since Θ is a neutral element for intersection (conflict), therefore Θ gets no mass after the redistribution of the conflicting mass. We have also proved the continuity property of the PCR5 result with continuous variations of bbas to combine in [11].

7.2 The PCR5 formula

PCR5 fusion rule for two sources is: $\forall X \in G \setminus \{\emptyset\}$,

$$m_{PCR5}(X) = m_{12}(X) + \sum_{\substack{Y \in G \\ c(Y \cap X) = \emptyset}} [\frac{m_1(X)^2 m_2(Y)}{m_1(X) + m_2(Y)} + \frac{m_2(X)^2 m_1(Y)}{m_2(X) + m_1(Y)}] \quad (12)$$

where $c(x)$ represents the canonical form of x, $m_{12}(X)$ corresponds to the conjunctive consensus on X, and where all denominators are *different from zero*. If a denominator is zero, that fraction is discarded. The general (but more complex) PCR5 formula for $s \geq 2$ sources is given in [11].

7.3 Some examples for PCR5

7.3.1 A two-source example 1

Let's take $\Theta = \{A, B\}$ of exclusive elements, and the following bbas

	A	B	$A \cup B$
$m_1(.)$	0.6	0	0.4
$m_2(.)$	0	0.3	0.7
$m_{12}(.)$	0.42	0.12	0.28

The conflicting mass is $k_{12} = m_{12}(A \cap B) = m_1(A)m_2(B) + m_1(B)m_2(A) = 0.18$. Therefore A and B are involved in the conflict ($A \cup B$ is not involved), hence only A and B deserve a part of the conflicting mass, $A \cup B$ does not deserve. With PCR5, one redistributes the conflicting mass $k_{12} = 0.18$ to A and B proportionally with the masses $m_1(A)$ and $m_2(B)$ assigned to A and B respectively. Let x be the conflicting mass to be redistributed to A, and y the conflicting mass redistributed to B, then $x/0.6 = y/0.3 = (x + y)/(0.6 + 0.3) = 0.18/0.9 = 0.2$ whence $x = 0.6 \cdot 0.2 = 0.12$, $y = 0.3 \cdot 0.2 = 0.06$. Thus, the final result is

With PCR5
$m_{PCR5}(A) = 0.42 + 0.12 = 0.54$
$m_{PCR5}(B) = 0.12 + 0.06 = 0.18$
$m_{PCR5}(A \cup B) = 0.28$

It has been proved in [11] that this result is equal to that of PCR3 and even PCR2, but is different from PCR1 and PCR4 which yield in this specific case to

With PCR1	With PCR4
$m_{PCR1}(A) = 0.474$	$m_{PCR4}(A) = 0.56$
$m_{PCR1}(B) = 0.147$	$m_{PCR4}(B) = 0.16$
$m_{PCR1}(A \cup B) = 0.379$	$m_{PCR4}(A \cup B) = 0.28$

7.3.2 A two-source example 2

Let's modify a little the previous example and consider now

	A	B	$A \cup B$
$m_1(.)$	0.6	0	0.4
$m_2(.)$	0.2	0.3	0.5
$m_{12}(.)$	0.50	0.12	0.20

The conflicting mass $k_{12} = m_{12}(A \cap B) = m_1(A)m_2(B) + m_1(B)m_2(A) = 0.18$ remains the same as in previous example, which means that $m_2(A) = 0.2$ did not have any impact on the conflict; why?, because $m_1(B) = 0$. Therefore A and B are involved in the conflict ($A \cup B$ is not involved), hence only A and B deserve a part of the conflicting mass, $A \cup B$ does not deserve. With PCR5, one redistributes the conflicting mass 0.18 to A and B proportionally with the masses $m_1(A)$ and $m_2(B)$ assigned to A and B respectively. The mass

$m_2(A) = 0.2$ is not considered to the weighting factors of the redistribution. Let x be the conflicting mass to be redistributed to A, and y the conflicting mass redistributed to B. By the same calculations one has: $x/0.6 = y/0.3 = (x+y)/(0.6+0.3) = 0.18/0.9 = 0.2$ whence $x = 0.6 \cdot 0.2 = 0.12$, $y = 0.3 \cdot 0.2 = 0.06$. Thus, one gets now:

With PCR5
$m_{PCR5}(A) = 0.50 + 0.12 = 0.62$
$m_{PCR5}(B) = 0.12 + 0.06 = 0.18$
$m_{PCR5}(A \cup B) = 0.20 + 0 = 0.20$

We did not take into consideration the sum of masses of column A, i.e. $m_1(A) + m_2(A) = 0.6 + 0.2 = 0.8$, since clearly $m_2(A) = 0.2$ has no impact on the conflicting mass. In this second example, the result obtained by PCR5 is different from WAO, PCR1, PCR2, PCR3 and PCR4 which are given by

With PCR1 or WAO	With PCR2
$m_{PCR1}(A) = 0.572$	$m_{PCR2}(A) \approx 0.631$
$m_{PCR1}(B) = 0.147$	$m_{PCR2}(B) \approx 0.169$
$m_{PCR1}(A \cup B) = 0.281$	$m_{PCR2}(A \cup B) = 0.20$

With PCR3	With PCR4
$m_{PCR3}(A) \approx 0.631$	$m_{PCR4}(A) \approx 0.645$
$m_{PCR3}(B) \approx 0.169$	$m_{PCR4}(B) \approx 0.155$
$m_{PCR3}(A \cup B) = 0.20$	$m_{PCR4}(A \cup B) = 0.20$

7.3.3 A two-source example 3

Let's go further modifying this time the previous example and considering:

	A	B	$A \cup B$
$m_1(.)$	0.6	0.3	0.1
$m_2(.)$	0.2	0.3	0.5
$m_{12}(.)$	0.44	0.27	0.05

The conflicting mass $k_{12} = m_{12}(A \cap B) = m_1(A)m_2(B) + m_1(B)m_2(A) = 0.18 + 0.06 = 0.24$ is now different from the two previous examples, which means that $m_2(A) = 0.2$ and $m_1(B) = 0.3$ did make an impact on the conflict; why?, because $m_2(A)m_1(B) = 0.2 \cdot 0.3 = 0.06$ was added to the conflicting mass. Therefore A and B are involved in the conflict ($A \cup B$ is not involved), hence only A and B deserve a part of the conflicting mass, $A \cup B$ does not deserve. With PCR5, one redistributes the partial conflicting mass 0.18 to A and B proportionally with the masses $m_1(A)$ and $m_2(B)$ assigned to A and B respectively, and also the partial conflicting mass 0.06 to A and B proportionally with the masses $m_2(A)$ and $m_1(B)$ assigned to A and B respectively, thus one gets two weighting factors of the redistribution for each corresponding set A and B respectively. Let x_1 be the conflicting mass to be redistributed to A, and y_1 the conflicting mass redistributed to B from the first partial conflicting mass 0.18. This first partial proportional redistribution is then done according $x_1/0.6 = y_1/0.3 = (x_1 + y_1)/(0.6 + 0.3) = 0.18/0.9 = 0.2$ whence $x_1 = 0.6 \cdot 0.2 = 0.12$, $y_1 = 0.3 \cdot 0.2 = 0.06$. Now let x_2 be the conflicting mass to be redistributed to A, and y_2 the conflicting mass redistributed to B from second the partial conflicting mass 0.06. This second partial proportional redistribution is then done according $x_2/0.2 = y_2/0.3 = (x_2 + y_2)/(0.2 + 0.3) = 0.06/0.5 = 0.12$ whence $x_2 = 0.2 \cdot 0.12 = 0.024$, $y_2 = 0.3 \cdot 0.12 = 0.036$. Thus, one gets now:

With PCR5
$m_{PCR5}(A) = 0.44 + 0.12 + 0.024 = 0.584$
$m_{PCR5}(B) = 0.27 + 0.06 + 0.036 = 0.366$
$m_{PCR5}(A \cup B) = 0.05 + 0 = 0.05$

