Qualitative Belief Conditioning Rules

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Introduction

Purpose: Given a prior qualitative mass of belief qm(.) and a conditioning event A, compute the posterior belief mass qm(.|A)

We introduce the notion of Dedekind's lattice (hyper-power set).

We justify the introduction of Belief Conditioning Rules (BCRs).

We propose a simple arithmetic to compute with equidistant and non-equidistant linguistic labels, operations with labels, normlization and quasi-normalization of qualitative masses.

We extend (BCRs) developed in 2006 from quantitative to qualitative Belief Conditioning Rules (QBCR)

Simple examples will show how two QBCRs work.

Hyper-power set and quantitative belief mass

Frame of the problem $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$. Finite set of exhaustive elements (discrete/continuous/fuzzy/relative concepts)

Hyper-power set (Dedekind's lattice) $|D^\Theta| > |2^\Theta|$

1) $\emptyset, \theta_1, \ldots, \theta_n \in D^\Theta$.

2) If $X, Y \in D^\Theta$, then $X \cap Y$ and $X \cup Y$ belong to D^Θ.

3) No other elements belong to D^Θ, except those obtained by using rules 1) or 2).

Example for n=3 $|D^\Theta| = 19$

$\alpha_0 \triangleq \emptyset$
$\alpha_1 \triangleq \theta_1 \cap \theta_2 \cap \theta_3$
$\alpha_2 \triangleq \theta_1 \cap \theta_2$
$\alpha_3 \triangleq \theta_1 \cap \theta_3$
$\alpha_4 \triangleq \theta_2 \cap \theta_3$
$\alpha_5 \triangleq (\theta_1 \cup \theta_2) \cap \theta_3$
$\alpha_6 \triangleq (\theta_1 \cup \theta_3) \cap \theta_2$
$\alpha_7 \triangleq (\theta_2 \cup \theta_3) \cap \theta_1$
$\alpha_8 \triangleq (\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3)$
$\alpha_9 \triangleq \theta_1$
$\alpha_{10} \triangleq \theta_2$
$\alpha_{11} \triangleq \theta_3$
$\alpha_{12} \triangleq (\theta_1 \cap \theta_2) \cup \theta_3$
$\alpha_{13} \triangleq (\theta_1 \cap \theta_3) \cup \theta_2$
$\alpha_{14} \triangleq (\theta_2 \cap \theta_3) \cup \theta_1$
$\alpha_{15} \triangleq \theta_1 \cup \theta_2$
$\alpha_{16} \triangleq \theta_1 \cup \theta_3$
$\alpha_{17} \triangleq \theta_2 \cup \theta_3$
$\alpha_{18} \triangleq \theta_1 \cup \theta_2 \cup \theta_3$

DSm models : Free model, Hybrid model, Shafer's model

Quantitative basic belief assignment/mass: $m(.) : D^\Theta \to [0, 1]$

$$m(\emptyset) = 0 \qquad \text{and} \qquad \sum_{A \in D^\Theta} m(A) = 1$$

Quantitative Belief Conditioning Rules (BCR)

One makes a fundamental distinction between **fusion** of a prior bba $m_1(.)$ with a source focused on a given set A (Shafer's approach) and belief revision conditioned by the fact that absolute truth is in A (BCRs approach).

Shafer's "conditioning" rule (SCR)

subjective certainty committed to A by source # 2

$$m_1(.|A) = [m_1 \oplus m_2](.) \quad \textbf{with} \quad \begin{cases} m_2(A) = 1 \\ \oplus = \text{Dempster's rule} \end{cases}$$

Note: We could replace $\oplus$ Dempster's rule by any other $\oplus$ fusion rules.

Belief Conditioning/revision Rules (BCR)

To compute $m_1(.|A)$, and because the conditioning event A contains absolute truth, one proposes to revise the prior bba $m_1(.)$ based on NEW mass transfer, NOT based on the fusion of $m_1(.)$ with specialized bba $m_2(A)=1$.

Many quantitative BCRs have been recently developed [Smarandache-Dezert, Fusion 2006]

They are all based on Hyper-Power Set Decomposition (HPSD)

Hyper-power set decomposition (HPSD)

The HPSD is imposed by the conditioning event, say A.

$$D^\Theta \setminus \{\emptyset\} = D_1 \cup D_2 \cup D_3$$

If A = conditioning event then

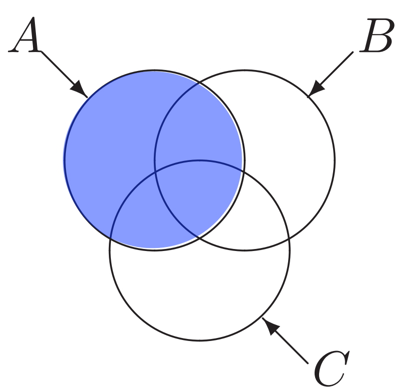
- $D_1 \triangleq \mathcal{P}_{\mathcal{D}}(A) = 2^A \cap D^\Theta \setminus \{\emptyset\}$ = all non-empty parts of A which are included in D^Θ;
- $D_2 \triangleq \{(\Theta \setminus s(A)), \cup, \cap\} \setminus \{\emptyset\}$ = the sub-hyper-power set generated by $\Theta \setminus s(A)$ under $\cup$ and $\cap$, without the empty set.
- $D_3 \triangleq (D^\Theta \setminus \{\emptyset\}) \setminus (D_1 \cup D_2)$.

where $s(A) = \{\theta_{i_1}, \theta_{i_2}, \dots, \theta_{i_p}\}$, $1 \leq p \leq n$, be the singletons/atoms that compose A.

