Enrichment of Qualitative Beliefs for Reasoning under Uncertainty

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***Abstract*—This paper deals with enriched qualitative belief functions for reasoning under uncertainty and for combining information expressed in natural language through linguistic labels. In this work, two possible enrichments (quantitative and/or qualitative) of linguistic labels are considered and operators (addition, multiplication, division, etc) for dealing with them are proposed and explained. We denote them qe-operators, qe standing for "qualitative-enriched" operators. These operators can be seen as a direct extension of the classical qualitative operators (q-operators) proposed recently in the Dezert-Smarandache Theory of plausible and paradoxist reasoning (DSmT). q-operators are also justified in details in this paper. The quantitative enrichment of linguistic label is a numerical supporting degree in $[0,\infty)$, while the qualitative enrichment takes its values in a finite ordered set of linguistic values. Quantitative enrichment is less precise than qualitative enrichment, but it is expected more close with what human experts can easily provide when expressing linguistic labels with supporting degrees. Two simple examples are given to show how the fusion of qualitative-enriched belief assignments can be done, and a simulation application is given to show its advantage in rough navigation map building of mobile robot.**

Keywords: Information fusion, Qualitative beliefs, DSmT, DST.

I. Introduction

Qualitative methods for reasoning under uncertainty have gained more and more attention by Information Fusion community, especially by the researchers and system designers working in the development of modern multi-source systems for defense, robotics and so on. This is because traditional methods based only on quantitative representation and analysis are not able to completely satisfy adequately the need of the development of science and technology integrating at higher fusion levels human beliefs and reports in complex systems. Therefore qualitative knowledge representation becomes more and more important and necessary in next generations of (semi) intelligent automatic and autonomous systems.

For example, Wagner et al. [16] consider that although recent robots have powerful sensors and actuators, their abilities to show intelligent behavior is often limited because of lacking of appropriate spatial representation. Ranganathan et al. [11] describe a navigation system for a mobile robot which must execute motions in a building, the environment is represented by a topological model based on a Generalized Voronoi Graph (GVG) and by a set of visual landmarks. A qualitative self-localization method for indoor environment using a belt of ultrasonic sensors and a camera is proposed. Moratz et al. [6] point out that qualitative spatial reasoning (QSR) abstracts metrical details of the physical world, of which two main directions are topological reasoning about regions and reasoning about orientations of point configurations. So, because concrete problems need a combination of qualitative knowledge of orientation and qualitative knowledge of distance, they present a calculus based on ternary relations where they introduce a qualitative distance measurement based on two of the three points. Duckham et al. [4] explore the development and the use of a qualitative reasoning system based on a description logic for providing the consistency between different geographic data sets. Their research results suggest that further work could significantly increase the level of automation for many geographic data integration tasks.

Recently, Smarandache and Dezert in [14] (Chap. 10) give a detailed introduction of major works for qualitative reasoning under uncertainty. Among important works in this field, one must mention George Polya who first attempted in 1954 to find a formal characterization of qualitative human reasoning [10], then followed by Lotfi Zadeh's works [19], [20]. Later, Wellman [17] proposed a general characterization of qualitative probability to relax precision in representation and reasoning within the probabilistic framework, in order to develop Qualitative Probabilistic Networks (QPN). Wong and Lingras [18] have proposed a method for generating basic belief functions from preference relations between each pair of propositions be specified qualitatively based on Dempster-Shafer Theory (DST) [12]. Parsons [7], [8] then proposed a qualitative Dempster-Shafer Theory, called Qualitative Evidence Theory (QET) by using techniques from qualitative reasoning. This approach seems however to have been abandoned by Parsons in favor of qualitative probabilistic reasoning (QPR) [9]. In 2004, Brewka et al. [2] have proposed a Qualitative Choice Logic (QCL), which is a propositional logic for representing alternative, ranked options for problem solutions. This logic adds to classical propositional logic a new connective called ordered disjunction, that is, if possible$A$, but if A is not possible then at least B. The semantics of qualitative choice logic is based on a preference relation among models. Very recently, Badaloni and Giacomin [1] integrate the ideas of flexibility and uncertainty into Allen's interval-based temporal framework and define a new formalism, called IA^{fuz}, which extends classical Interval Algebra (IA) to express qualitative fuzzy constraints between intervals.

In [14], Smarandache and Dezert introduce a definition of qualitative basic belief assignment (qbba or just qm - standing for qualitative mass), and they propose an extension of quantitative fusion rules developed in DSmT framework for combining directly qbba's without mapping linguistic labels into numbers, and thus computing directly with words. Such extension (mainly the qualitative extension of DSmC, DSmH and PCR5 rules - see [14]) is based on the definition of new operators (addition, multiplication, etc) on linguistic labels which are called q-operators. In this work, we propose to enrich the original definition of qualitative basic belief assignment (qbba) into two possible different ways, quantitatively and qualitatively. These enrichments yields to the definition new linguistic operators for these new types of enriched qbba's. We will denote them qe-operators.

The first qbba enrichment consists in associating a quantitative (numerical) supporting degree in $[0,\infty)$ given a body of evidence/source to each linguistic label. Such enrichment allows to take into account and mix (when available) some numerical extra knowledge about the reliability/trustability of the linguistic label committed to propositions of the frame of discernment. The second possible enrichment is purely qualitative in order to fit more closely with what human experts are expected to provide in reality when enriching their linguistic labels using natural language.

This paper is organized as follows: In section II, we remind briefly the basics of DSmT. In section III we present and justify in details the q-operators, in order to get ready for introducing new enriched qualitative-enriched (qe) operators in sections IV. In section V, we illustrate through very simple examples how these operators can be used for combining enriched qualitative beliefs. Concluding remarks are then given in VII.