The result is different from PCR1, PCR2, PCR3 and PCR4 since one has[4]:

With PCR1	With PCR2 ∼PCR3
$m_{PCR1}(A) = 0.536$	$m_{PCR2}(A) \approx 0.577$
$m_{PCR1}(B) = 0.342$	$m_{PCR2}(B) \approx 0.373$
$m_{PCR1}(A \cup B) = 0.122$	$m_{PCR2}(A \cup B) = 0.05$

With PCR4	With Dempster's rule
$m_{PCR4}(A) \approx 0.589$	$m_{DS}(A) \approx 0.579$
$m_{PCR4}(B) \approx 0.361$	$m_{DS}(B) \approx 0.355$
$m_{PCR4}(A \cup B) = 0.05$	$m_{DS}(A \cup B) \approx 0.066$

One clearly sees that $m_{DS}(A \cup B)$ gets some mass from the conflicting mass although $A \cup B$ does not deserve any part of the conflicting mass since $A \cup B$ is not involved in the conflict (only A and B are involved in the conflicting mass). Dempster's rule appears to us less exact than PCR5.

8 Application of fusion on Zadeh's example

In this section, we just present the comparison of the different rules of combinations on the well-known Zadeh's example[5] [14]. More examples including hybrid DSm models can be found in [11]. So let's take $\Theta = \{A, B, C\}$, Shafer's model and the two following belief assignments

	A	B	C
$m_1(.)$	0.9	0	0.1
$m_2(.)$	0	0.9	0.1
$m_{12}(.)$	0	0	0.01

The masses committed to partial conflicts are given by $m_{12}(A \cap B) = 0.81$, $m_{12}(A \cap C) = m_{12}(B \cap C) = 0.09$ and the conflicting mass by $k_{12} = m_1(A)m_2(B) + m_1(A)m_2(C) + m_2(B)m_1(C) = 0.81 + 0.09 + 0.09 = 0.99$. We denote by indexes DS, S, DP, Y, DSmC the fusion rules based respectively on the Demspter's rule, Smets' rule (in open world), Dubois and Prade's rule, Yager's rule, Dezert-Smarandache classic rule (based on free model). The DSmH (Dezert-Smarandache hybrid rule of combination) based on the Shafer's model coincides here in this static fusion problem with the Dubois and Prade's result and will not be reported. The following table summarizes the results of all these different rules for the Zadeh's example above.

[4] The verification is left to the reader.

[5] A detailed discussion on this example can be found in [9] (Chap. 5).

	m_{DS}	m_S	m_{DP}	m_Y	m_{DSmC}
$\emptyset$		0.99			
$A \cap B$					0.81
$A \cap C$					0.09
$B \cap C$					0.09
C	1	0.01	0.01	0.01	0.01
$A \cup B$			0.81		
$A \cup C$			0.09		
$B \cup C$			0.09		
$A \cup B \cup C$				0.99	

We present final results obtained now with PCR1-PCR5 fusion rules for this same Zadeh's example based on the Shafer's model. All details of derivations are given in [11]

	m_{PCR1}	m_{PCR2}	m_{PCR3}	m_{PCR4}	m_{PCR5}
A	0.4455	0.4455	0.47864	0.405	0.486
B	0.4455	0.4455	0.47864	0.405	0.486
C	0.1090	0.1090	0.04272	0.190	0.028

The WAO gives same result as PCR1 and PCR2. For this Zadeh's example, $m_{PCR4}(.) = m_{minC}(.)$, although minC calculates in a different way.