Example: if $A = \theta_1 \cup (\theta_3 \cap \theta_4)$ then $s(A) = \{\theta_1, \theta_3, \theta_4\}$.

The quantitative (or qualitative) masses of elements of D_2 and D_3 are redistributed to non-empty elements of D_1 **in many possible ways** by BCRs (or QBCR).

Simple examples of HPSD

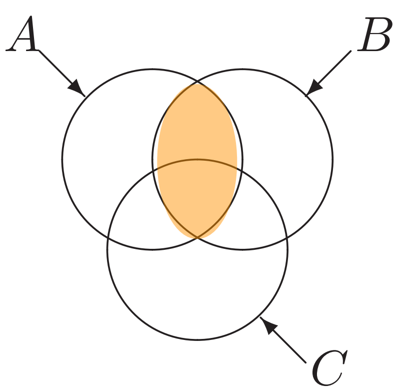


Example 1 : If the truth is in A

$$D_1 = \{A, A\cap B, A\cap C, A\cap B\cap C\} \equiv \mathcal{P}(A) \cap (D^\Theta \setminus \emptyset)$$

$$D_2 = (\{B, C\}, \cup, \cap) = D^{\{B,C\}} = \{B, C, B\cup C, B\cap C\}$$

$$D_3 = \{A\cup B, A\cup C, A\cup B\cup C, A\cup(B\cap C)\}$$

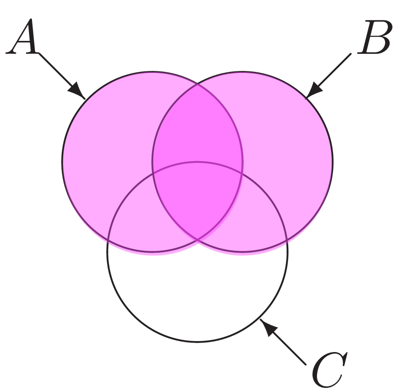


Example 2 : If the truth is in $A\cap B$

$$D1 = \{A\cap B, A\cap B\cap C\}$$

$$D_2 = \{C\}$$

$$D_3 = \{A, B, A\cup B, A\cap C, B\cap C, \ldots\} = (D^\Theta \setminus \{\emptyset\}) \setminus (D_1 \cup D_2)$$

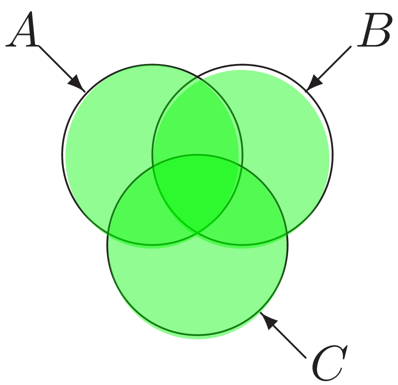


Example 3 : If the truth is in $A\cup B$

$$D_1 = \{A, B, A\cap B, A\cup B, \ldots\}$$ all other sets included in these four ones, i.e. $A\cap C$, $B\cap C$, $A\cap B\cap C$, $A\cup(B\cap C)$, $B\cup(A\cap C)$, etc.

$$D_2 = \{C\}$$

$$D_3 = \{A\cup C, B\cup C, A\cup B\cup C, C\cup(A\cap B)\}$$



Example 4 : If the truth is in $A\cup B\cup C$

$$D_1 = D^\Theta \setminus \{\emptyset\}$$

D_2 and D_3 do not exist.

Qualitative belief mass qm(.)

Ordered Linguistic labels/mass: $L = \{L_0, L_1, L_2, \ldots, L_n, L_{n+1}\}$

very low, low, medium, etc

Linguistic operators (with equidistant labels) $L_i \mapsto i/(n+1)$

$$L = \{L_0, L_1, L_2, \ldots, L_n, L_{n+1}\} \longmapsto \{0, 1/(n+1), 2/(n+1), \ldots, n/(n+1), 1\}$$

$$L_i + L_j = \frac{i}{n+1} + \frac{j}{n+1} = \frac{i+j}{n+1} = L_{i+j}$$

$$L_i - L_j = \frac{i}{n+1} - \frac{j}{n+1} = \frac{i-j}{n+1}$$

$$L_i \times L_j = \frac{i}{n+1} \cdot \frac{j}{n+1} = \frac{(i \cdot j)/(n+1)}{n+1} \approx L_{[(i \cdot j)/(n+1)]}$$

$$L_i / L_j = \frac{i/(n+1)}{j/(n+1)} = \frac{(i/j) \cdot (n+1)}{n+1} \approx L_{[(i/j) \cdot (n+1)]}$$

Linguistic operators (by extrapolation)

$$L_i + L_j = \begin{cases} L_{i+j}, & \text{if } i+j < n+1, \\ L_{n+1}, & \text{if } i+j \geq n+1. \end{cases}$$

$$L_i - L_j = \begin{cases} L_{i-j} & \text{if} \quad i \geq j, \\ -L_{j-i} & \text{if} \quad i < j. \end{cases}$$

$$L_i \times L_j = L_{[(i \cdot j)/(n+1)]}$$

$$L_i / L_j = \begin{cases} L_{[(i/j) \cdot (n+1)]} & \text{if} [(i/j) \cdot (n+1)] < n+1, \\ L_{n+1} & \text{otherwise.} \end{cases}$$

Qualitative belief mass: $\boxed{qm(.) : X \in G^{\Theta} \mapsto qm(X) = L_i \in L}$

Quasi-normalized qualitative belief mass

a) **Qualitatitive normalization**: If the labels L_0, L_1, L_2, ..., L_n, L_{n+1} are equidistant, we make an isomorphism between L and a set of sub-unitary numbers from the interval $[0, 1]$ in the following way:

$$L_i = i/(n+1), \qquad \text{for all } i \in \{0, 1, 2, \ldots, n+1\}$$

and therefore the interval $[0, 1]$ is divided into $n+1$ equal parts.