II. Basics of DSmT for quantitative beliefs

Let $\Theta = \{\theta_1, \theta_2, \cdots, \theta_n\}$ be a finite set of n elements θ_i, $i = 1, \ldots, n$ assumed to be exhaustive. Θ corresponds to the frame of discernment of the problem under consideration. In general (unless introducing some integrity constraints), we assume that elements of Θ are non exclusive in order to deal with vague/fuzzy and relative concepts [13]. This is the so-called *free-DSm model* which is denoted by $\mathcal{M}^f(\Theta)$. In DSmT framework, there is no need to work on refined frame Θ_{ref} consisting in a (possibly finer) discrete finite set of exclusive and exhaustive hypotheses which is usually referred as *Shafer's model* $\mathcal{M}^0(\Theta)$ in literature, because DSm rules of combination work for any models of the frame, i.e. the free DSm model, Shafer's model or any hybrid model. The hyper-power set (Dedekind's lattice) D^Θ is defined as the set of all compositions built from elements of Θ with $\cup$ and $\cap$ (Θ generates D^Θ under $\cup$ and $\cap$) operators such that

a) $\emptyset, \theta_1, \theta_2, \cdots, \theta_n \in D^\Theta$.
b) If $A, B \in D^\Theta$, then $A \cap B \in D^\Theta$ and $A \cup B \in D^\Theta$.
c) No other elements belong to D^Θ, except those obtained by using rules a) or b).

A (quantitative) basic belief assignment (bba) expressing the belief committed to the elements of D^Θ by a given source/body of evidence S is a mapping function $m(\cdot)$: $D^\Theta \to [0,1]$ such that:

$$m(\emptyset) = 0 \qquad \text{and} \qquad \sum_{A \in D^\Theta} m(A) = 1 \tag{1}$$

Elements $A \in D^\Theta$ having $m(A) > 0$ are called *focal elements* of the bba $m(.)$. The general belief function and plausibility functions are defined respectively in almost the same manner as within the DST [12], i.e.

$$Bel(A) = \sum_{B \in D^\Theta, B \subseteq A} m(B) \tag{2}$$

$$Pl(A) = \sum_{B \in D^\Theta, B \cap A \neq \emptyset} m(B) \tag{3}$$

The main concern in information fusion is the combination of sources of evidence and the efficient management of conflicting and uncertain information. DSmT offers several fusion rules, denoted by the generic symbol $\oplus$, for combining basic belief assignments. The simplest one, well adapted when working with the free-DSm[1] model $\mathcal{M}^f(\Theta)$ and called DSmC (standing for *DSm Classical rule*) is nothing but the conjunctive fusion operator of bba's defined over the hyper-power set D^Θ. Mathematically, DSmC for the fusion of $k \geq 2$ sources of evidence is defined by $m_{\mathcal{M}^f(\Theta)}(\emptyset) = 0$ and $\forall A \neq \emptyset \in D^\Theta$,

$$\begin{aligned} m_{\mathcal{M}^f(\Theta)}(A) &\triangleq [m_1 \oplus \cdots \oplus m_k](A) \\ m_{\mathcal{M}^f(\Theta)}(A) &= \sum_{\substack{X_1, \cdots, X_k \in D^\Theta \\ X_1 \cap \cdots \cap X_k = A}} \prod_{s=1}^{k} m_s(X_s) \end{aligned} \tag{4}$$

When working with hybrid models and/or Shafer's model $\mathcal{M}^0(\Theta)$, other rules for combination must be used for taking into account integrity constraints of the model (i.e. some exclusivity constraints and even sometimes no-existing constraints in dynamical problems of fusion where the model and the frame can change with time). For managing efficiently

[1] We call it *free* because no integrity constraint is introduced in such model.

the conflicts between sources of evidence, DSmT proposes mainly two alternatives to the classical Dempster's rule of combination [12] for working efficiently with (possibly) high conflicting sources. The first rule proposed in [13] was the DSm hybrid rule (DSmH) of combination which offers a prudent/pessimistic way of redistributing partial conflicting mass. The basic idea of DSmH is to redistribute the partial conflicting mass to corresponding partial ignorance. For example: let's consider only two sources with two bba's $m_1(.)$ and $m_2(.)$, if $A \cap B = \emptyset$ is an integrity constraint of the model of Θ and if $m_1(A)m_2(B) > 0$, then $m_1(A)m_2(B)$ will be transferred to $A \cup B$ through DSmH. The general formula for DSmH is quite complicated and can be found in [13] and is not reported here due to space limitation. DSmH is actually a natural extension of Dubois & Prade's rule of combination [3] which allows also to work with dynamical changes of the frame and its model. A much more precise fusion rule, called Proportional Conflict Redistribution rule no. 5 (PCR5) has been developed recently in [14] for transferring more efficiently all partial conflicting masses. Basically, the idea of PCR5 is to transfer the conflicting mass only to the elements involved in the conflict and proportionally to their individual masses. For example: let's assume as before only two sources with bba's $m_1(.)$ and $m_2(.)$, $A \cap B = \emptyset$ for the model of Θ and $m_1(A) = 0.6$ and $m_2(B) = 0.3$. Then with PCR5, the partial conflicting mass $m_1(A)m_2(B) = 0.6 \cdot 0.3 = 0.18$ is redistributed to A and B only with the following proportions respectively: $x_A = 0.12$ and $x_B = 0.06$ because the proportionalization requires

$$\frac{x_A}{m_1(A)} = \frac{x_B}{m_2(B)} = \frac{m_1(A)m_2(B)}{m_1(A) + m_2(B)} = \frac{0.18}{0.9} = 0.2$$

General PCR5 fusion formula for the combination of $k \geq 2$ sources of evidence can be found in [14].