9 Conclusion

We have presented in this article five versions of the Proportional Conflict Redistribution rule of combination in information fusion, which are implemented as follows: first one uses the conjunctive rule, then one redistributes the conflicting mass to non-empty sets proportionally with respect to either the non-zero column sum of masses (for PCR1, PCR2, PCR3) or with respect to the non-zero masses (of the corresponding non-empty set) that enter in the composition of each individual product in the partial conflicting masses (PCR5). PCR1 restricted from the hyper-power set to the power set and without degenerate cases gives the same result as WAO. PCR1 and PCR2 redistribute the total conflicting mass, while PCR3 and PCR5 redistribute partial conflicting masses. PCR1-3 uses the proportionalization with respect to the sum of mass columns, PCR4 with respect to the results of the conjunctive rule, and PCR5 with respect to the masses entered in the sum products of the conflicting mass. PCR4 is an improvement of minC and Dempster's rules. From PCR1 up to PCR5, one increases the complexity of the rules and also the exactitude of the redistribution of conflicting masses. All the PCR rules proposed in this paper preserve the neutral impact of the vacuous belief assignment (but PCR1) and work for any hybrid model (including the Shafer's model). For the free model, i.e. when all intersections are not empty only DSm classic rule (conjunctive consensus) on hyper-power set applies. Inagaki and Lefèvre and al. already proved that there are infinitely many possible fusion rules based on the conjunctive consensus and then on the transfer of the conflicting mass, all of them depending on the weighting coefficients/factors that transfer that conflicting mass. How to choose them, what parameters should they rely on ? That is today the open and fundamental question! In authors' opinion, neither DSm hybrid rule, nor PCR rules are not more ad-hoc than other fusion rules proposed in literature so far.

References

[1] Daniel M., *Distribution of Contradictive Belief Masses in Combination of Belief Functions*, Inf., Uncert. and Fusion, Bouchon-Meunier, Yager and Zadeh (Eds.), Kluwer Acad. Publ., 2000, pp. 431-446.

[2] Daniel M., *Associativity in Combination of belief functions; a derivation of minC combination*, Soft Computing, 7(5), pp. 288–296, 2003.

[3] Dubois D., Prade H., *A Set-Theoretic View of Belief Functions*, Int. J. of Gen. Syst., pp.193-226, Vol.12, 1986.

[4] Inagaki T., *Interdependence between safety-control policy and multiple-sensor schemes via Dempster-Shafer theory*, IEEE Trans. on Reliability, Vol. 40, no. 2, pp. 182-188, 1991.

[5] Jøsang, A., Daniel, M., Vannoorenberghe, P., *Strategies for Combining Conflicting Dogmatic Beliefs*, Proceedings of the 6th International Conference on International Fusion, Vol. 2, pp. 1133-1140, 2003.

[6] Lefèvre E., Colot O., Vannoorenberghe P., *Belief functions combination and conflict management*, Information Fusion Journal, Elsevier Publisher, Vol. 3, No. 2, pp. 149-162, 2002.

[7] Sentz K., Ferson S., *Combination of evidence in Dempster-Shafer Theory*, SANDIA Tech. Report, SAND2002-0835, 96 pages, April 2002.

[8] Shafer G., *A Mathematical Theory of Evidence*, Princeton Univ. Press, Princeton, NJ, 1976.

[9] Smarandache F., Dezert J. (Editors), *Applications and Advances of DSmT for Information Fusion*, Am. Res. Press, Rehoboth, 2004, http://www.gallup.unm.edu/˜smarandache/DSmT-book1.pdf.

[10] Smarandache F., Dezert J., *A Simple Proportional Conflict Redistribution Rule*, arXiv Archives, Los Alamos National Laboratory, July-August 2004; the paper is available at http://arxiv.org/PS_cache/cs/pdf/0408/0408010.pdf.

[11] Smarandache F., Dezert J., *Proportional Conflict Redistribution Rules for Information Fusion*, submitted paper to ISIF JAIF (Journal of Advances in Information Fusion), 26 pages, Dec 2004 (available upon request to authors).

[12] Smets Ph., *Data Fusion in the Transferable Belief Model*, Proc. of Fusion 2000 Intern. Conf. on Information Fusion, Paris, July 2000.

[13] Yager R.R., *On the Dempster-Shafer framework and new combination rules*, Information Sciences, Vol. 41, pp. 93–138, 1987.

[14] Zadeh L., *On the validity of Dempster's rule of combination*, Memo M 79/24, Univ. of California, Berkeley, 1979.