Hence $qm(X_i) = L_i$ is equivalent to a quantitative mass $m(X_i) = i/(n+1)$ which is normalized when

$$\sum_{X \in D^\Theta} m(X) = \sum_k i_k/(n+1) = 1 \Leftrightarrow \sum_{X \in D^\Theta} qm(X) = \sum_k L_{i_k} = L_{n+1}$$

b) **Qualitative quasi-normalization**: But, if labels L_0, L_1, L_2, ..., L_n, L_{n+1} are not equidistant, then it makes sense to consider a qualitative quasi-normalization, the approximation of the (classical) numerical normalization for the qualitative masses in the same way:

$$\sum_{X \in D^\Theta} qm(X) = L_{n+1}$$

Qualitative Belief Conditioning Rules (QBCR)

Purpose: Compute qm(.|A) from qm(.) and event A using the extension of quantitative Belief Conditioning Rules based on HPSD.

Qualitative Belief Conditioning Rule # 1 (QBCR1)

QBCR1 proposes a redistribution of masses in a pessimistic/prudent way, i.e.

- transfer the mass of each Y in $D_2 \cup D_3$ to the largest element X in D_1 which is contained by Y;
- if no such X element exists, then the mass of Y is transferred to A.

Qualitative Belief Conditioning Rule # 2 (QBCR2)

QBCR2 does a uniform redistribution of masses, as follows:

- transfer the mass of each element Y in $D_2 \cup D_3$ to the largest element X in D_1 which is contained by Y (as QBCR1 does);
- if no such X element exists, then the mass of Y is uniformly redistributed to all subsets of A whose (qualitative) masses are not L_0 (i.e. to all qualitative focal elements included in A).
- if there is no qualitative focal element included in A, then the mass of Y is transferred to A.

Mathematical expression for QBCR1

- If $X \notin D_1$, $$\boxed{qm_{QBCR1}(X|A) = L_{\min} \equiv L_0}$$
- If $X \in D_1$, $$\boxed{qm_{QBCR1}(X|A) = qm(X) + qS_1(X,A) + qS_2(X,A)}$$

where

$$qS_1(X,A) \triangleq \sum_{\substack{Y \in D_2 \cup D_3 \\ X \subset Y \\ X = \max}} qm(Y)$$

This sum transfers the qualitative mass of each Y in $D_2 \cup D_3$ to the largest X in D_1.

$$qS_2(X,A) \triangleq \sum_{\substack{Y \in D_2 \cup D_3 \\ Y \cap A = \emptyset \\ X = A}} qm(Y)$$

This sum transfers the qualitative mass of Y in $D_2 \cup D_3$ to A if no largest X in D_1 exists.

Mathematical expression for QBCR2

- If $X \notin D_1$, $\boxed{qm_{QBCR2}(X|A) = L_{\min} \equiv L_0}$
- If $X \in D_1$, $\boxed{qm_{QBCR2}(X|A) = qm(X) + qS_1(X,A) + qS_3(X,A) + qS_4(X,A)}$

where

$$qS_1(X,A) \triangleq \sum_{\substack{Y \in D_2 \cup D_3 \\ X \subset Y \\ X = \max}} qm(Y)$$

This sum transfers the qualitative mass of each Y in D_2UD_3 to the largest X in D_1.

$$qS_3(X,A) \triangleq \sum_{\substack{Y \in D_2 \cup D_3 \\ Y \cap A = \emptyset \\ q_F \neq 0}} \frac{qm(Y)}{q_F}$$

This sum uniformly transfers the qualitative mass of each Y in D_2UD_3 with Y outside of A, to all focal elements of P(A)\A, if there exist at least one such focal element.

$$qS_4(X,A) \triangleq \sum_{\substack{Y \in D_2 \cup D_3 \\ Y \cap A = \emptyset \\ X = A, q_F = 0}} qm(Y),$$