III. Extension of DSmT for qualitative beliefs

In order to compute with words (i.e. linguistic labels) and qualitative belief assignments instead of quantitative belief assignments[2] over G^Θ, Smarandache and Dezert have defined in [14] a *qualitative basic belief assignment* $qm(.)$ as a mapping function from G^Θ into a set of linguistic labels $L = \{L_0, \tilde{L}, L_{n+1}\}$ where $\tilde{L} = \{L_1, \cdots, L_n\}$ is a finite set of linguistic labels and where $n \geq 2$ is an integer. For example, L_1 can take the linguistic value "poor", L_2 the linguistic value "good", etc. $\tilde{L}$ is endowed with a total order relationship $\prec$, so that $L_1 \prec L_2 \prec \cdots \prec L_n$. To work on a true closed linguistic set L under linguistic addition and multiplication operators, Smarandache and Dezert extended naturally $\tilde{L}$ with two extreme values $L_0 = L_{\min}$ and $L_{n+1} = L_{\max}$, where L_0 corresponds to the minimal qualitative value and L_{n+1} corresponds to the maximal qualitative value, in such a way that $L_0 \prec L_1 \prec L_2 \prec \cdots \prec L_n \prec L_{n+1}$, where $\prec$ means inferior to, or less (in quality) than, or smaller than, etc. Labels L_0, L_1, L_2, ..., L_n, L_{n+1} are said *linguistically equidistant* if: $L_{i+1} - L_i = L_i - L_{i-1}$ for all $i = 1, 2, \ldots, n$ where the definition of subtraction of labels is given in the sequel by (11). In the sequel $L_i \in L$ are assumed linguistically equidistant[3] labels such that we can make an isomorphism between $L = \{L_0, L_1, L_2, \ldots, L_n, L_{n+1}\}$ and $\{0, 1/(n+1), 2/(n+1), \ldots, n/(n+1), 1\}$, defined as $L_i = i/(n+1)$ for all $i = 0, 1, 2, \ldots, n, n+1$. Using this isomorphism, and making an analogy to the classical operations of real numbers, we are able to define the following qualitative operators (or q-operators for short):

- q-addition of linguistic labels

$$L_i + L_j = \frac{i}{n+1} + \frac{j}{n+1} = \frac{i+j}{n+1} = L_{i+j} \quad (5)$$

  but of course we set the restriction that $i + j < n + 1$; in the case when $i + j \geq n + 1$ we restrict $L_{i+j} = L_{n+1}$. So this is the justification of the qualitative addition we have defined.
- q-multiplication of linguistic labels
  a) Since $L_i \times L_j = \frac{i}{n+1} \cdot \frac{j}{n+1} = \frac{(i \cdot j)/(n+1)}{n+1}$, the best approximation would be $L_{[(i \cdot j)/(n+1)]}$, where $[x]$ means the closest integer to x, i.e.

$$L_i \times L_j = L_{[(i \cdot j)/(n+1)]} \quad (6)$$

  For example, if we have L_0, L_1, L_2, L_3, L_4, L_5, corresponding to respectively 0, 0.2, 0.4, 0.6, 0.8, 1, then $L_2 \cdot L_3 = L_{[(2 \cdot 3)/5]} = L_{[6/5]} = L_{[1.2]} = L_1$; using numbers: $0.4 \cdot 0.6 = 0.24 \approx 0.2 = L_1$; also $L_3 \cdot L_3 = L_{[(3 \cdot 3)/5]} = L_{[9/5]} = L_{[1.8]} = L_2$; using numbers $0.6 \cdot 0.6 = 0.36 \approx 0.4 = L_2$.
  b) A simpler approximation of the multiplication, but less accurate (as proposed in [14]) is thus

$$L_i \times L_j = L_{\min\{i,j\}} \quad (7)$$

- Scalar multiplication of a linguistic label
  Let a be a real number. We define the multiplication of a linguistic label by a scalar as follows:

$$a \cdot L_i = \frac{a \cdot i}{n+1} \approx \begin{cases} L_{[a \cdot i]} & \text{if } [a \cdot i] \geq 0, \\ L_{-[a \cdot i]} & \text{otherwise.} \end{cases} \quad (8)$$

- Division of linguistic labels
  a) Division as an internal operator: $/ : L \times L \to L$. Let $j \neq 0$, then

$$L_i / L_j = \begin{cases} L_{[(i/j) \cdot (n+1)]} & \text{if} [(i/j) \cdot (n+1)] < n+1, \\ L_{n+1} & \text{otherwise.} \end{cases} \quad (9)$$

  The first equality in (9) is well justified because when $[(i/j) \cdot (n+1)] < n + 1$, one has

$$L_i / L_j = \frac{i/(n+1)}{j/(n+1)} = \frac{(i/j) \cdot (n+1)}{n+1} = L_{[(i/j) \cdot (n+1)]}$$

[2] G^Θ is the generic notation for the hyper-power set taking into account all integrity constraints (if any) of the model. For example, if one considers a free-DSm model for Θ then $G^\Theta = D^\Theta$. If Shafer's model is used instead then $G^\Theta = 2^\Theta$ (the classical power-set).

[3] If the labels are not equidistant, the q-operators still work, but they are less accurate.

For example, if we have L_0, L_1, L_2, L_3, L_4, L_5, corresponding to respectively 0, 0.2, 0.4, 0.6, 0.8, 1, then: $L_1/L_3 = L_{[(1/3)\cdot 5]} = L_{[5/3]} = L_{[1.66]} \approx L_2$. $L_4/L_2 = L_{[(4/2)\cdot 5]} = L_{[2\cdot 5]} = L_{\max} = L_5$ since $10 > 5$.

b) Division as an external operator: $\oslash : L \times L \to \mathbb{R}^+$. Let $j \neq 0$. Since $L_i \oslash L_j = (i/(n+1))/(j/(n+1)) = i/j$, we simply define

$$L_i \oslash L_j = i/j \tag{10}$$

Justification of b): when we divide say L_4/L_1 in the above example, we get $0.8/0.2 = 4$, but no label is corresponding to number 4 which is not even in the interval $[0, 1]$, hence in the division as an internal operator we need to get as response a label, so in our example we approximate it to $L_{\max} = L_5$, which is a very rough approximation! So, depending on the fusion combination rules, it might better to consider the qualitative division as an external operator, which gives us the exact result.