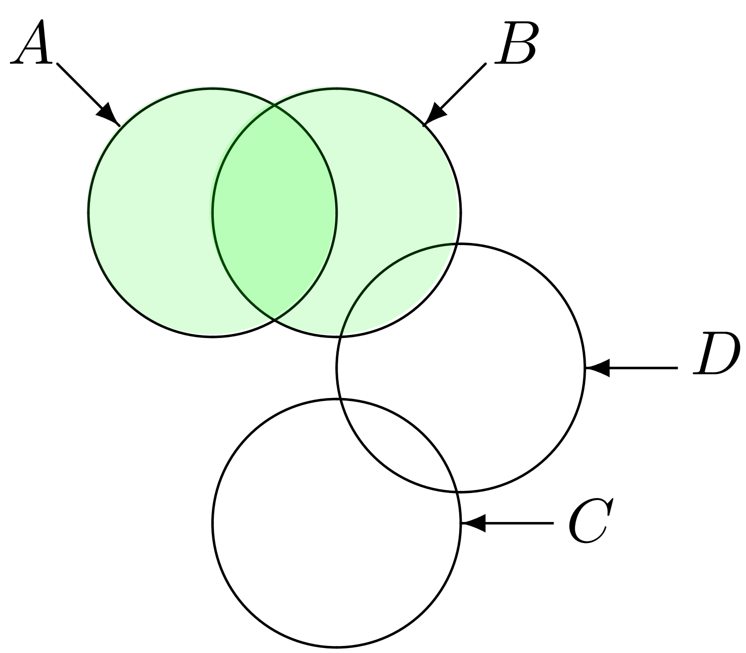
This sum transfers the qualitative mass of each Y in D_2UD_3 , with Y outside of A, to A only, if there exist non focal element in P(A)\A.

where $q_F \triangleq \mathrm{Card}\{Z | Z \subset A, qm(Z) \neq L_0\} =$ number of qualitative focal elements of A.

Example # 1

Regions of interest

$\Theta = \{A, B, C, D\}$



Labels: $L = \{L_0, L_1, L_2, L_3, L_4, L_5, L_6\}$

very poor poor medium good very good

Input (prior qualitative masses):

$$qm(A) = L_1, \quad qm(C) = L_1, \quad qm(D) = L_4$$

Conditioning event: $A \cup B$

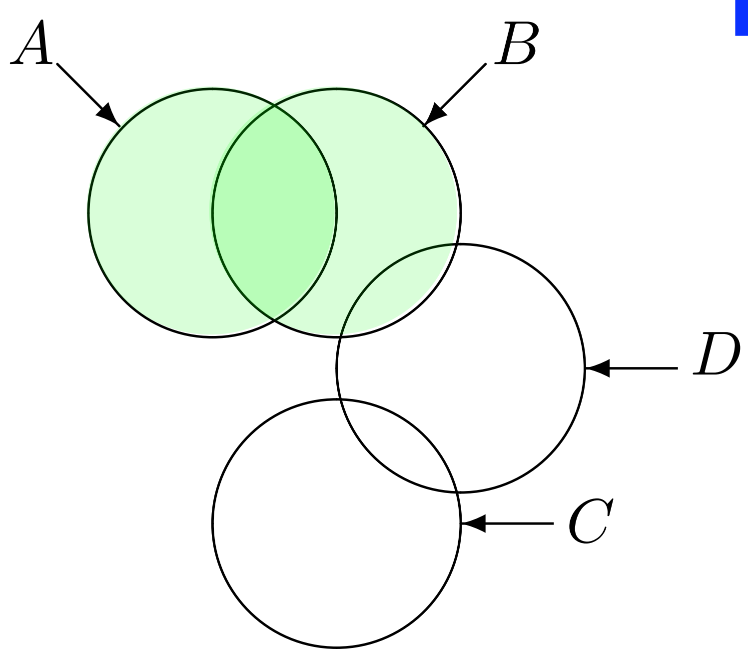
HPSD: $D^\Theta \setminus \{\emptyset\} = D_1 \cup D_2 \cup D_3$

$$D_1 = \{A \cap B, A, B, A \cup B, B \cap D, A \cup (B \cap D), (A \cap B) \cup (B \cap D)\}$$

$$D_2 = \{(C, D), \cup, \cap\} \setminus \emptyset = \{C, D, C \cup D, C \cap D\}$$

$$D_3 = \{A \cup C, A \cup D, B \cup C, B \cup D, A \cup B \cup C, A \cup (C \cap D), \ldots\}$$

Example # 1 (cont'd)



$$qm(A) = L_1, \quad qm(C) = L_1, \quad qm(D) = L_4$$

Conditioning event: $A \cup B$

$$D_1 = \{A \cap B, A, B, A \cup B, B \cap D, A \cup (B \cap D), (A \cap B) \cup (B \cap D)\}$$
$$D_2 = \{(C, D), \cup, \cap\} \setminus \emptyset = \{C, D, C \cup D, C \cap D\}$$
$$D_3 = \{A \cup C, A \cup D, B \cup C, B \cup D, A \cup B \cup C, A \cup (C \cap D), \dots\}$$

With QBCR1

$qm(D) = L_4$ is transferred to $D \cap (A \cup B) = B \cap D$, since D is in the set $D_2 \cap D_3$ and the largest element X in D_1 which is contained by element D is $B \cap D$.

$qm(C) = L_1$, which is in $D_2 \cup D_3$ (but C has an empty intersection with $A \cup B$), is transferred to $A \cup B$.

$qm(A) = L_1$, which is already in D_1 is unchanged by the conditioning.

$$qm_{QBCR1}(B \cap D | A \cup B) = L_4$$
$$qm_{QBCR1}(A \cup B | A \cup B) = L_1$$
$$qm_{QBCR1}(A | A \cup B) = L_1$$

All other masses take the value L_0

With QBCR2

Analogously to QBCR1, $qm(D) = L_4$ is transferred to $D \cap (A \cup B) = B \cap D$.

Differently from QBCR1, $qm(C) = L_1$, which is in $D_2 \cup D_3$, but $C \cap A \cup B) = \emptyset$, is transferred to A only since $A \subset A \cup B$ and $qm(A) \neq L_0$ while other sets included in $A \cup B$ have the qualitative mass equal to L_0.