- q-subtraction of linguistic labels: $- : L \times L \to \{L, -L\}$,

$$L_i - L_j = \begin{cases} L_{i-j} & \text{if} \quad i \geq j, \\ -L_{j-i} & \text{if} \quad i < j. \end{cases} \tag{11}$$

where $-L = \{-L_1, -L_2, \ldots, -L_n, -L_{n+1}\}$. The q-substraction above is well justified since when $i \geq j$, one has $L_i - L_j = \frac{i}{n+1} - \frac{j}{n+1} = \frac{i-j}{n+1}$.

The above qualitative operators are logical, justified due to the isomorphism between the set of linguistic equidistant labels and a set of equidistant numbers in the interval $[0, 1]$. These qualitative operators are built exactly on the track of their corresponding numerical operators, so they are more mathematical than the ad-hoc definition of qualitative operators proposed so far in the literature. They are similar to the PCR5 combination numerical rule with respect to other fusion combination numerical rules based on the conjunctive rule. But moving to the enriched label qualitative operators the accuracy decreases.

There is no way to define a normalized $qm(.)$, but a qualitative quasi-normalization [14], [15] is nevertheless possible when considering equidistant linguistic labels because in such case, $qm(X_i) = L_i$, is equivalent to a quantitative mass $m(X_i) = i/(n + 1)$ which is normalized if

$$\sum_{X \in D^\Theta} m(X) = \sum_k i_k/(n + 1) = 1$$

but this one is equivalent to

$$\sum_{X \in D^\Theta} qm(X) = \sum_k L_{i_k} = L_{n+1}$$

In this case, we have a *qualitative normalization*, similar to the (classical) numerical normalization. But, if the previous labels L_0, L_1, L_2, …, L_n, L_{n+1} from the set L are not equidistant, so the interval $[0, 1]$ cannot be split into equal parts according to the distribution of the labels, then it makes sense to consider a *qualitative quasi-normalization*, i.e. an approximation of the (classical) numerical normalization for the qualitative masses in the same way:

$$\sum_{X \in D^\Theta} qm(X) = L_{n+1}$$

In general, if we don't know if the labels are equidistant or not, we say that a qualitative mass is quasi-normalized when the above summation holds. In the sequel, for simplicity, one assumes to work with quasi-normalized qualitative basic belief assignments.

From these very simple qualitative operators, it is thus possible to extend directly the DSmH fusion rule for combining qualitative basic belief assignments by replacing classical addition and multiplication operators on numbers with those for linguistic labels in DSmH formula. In a similar way, it is also possible to extend PCR5 formula as shown with detailed examples in [14] and in section V of this paper. In the next section, we propose new qualitative-enriched (qe) operators for dealing with enriched linguistic labels which mix the linguistic value with either quantitative/numerical supporting degree or qualitative supporting degree as well. The direct qualitative discounting (or reinforcement) is motivated by the fact that in general human experts provide more easily qualitative values than quantitative values when analyzing complex situations.

In this paper, both *quantitative enrichments* and *qualitative enrichments* of linguistic labels are considered and unified through same general qe-operators. The quantitative enrichment is based directly on the percentage of discounting (or reinforcement) of any linguistic label. This is what we call a Type 1 of enriched labels. The qualitative enrichment comes from the idea of direct qualitative discounting (or reinforcement) and constitutes the Type 2 of enriched labels.

IV. qe-OPERATORS

We propose to improve the previous q-operators for dealing now with enriched qualitative beliefs provided from human experts. We call these operators the qe-operators. The basic idea is to use "enriched"-linguistic labels denoted $L_i(\epsilon_i)$, where ϵ_i can be either a numerical supporting degree in $[0, \infty)$ or a qualitative supporting degree taken its value in a given (ordered) set X of linguistic labels. $L_i(\epsilon_i)$ is called the qualitative enrichment[4] of L_i. When $\epsilon_i \in [0, \infty)$, $L_i(\epsilon_i)$ is called an enriched label of Type 1, whereas when $\epsilon_i \in X$, $L_i(\epsilon_i)$ is called an enriched label of Type 2. The (quantitative or qualitative) quantity ϵ_i characterizes the weight of reinforcing or discounting expressed by the source when using label L_i for committing its qualitative belief to a given proposition $A \in G^\Theta$. It can be interpreted as a second order type of linguistic label which includes both

[4]Linguistic labels without enrichment (discounting or reinforcement) as those involved in q-operators are said *classical* or being of Type 0.

the linguistic value itself but also the associated degree of confidence expressed by the source. The values of ϵ_i express the expert's attitude (reinforcement, neutral, or discounting) to a certain proposition when using a given linguistic label for expressing its qualitative belief assignment.

For example with enriched labels of Type 1, if the label $L_1 \triangleq L_1(1)$ represents the linguistic variable *Good*, then $L_1(\epsilon_1)$ represents either the reinforced or discounted L_1 value which depends on the value taken by ϵ_1. In this example, ϵ_1 represents the (numerical) supporting degree of the linguistic value $L_1 =$ Good. If $\epsilon_1 = 1.2$, then we say that the linguistic value $L_1 =$ Good has been reinforced by 20% with respect to its nominal/neutral supporting degree. If $\epsilon_1 = 0.4$, then it means that the linguistic value L_1 is discounted 60% by the source.