Whence $qm_{QBCR2}(A | A \cup B) = \underbrace{L_1}_{qm(A)} + \underbrace{L_1}_{qm(C)} = L_2$

$$qm_{QBCR2}(B \cap D | A \cup B) = L_4$$
$$qm_{QBCR2}(A | A \cup B) = L_2$$

Example # 2

4 Regions of interest

$$\Theta = \{A, B, C, D\}$$

Labels: $L = \{L_0, L_1, L_2, L_3, L_4, L_5, L_6\}$

very poor poor medium good very good

Input (prior qualitative masses): same values

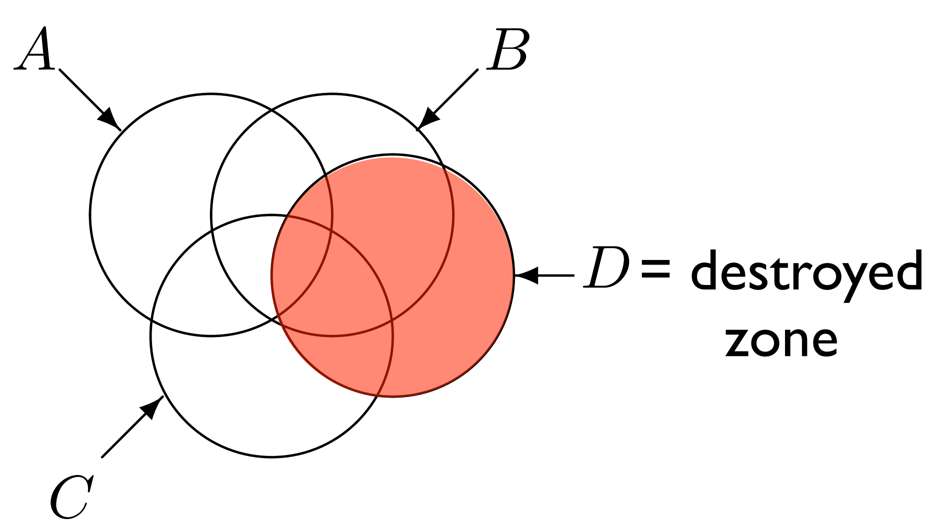
$$qm(A) = L_1, \quad qm(C) = L_1, \quad qm(D) = L_4$$

Scenario: Military headquarter bombs region D.

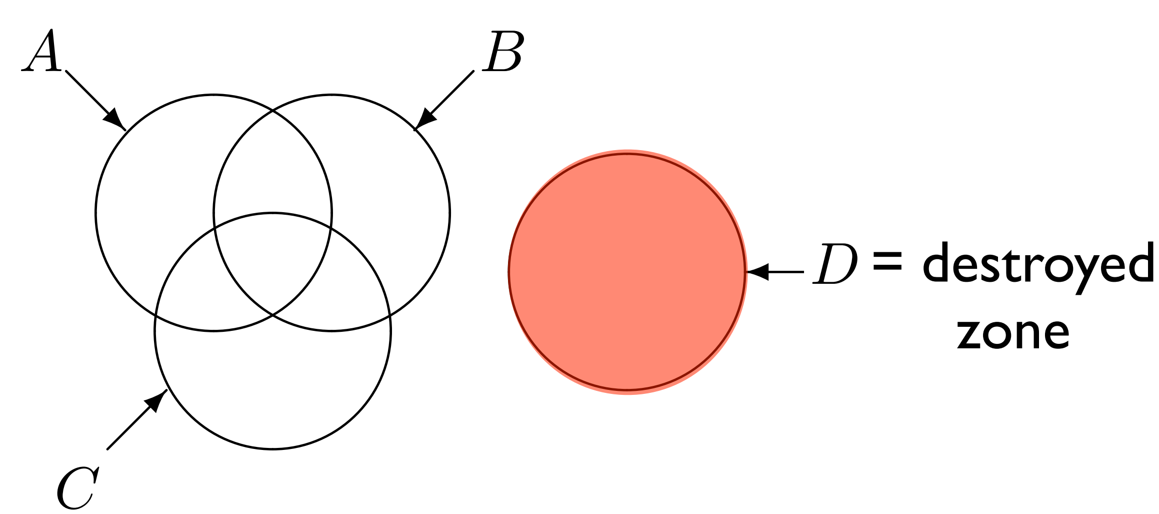
After bombing, it turns out that enemies were not in D.

Question: How to compute qm(.|not in D) ?

Two cases are examined:

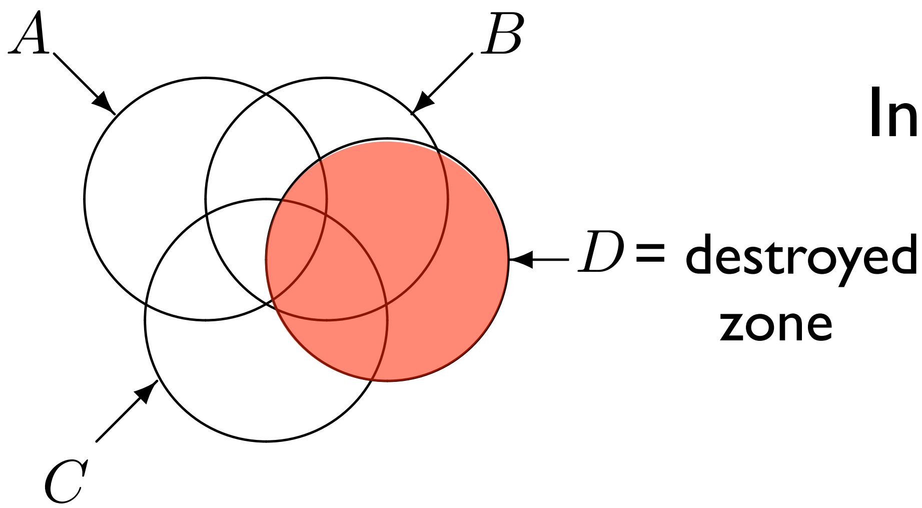


Case 1: $\bar{D} \neq A \cup B \cup C$.