With enriched labels of Type 2, if one chooses by example $X = \{NB, NM, NS, O, PS, PM, PB\}$, where elements of X have the following meaning: $NB \triangleq$ "negative big", $NM \triangleq$ "negative medium", $NS \triangleq$ "negative small", $O \triangleq$ "neutral" (i.e. no discounting, neither reinforcement), $PS \triangleq$ "positive small", $PM \triangleq$ "positive medium" and $PB \triangleq$ "positive big". Then, if the label $L_1 \triangleq L_1(O)$ represents the linguistic variable *Good*, then $L_1(\epsilon_1)$, $\epsilon_1 \in X$, represents either the qualitative reinforced or discounted L_1 value which depends on the value taken by ϵ_1 in X. $\epsilon_1 = O$ means a neutral qualitative supporting degree corresponding to $\epsilon_1 = 1$ for enriched label of Type 1. ϵ_1 represents the qualitative supporting degree of the linguistic value $L_1 =$ Good. If $\epsilon_1 = PS$, then we say that the linguistic value $L_1 =$ Good has been reinforced a little bit positively with respect to its nominal/neutral supporting degree. If $\epsilon_1 = NB$, then it means that the linguistic value L_1 is discounted slightly and negatively by the source.

We denote by $\tilde{L}(\epsilon)$ any given set of (classical/pure) linguistic labels $\tilde{L} = \{L_1, L_2, \ldots, L_n\}$ endowed with the supporting degree property (i.e. discounting, neutral and/or reinforcement). In other words,

$$\tilde{L}(\epsilon) = \{L_1(\epsilon_1), L_2(\epsilon_2), \ldots, L_n(\epsilon_n)\}$$

represents a given set of enriched linguistic labels[5]. We assume the same order relationship $\prec$ on $\tilde{L}(\epsilon)$ as the one defined on $\tilde{L}$. Moreover we extend $\tilde{L}(\epsilon)$ with two extreme (minimal and maximal) enriched qualitative values $L_0(\epsilon)$ and $L_{n+1}(\epsilon)$ in order to get closed under qe-operators on $L(\epsilon) \triangleq \{L_0(\epsilon), \tilde{L}(\epsilon), L_{n+1}(\epsilon)\}$. For working with enriched labels (and then with qualitative enriched basic belief assignments), it is necessary to extend the previous q-operators in a consistent way. This is the purpose of our new qe-operators.

[5]In this formal notation, the quantities $\epsilon_1, \ldots, \epsilon_n$ represent any values in $[0, \infty)$ if the enrichment is quantitative (Type 1), or values in X is we consider an qualitative enrichment (Type 2).

An enriched label $L_i(\epsilon_i)$ means that the source has discounted (or reinforced) the label L_i by a quantitative or qualitative factor ϵ_i. Similarly for $L_j(\epsilon_j)$. So we use the q-operators for L_i, L_j labels, but for confidences we propose three possible versions: If the confidence in L_i is ϵ_i and the confidence in L_j is ϵ_j, then the confidence in combining L_i with L_j can be:

a) either the average, i.e. $(\epsilon_i + \epsilon_j)/2$;
b) or $\min\{\epsilon_i, \epsilon_j\}$;
c) or we may consider a confidence interval as in statistics, so we get $[\epsilon_{\min}, \epsilon_{\max}]$, where $\epsilon_{\min} \triangleq \min\{\epsilon_i, \epsilon_j\}$ and $\epsilon_{\max} \triangleq \max\{\epsilon_i, \epsilon_j\}$; if $\epsilon_i = \epsilon_j$ then the confidence interval is reduced to a single point, ϵ_i.

In the sequel, we denote by "c" any of the above resulting confidence of combined enriched labels. All these versions coincide when $\epsilon_i = \epsilon_j = 1$ (for Type 1) or when $\epsilon_i = \epsilon_j = O$ (for Type 2), i.e. where there is no reinforcement or no discounting of the linguistic label. However the confidence degree average operator (case a)) is not associative, so in many cases it's inconvenient to use it. The best among these three, more prudent and easier to use, is the $\min$ operator. The confidence interval operator provides both a lower and a upper confidence level, so in an optimistic way, we may take at the end the midpoint of this confidence interval as a confidence level.

The new extended operators allowing working with enriched labels of Type 1 or Type 2 are then defined by:

- qe-addition of enriched labels

$$L_i(\epsilon_i) + L_j(\epsilon_j) = \begin{cases} L_{n+1}(c) & \text{if} \quad i+j \geq n+1, \\ L_{i+j}(c) & \text{otherwise.} \end{cases} \tag{12}$$

- qe-multiplication of linguistic labels
  a) As direct extension of (6), the multiplication of enriched labels is defined by

$$L_i(\epsilon_i) \times L_j(\epsilon_j) = L_{[(i \cdot j)/(n+1)]}(c) \tag{13}$$

  b) as another multiplication of labels, easier, but less exact:

$$L_i(\epsilon_i) \times L_j(\epsilon_j) = L_{\min\{i,j\}}(c) \tag{14}$$

- Scalar multiplication of a enriched label Let a be a real number. We define the multiplication of an enriched linguistic label by a scalar as follows:

$$a \cdot L_i(\epsilon_i) \approx \begin{cases} L_{[a \cdot i]}(\epsilon_i) & \text{if } [a \cdot i] \geq 0, \\ L_{-[a \cdot i]}(\epsilon_i) & \text{otherwise.} \end{cases} \tag{15}$$

- qe-division of enriched labels

a) Division as an internal operator: Let $j \neq 0$, then

$$\frac{L_i(\epsilon_i)}{L_j(\epsilon_j)} = \begin{cases} L_{n+1}(c) & \text{if } [(i/j)\cdot(n+1)] \geq n+1, \\ L_{[(i/j)\cdot(n+1)]}(c) & \text{otherwise.} \end{cases} \tag{16}$$

b) Division as an external operator: Let $j \neq 0$, then we can also consider the division of enriched labels as external operator too as follows:

$$L_i(\epsilon_i) \oslash L_j(\epsilon_j) = (i/j)_{\text{supp}(c)} \tag{17}$$

The notation $(i/j)_{\text{supp}(c)}$ means that the numerical value (i/j) is supported with the degree c.