Case 2: $\bar{D} = A \cup B \cup C$.

Example # 2 (case 1)



Case 1: $\bar{D} \neq A \cup B \cup C$.

Input: $qm(A) = L_1, \quad qm(C) = L_1, \quad qm(D) = L_4$

$D^\Theta \setminus \{\emptyset\} = D_1 \cup D_2 \cup D_3$ **with**

$$D_1 = \{\text{all non-empty parts of } \bar{D} \text{ only}\}$$

$$D_2 = \{\text{all non-empty parts of } D \text{ only}\}$$

$$D_3 = \{A, B, C, A \cup D, B \cup D, A \cup B, \ldots\}$$

a) Using QBCR1: one gets:

$$qm_{QBCR1}(A \cap \bar{D}|\bar{D}) = L_1$$
$$qm_{QBCR1}(C \cap \bar{D}|\bar{D}) = L_1$$
$$qm_{QBCR1}(\bar{D}|\bar{D}) = L_4$$

b) Using QBCR2: one gets

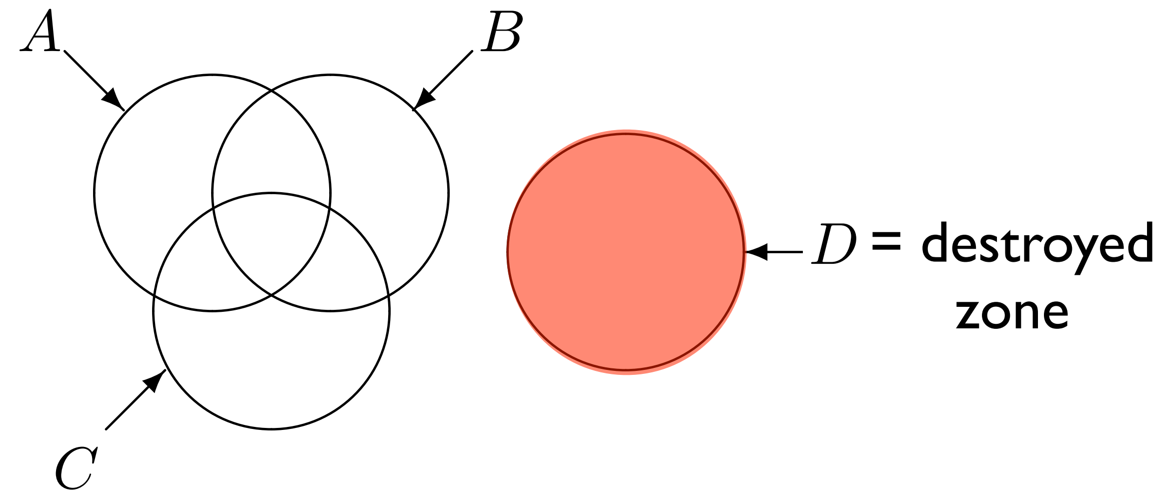
$$\begin{aligned} qm_{QBCR2}(A \cap \bar{D}|\bar{D}) &= L_1 + \frac{1}{2}L_4 \\ &= L_1 + L_{[\frac{4}{2}]} = L_3 \\ qm_{QBCR2}(C \cap \bar{D}|\bar{D}) &= L_1 + \frac{1}{2}L_4 = L_3 \end{aligned}$$

In QBCR2, we considered a dynamic qualitative set of focal elements.

With QBCR1-2, one gets quasi-normalized qualitative belief masses. The results indicate that zones A and C have the same level of qualitative belief after the conditioning which is normal.

QBRC1 is more prudent tahn QBCR2 since it commits the higher belief to the whole non destroyed zone of $A \cup B \cup C$ (i.e. the less specific information), while QBRC2 commits equal beliefs to the restricted zones $A \cap \bar{D}$ and $C \cap \bar{D}$ only.

Example # 2 (case 2)



Case 2: $\bar{D} = A \cup B \cup C$.

Input: $qm(A) = L_1, \quad qm(C) = L_1, \quad qm(D) = L_4$

$D^\Theta \setminus \{\emptyset\} = D_1 \cup D_2 \cup D_3$ with

$D_1 = \{\text{all non-empty parts of } \bar{D} = A \cup B \cup C\}$

$D_2 = \{D\}$

$D_3 = \{A \cup D, B \cup D, C \cup D, A \cup B \cup D, A \cup C \cup D, B \cup C \cup D, A \cup B \cup C \cup D, (A \cap B) \cup D, (A \cap B \cap C) \cup D, \ldots\}$

a) Using QBCR1: one gets

$$qm_{QBCR1}(A|\bar{D}) = L_1$$
$$qm_{QBCR1}(C|\bar{D}) = L_1$$
$$qm_{QBCR1}(A \cup B \cup C|\bar{D}) = L_4$$

b) Using QBCR2: one gets

$$qm_{QBCR2}(A|\bar{D}) = L_3$$
$$qm_{QBCR2}(C|\bar{D}) = L_3$$

Same conclusion as for case 1. There is uncertainty in the decision to next zone to destroy (A or C) because they have the same supporting belief. The only difference with respect to case 1, it that the zone to be bombed will remain larger than in case 1 because D has no intersection with A, B and C for this model.

Example # 3 (unconventional bombing strategy)

4 Regions of interest

$$\Theta = \{A, B, C, D\}$$

Labels: $L = \{L_0, L_1, L_2, L_3, L_4, L_5, L_6\}$

very poor poor medium good very good

Input (prior qualitative masses):

different values

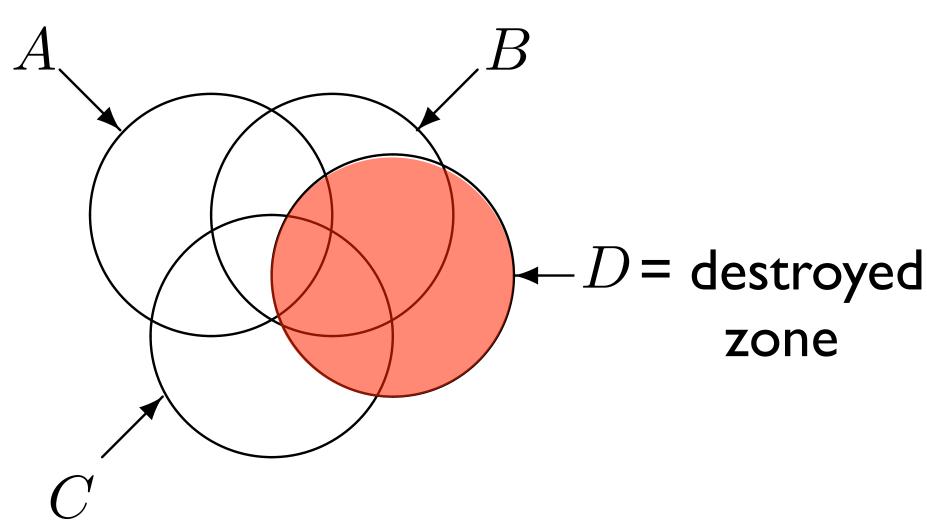
$$qm(A) = L_1, qm(C) = L_3, qm(D) = L_2$$

Scenario: Military headquarter bombs region D although qm(D) < qm(C), i.e unconventional decision.

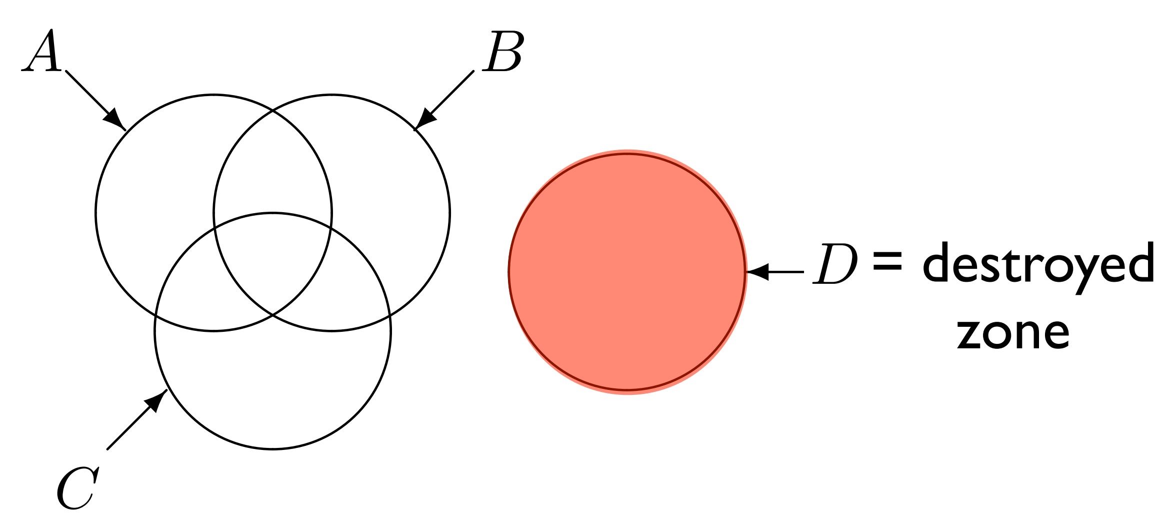
After bombing, it turns out that enemies were not in D.

Question: How to compute qm(.|not in D) ?

Two cases are examined:



Case 1: $\bar{D} \neq A \cup B \cup C$.