- qe-subtraction of enriched labels

$$L_i(\epsilon_i) - L_j(\epsilon_j) = \begin{cases} L_{i-j}(c) & \text{if } i \geq j, \\ -L_{j-i}(c) & \text{if } i < j. \end{cases} \tag{18}$$

These qe-operators with numerical confidence degrees are consistent with the classical qualitative operators when $e_i = e_j = 1$ since $c = 1$ and $L_i(1) = L_i$ for all i, and the qe-operators with qualitative confidence degrees are also consistent with the classical qualitative operators when $e_i = e_j = O$ (this is letter "O", not zero, hence the neutral qualitative confidence degree) since $c = O$ (neutral).

V. Examples of qPCR5 fusion of qualitative belief assignments

A. Qualitative masses using quantitative enriched labels

Let's consider a simple frame $\Theta = \{A, B\}$ with Shafer's model (i.e. $A \cap B = \emptyset$), two qualitative belief assignments $qm_1(\cdot)$ and $qm_2(\cdot)$, the set of ordered linguistic labels $L = \{L_0, L_1, L_2, L_3, L_4, L_5, L_6\}$, $n = 5$, enriched with quantitative support degree (i.e. enriched labels of Type 1). For this example the (prudent) min operator for combining confidences proposed in section IV (case b)) is used, but other methods a) and c) can also be applied.We consider the following qbba summarized in the Table I: Note that $qm_1(\cdot)$ and $qm_2(\cdot)$ are

	A	B	$A \cup B$	$A \cap B$
$qm_1(\cdot)$	$L_1(0.3)$	$L_2(1.1)$	$L_3(0.8)$	
$qm_2(\cdot)$	$L_4(0.6)$	$L_2(0.7)$	$L_0(1)$	
$qm_{12}(\cdot)$	$L_3(0.3)$	$L_2(0.7)$	$L_0(0.8)$	$L_1(0.3)$

Table I
$qm_1(\cdot)$, $qm_2(\cdot)$ and $qm_{12}(\cdot)$ with quantitative enriched labels

quasi-normalized since $L_1 + L_2 + L_3 = L_4 + L_2 + L_0 = L_6 = L_{\max}$. The last raw of Table I, corresponds to the result $qm_{12}(\cdot)$ obtained when applying the qualitative conjunction rule. The values for $qm_{12}(\cdot)$ are obtained as follows:

$$\begin{aligned} qm_{12}(A) &= qm_1(A)qm_2(A) + qm_1(A)qm_2(A \cup B) \\ &\quad + qm_2(A)qm_1(A \cup B) \\ &= L_1(0.3)L_4(0.6) + L_1(0.3)L_0(1) \\ &\quad + L_4(0.6)L_3(0.8) \\ &= L_{[(1\cdot4)/6]}(\min\{0.3, 0.6\}) + L_{[(0\cdot1)/6]}(\min\{0.3, 1\}) \\ &\quad + L_{[(4\cdot3)/6]}(\min\{0.6, 0.8\}) \\ &= L_1(0.3) + L_0(0.3) + L_2(0.6) \\ &= L_{1+0+2}(\min\{0.3, 0.3, 0.6\}) = L_3(0.3) \end{aligned}$$

$$\begin{aligned} qm_{12}(B) &= qm_1(B)qm_2(B) + qm_1(B)qm_2(A \cup B) \\ &\quad + qm_2(B)qm_1(A \cup B) \\ &= L_2(1.1)L_2(0.7) + L_2(1.1)L_0(1) \\ &\quad + L_2(0.7)L_3(0.8) \\ &= L_{[(2\cdot2)/6]}(\min\{1.1, 0.7\}) + L_{[(2\cdot0)/6]}(\min\{1.1, 1\}) \\ &\quad + L_{[(2\cdot3)/6]}(\min\{0.7, 0.8\}) \\ &= L_1(0.7) + L_0(1) + L_1(0.7) \\ &= L_{1+0+1}(\min\{0.7, 1, 0.7\}) = L_2(0.7) \end{aligned}$$

$$\begin{aligned} qm_{12}(A \cup B) &= qm_1(A \cup B)qm_2(A \cup B) \\ &= L_3(0.8)L_0(1) \\ &= L_{[(3\cdot0)/6]}(\min\{0.8, 1\}) = L_0(0.8) \end{aligned}$$

and the conflicting qualitative mass by

$$\begin{aligned} qm_{12}(\emptyset) &= qm_{12}(A \cap B) \\ &= qm_1(A)qm_2(B) + qm_2(A)qm_1(B) \\ &= L_1(0.3)L_2(0.7) + L_4(0.6)L_2(1.1) \\ &= L_{[(1\cdot2)/6]}(\min\{0.3, 0.7\}) \\ &\quad + L_{[(4\cdot2)/6]}(\min\{0.6, 1.1\}) \\ &= L_0(0.3) + L_1(0.6) \\ &= L_{0+1}(\min\{0.3, 0.6\}) = L_1(0.3) \end{aligned}$$

The resulted qualitative mass, $qm_{12}(\cdot)$, is also quasi-normalized since $L_3 + L_2 + L_0 + L_1 = L_6 = L_{\max}$.