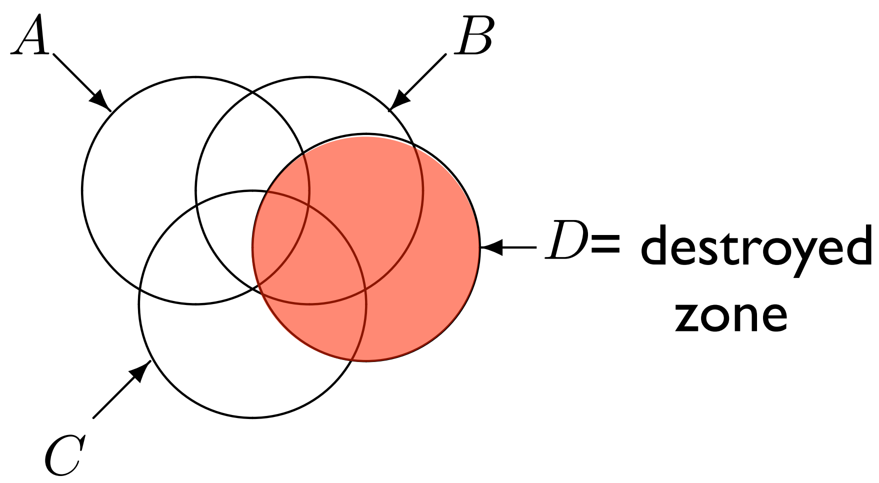


Case 2: $\bar{D} = A \cup B \cup C$.

Example # 3 (cont'd)

$$qm(A) = L_1, qm(C) = L_3, qm(D) = L_2$$

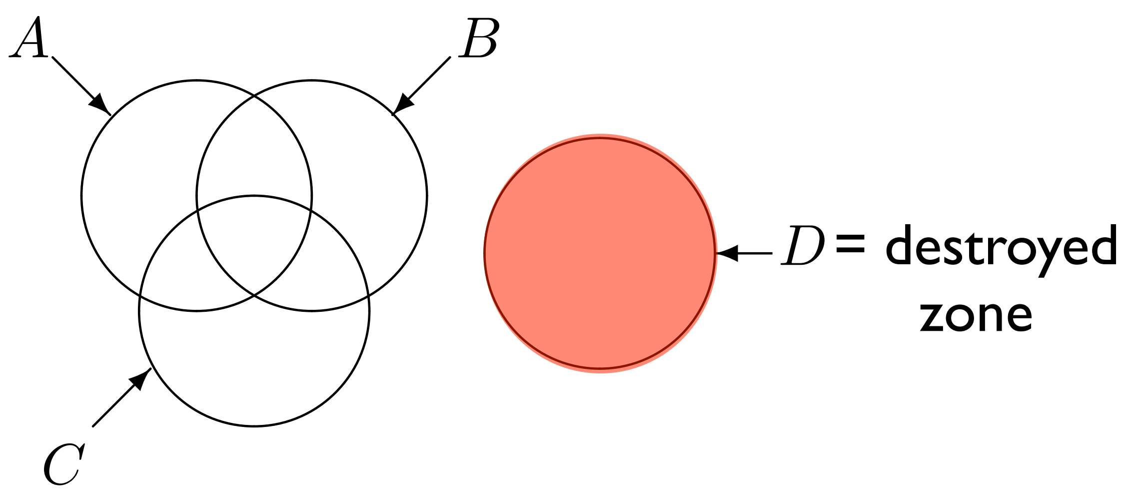
Case 1: $\bar{D} \neq A \cup B \cup C$.



In this case, using both QBCR1 and QBCR2, the mass $qm(A) = L_1$ is transferred to $A \cap \bar{D}$; the mass $qm(C) = L_3$ is transferred to $C \cap \bar{D}$; and the mass $qm_2(D) = L_2$ is transferred to $\bar{D}$.

$$qm_{QBCR1}(A \cap \bar{D}|\bar{D}) = qm_{QBCR2}(A \cap \bar{D}|\bar{D}) = L_1$$
$$qm_{QBCR1}(C \cap \bar{D}|\bar{D}) = qm_{QBCR2}(C \cap \bar{D}|\bar{D}) = L_3$$
$$qm_{QBCR1}(\bar{D}|\bar{D}) = qm_{QBCR2}(\bar{D}|\bar{D}) = L_2$$

Case 2: $\bar{D} = A \cup B \cup C$.



Using QBCR1, the masses of A, B, C do not change since they are included in $\bar{D} = A \cup B \cup C$.
$qm(D) = L_2$ is transferred to $\bar{D}$.

$$qm_{QBCR1}(A|\bar{D}) = L_1$$
$$qm_{QBCR1}(C|\bar{D}) = L_3$$
$$qm_{QBCR1}(A \cup B \cup C|\bar{D}) = L_2$$

Using QBCR2, $qm(D) = L_2$ is equally split to A and C since they are the only qualitative focal elements from D_1 which means all parts of $A \cup B \cup C$.
Therefore each of them A and C receive $(1/2)L_2 = L_1$.

$$qm_{QBCR2}(A|\bar{D}) = L_1 + (1/2)L_2 = L_1 + L_{2/2} = L_2$$
$$qm_{QBCR1}(C|\bar{D}) = L_3 + (1/2)L_2 = L_3 + L_{2/2} = L_4$$

Conclusions

Two Qualitative Belief Conditioning Rules (QBCRs) have been proposed to revise any qualitative belief masses qm(.) from a conditioning event and hyper-power set decomposition.

Simple examples have been presented to show how they work.

QBCR1 is more prudent than QBCR2 because the revision of the belief is done in a less specific transfer than for QBCR2.

We suggest to use QBCR1 when we are less confident in the source. When we are more confident in the source, QBCR2 is preferred.

Of course, the qualitative conditioning process is less precise than its quantitative counterparts (BCR) because it is based on a rough approximation, as it normally happens when working with linguistic labels.