According to qPCR5 (see [14]), we need to redistribute the conflicting mass $L_1(0.3)$ to the elements involved in the conflict, A and B, thus:

a) $qm_1(A)qm_2(B) = L_1(0.3)L_2(0.7) = L_0(0.3)$ is redistributed back to A and B proportionally with respect to their corresponding qualitative masses put in this partial conflict, i.e. proportionally with respect to $L_1(0.3)$ and $L_2(0.7)$. But, since $L_0(0.3)$ is the null qualitative label (equivalent to zero for numerical masses), both A and B get L_0 with the minimum confidence, i.e. $L_0(0.3)$.
b) $qm_2(A)qm_1(B) = L_4(0.6)L_2(1.1) = L_1(0.6)$ is redistributed back to A and B proportionally with respect to their corresponding qualitative masses put in this partial

conflict, i.e. proportionally with respect to $L_4(0.6)$ and $L_2(1.1)$, i.e.

$$\frac{x_A}{L_4(0.6)} = \frac{y_B}{L_2(1.1)} = \frac{L_1(0.6)}{L_4(0.6)+L_2(1.1)}$$
$$= \frac{L_1(0.6)}{L_6(0.6)} = L_{[(1/6)\cdot 6]}(\min\{0.6,0.6\})$$
$$= L_1(0.6)$$

whence

$$x_A = L_4(0.6)\cdot L_1(0.6)$$
$$= L_{[(4\cdot 1)/6]}(\min\{0.6,0.6\}) = L_1(0.6)$$

$$y_B = L_2(1.1)\cdot L_1(0.6)$$
$$= L_{[(2\cdot 1)/6]}(\min\{1.1,0.6\}) = L_0(0.6)$$

Thus, the result of the qPCR5 fusion of $qm_1(\cdot)$ with $qm_2(\cdot)$ is given by

$$qm_{PCR5}(A) = L_3(0.3)+L_0(0.3)+x_A$$
$$= L_3(0.3)+L_0(0.3)+L_1(0.6)$$
$$= L_{3+0+1}(\min\{0.3,0.3,0.6\}) = L_4(0.3)$$

$$qm_{PCR5}(B) = L_2(0.7)+L_0(0.3)+y_B$$
$$= L_2(0.7)+L_0(0.3)+L_0(0.6)$$
$$= L_{2+0+0}(\min\{0.7,0.3,0.6\}) = L_2(0.3)$$
$$qm_{PCR5}(A\cup B) = L_0(0.8)$$
$$qm_{PCR5}(A\cap B) = L_0 = L_0(1)$$

This qualitative PCR5-combined resulting mass is also quasi-normalized[6] since $L_4+L_2+L_0+L_0 = L_6 = L_{\max}$.

B. Qualitative masses with qualitative-enriched labels

Using qualitative supporting degrees (i.e. enriched labels of Type 2) taking their values in the linguistic set $X = \{NB, NM, NS, O, PS, PM, PB\}$, with $NB \prec NM \prec NS \prec O \prec PS \prec PM \prec PB$ we get similar result for this example. So, let's consider $\Theta = \{A, B\}$ with Shafer's model and $qm_1(\cdot)$ and $qm_2(\cdot)$ chosen as in Table II The

	A	B	$A\cup B$
$qm_1(\cdot)$	$L_1(NB)$	$L_2(PS)$	$L_3(NS)$
$qm_2(\cdot)$	$L_4(NM)$	$L_2(NS)$	$L_0(O)$

Table II
$qm_1(\cdot), qm_2(\cdot)$ WITH QUALITATIVE ENRICHED LABELS

qualitative conjunctive and PCR5 fusion rules are obtained with derivations identical to the previous ones, since $NB \prec NM \prec NS \prec O \prec PS \prec PM \prec PB$ and we associated $NB = 0.3$ or less, $NM = [0.5, 0.6]$, $NS = [0.7, 0.8]$, $O = 1$ and $PS = 1.1$. The minimum operator on X(qualitative degrees) works similarly as on $\mathbb{R}^+$(quantitative degrees). Thus, finally one gets results according to Table III.

[6]The confidence level/degree in the labels does not matter in the definition of quasi-normalization.

	A	B	$A\cup B$	$A\cap B$
$qm_{12}(\cdot)$	$L_3(NB)$	$L_2(NS)$	$L_0(NS)$	$L_1(NB)$
$qm_{PCR5}(\cdot)$	$L_4(NB)$	$L_2(NB)$	$L_0(NS)$	$L_0(O)$

Table III
RESULT OBTAINED WITH QUALITATIVE CONJUNCTIVE AND PCR5 FUSION RULES

VI. APPLICATION IN MOBILE ROBOT MAP BUILDING

Map building under unknown environments has been one of the principal issues in the field of intelligent mobile robot. In fact, a variety of map building methods based on more or less quantitative and precise measurements of self locations have been proposed for this purpose [5]. However, we humans are able to build a rough navigation map of the environment in our minds, even if we are given only qualitative and fragmented spatial information such as “Object A is near to B”, “A is seen on the right of B from C” and so on. So we can describe spatial information from the point of view of qualitative-enriched label.

To simplify the problem, here we suppose that in the system there are two focal elements included in the frame of discernment, that is, $\Theta = \{\theta_1, \theta_2\}$, then its hyper power-set is $D^\Theta = \{\theta_1, \theta_2, \theta_1\cap\theta_2, \theta_1\cup\theta_2\}$, hereinto, θ_1 represents the qualitative attribute “near” and θ_2 represents “far”, $\theta_1\cap\theta_2$ represents “near and far”, $\theta_1\cup\theta_2$ represents the ignorance of current status due to not enough information. Seven qualitative enriched ordered labels ordered are, that is, $L = \{L_0(\epsilon), L_1(\epsilon), \cdots, L_6(\epsilon)\}$ is used to express the qualitative belief degree. Let's consider ten sources of evidence with qualitative enriched labels obtained from the perceptive information according to Table IV. Here we adopted qualitative enriched DSmC combinational rule and PCR5 rule described in last section to combine all the ten sources. The fusion process is simulated through Matlab program. The final fusion result is shown in the last row of Table IV and on Fig.1. Here is our analysis of simulation results:

1) On Fig.1, X-axis represents 4 elements included in hyper power-set, where $1 \mapsto \Theta_1$, $2 \mapsto \Theta_2$, $3 \mapsto \theta_1\cap\theta_2$, $4 \mapsto \theta_1\cup\theta_2$. Y-axis represents the classical label value such as $0\mapsto L_0, 1\mapsto L_1, \cdots, 5\mapsto L_5, 6\mapsto L_6$. Z axis represents the enriched value $\epsilon\times 100$ after fusion[7].
2) The result shows that the qualitative enriched combination rule has very good convergence characters. Because $qm(\theta_1)$ is $L_6(0.4)$, which represents the biggest label, whereas, the other belief assignments all are $L_0(\epsilon)$, then we can know clearly the obstacle is near to robot.
3) The qualitative enriched combination rule has very good quasi-normalization character, this is because we add the quasi-normalization step, for example, $S1 : L_5, L_1, L_0, L_0; S2 : L_2, L_2, L_2, L_0$ in the above system[8],

[7]Here ϵ is magnified 100 multiples, so that it is shown more clearly on Fig.1

[8]to simplify the problem, ϵ is ignored temporarily

if no quasi-normalization step, then the combinational result between two sources is $S_c : L_6, L_1, L_1, L_0$. if adopting quasi-normalization step($while(\sum i > 6), \{i = [\frac{i*6}{\sum i}], \sum i\}$), i.e. $L_{[\frac{5\times6}{8}]}, L_{[\frac{1\times6}{8}]}, L_{[\frac{1\times6}{8}]}, L_0$, then the fusion result is $S_c : L_4, L_1, L_1, L_0$.

4) The qualitative enriched labels have more information than classical qualitative labels, which can reflect the qualitative information precisely.

	A	B	$A \cap B$	$A \cup B$
$qm_1(\cdot)$	$L_1(0.6)$	$L_2(1.4)$	$L_0(0.54)$	$L_3(0.6)$
$qm_2(\cdot)$	$L_4(1.3)$	$L_2(0.7)$	$L_0(1.4)$	$L_0(0.9)$
$qm_3(\cdot)$	$L_3(0.4)$	$L_1(1.3)$	$L_0(0.9)$	$L_2(0.6)$
$qm_4(\cdot)$	$L_2(0.7)$	$L_2(1.6)$	$L_2(0.9)$	$L_0(2.0)$
$qm_5(\cdot)$	$L_1(0.4)$	$L_0(1.8)$	$L_4(1.8)$	$L_1(1.5)$
$qm_6(\cdot)$	$L_2(0.5)$	$L_1(1.2)$	$L_2(1.2)$	$L_1(0.7)$
$qm_7(\cdot)$	$L_1(1.4)$	$L_3(1.3)$	$L_1(1.5)$	$L_1(0.8)$
$qm_8(\cdot)$	$L_5(1.3)$	$L_1(1.6)$	$L_0(1.2)$	$L_0(0.7)$
$qm_9(\cdot)$	$L_4(1.4)$	$L_1(1.5)$	$L_1(0.9)$	$L_0(0.8)$
$qm_{10}(\cdot)$	$L_3(1.8)$	$L_1(1.5)$	$L_1(0.9)$	$L_1(1.4)$
$qm(\cdot)$	$L_6(0.4)$	$L_0(0.4)$	$L_0(1.0)$	$L_0(0.4)$

Table IV
10 EVIDENCE SOURCES AND THE COMBINATIONAL RESULT WITH QUANTITATIVE ENRICHED LABELS

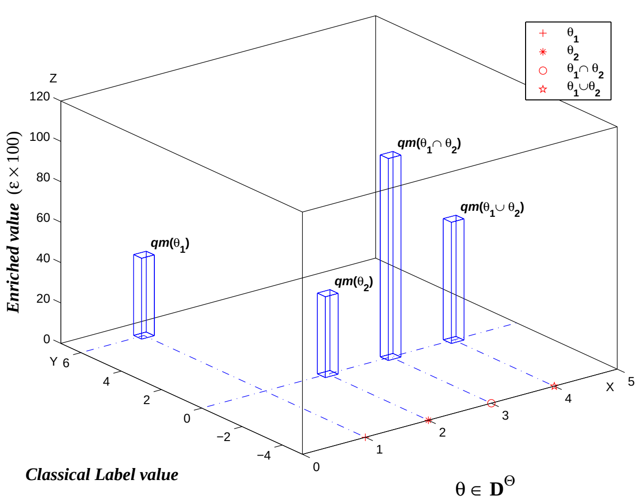

Figure 1. The final fusion result of qualitative evidence sources

VII. Conclusion

With the recent development of qualitative methods for reasoning under uncertainty developed in Artificial Intelligence, more and more experts and scholars have great interest on qualitative information fusion, especially those working in the development of modern multi-source systems for defense, robot navigation, mapping, localization and path planning and so on. In this paper, we have proposed two possible enrichments (quantitative and/or qualitative) of linguistic labels and a simple and direct extension of the q-operators developed in the DSmT framework. We have also shown how to fuse qualitative-enriched belief assignments which can be expressed in natural language by human experts. Two illustrating examples have been presented in details to explain how our qualitative-enriched operators (qe-operators) and qualitative PCR5 rule of combination work. A simulation application is given to show its advantage in mobile robot navigation map building. Some researches in real robotics of the application of qe-operators (with quantitative or qualitative supporting degrees) are under progress and will be presented in a forthcoming publication.

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