A Real Z-box Experiment for Testing Zadeh's Example

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***Abstract*—In this paper, we propose a real experiment for building and realizing the physical combination of basic belief assignments associated with two independent, informative, and equireliable sources of information, according to the famous Zadeh's example. This experiment is based on a particular electronic circuit box, called Z-box, enabling to observe and to check the fusion result experimentally. Our experimental results clearly invalidate the fusion result obtained by Dempster-Shafer's rule of combination and show that it is physically possible to consider in a natural fusion process two independent and equi-reliable sources of evidences at same time, even if they appear as highly conflicting in Shafer's sense.**

Keywords: Belief function; Zadeh's example; Z-box experiment; Information fusion; Dempster's rule.

I. Introduction

Dempster-Shafer Theory (DST), introduced by Shafer in 1976 [1] offers an elegant theoretical framework for modeling epistemic uncertainty and for combining distinct bodies of evidence collected from different sources. In DST, the combination (fusion) of several distinct sources of evidences is done with Dempster-Shafer (DS)[1] rule of combination, which corresponds to the normalized conjunctive consensus operator [1], assuming that the sources are not in total conflict[2]. Since 1976, DST has been used in many fields of applications, including information fusion, pattern recognition, decision making, etc, but it also has been seriously criticized by some authors [2]– [12].

In spite of it, starting from Zadeh's criticism [2]–[4], many questions have arisen about the validity and the consistency of this theory when combining uncertain and conflicting evidences expressed as basic belief assignments (BBAs). Zadeh's "paradox" [2] is the first example where DS rule gives an apparently counter-intuitive result in highly conflicting case. Another very interesting example showing the counter-intuitive behavior of DS rule in some very low conflicting cases has been discovered recently and discussed by the authors in [11].

Since the publication of Zadeh's example, many researchers and engineers [5]–[9], [14] working in applications with belief functions have observed and admitted that DS rule is problematic for evidence combination, especially when the sources of evidence are highly conflicting. A most recent detailed discussion on the validity of DS rule can be found in [10]–[12]. It is worth noting that the discussion of the choice of semantics for the justification of a rule of combination is not the purpose of this paper. We just want to revisit and discuss here the most well-known Zadeh's emblematic example only from a physical-based standpoint because we are very concerned with fusion in real applications, especially for defense and security.

This paper was inspired by our curiosity to revisit Zadeh's example on the base of a real experiment, in order to become aware of the authentic physical fusion process (validated by the Nature's physical laws) and to understand the way how this emblematic example is "resolved" in actual fact by the Nature. In this paper, we propose a real experiment for generating BBAs from physical quantities that are consistent with BBAs inputs given in Zadeh's example, and that can be fused automatically by a pure natural phenomenon. Our paper shows that in this Z-box experiment, Dempster's rule of combination is inconsistent with physical (fusion) law of Nature and thus it cannot be used to predict the experimental results. Our experiment can be reproduced and verified by any reader who wants to check by him/herself the validity of our results. In this experiment, we have considered and generated two independent Bayesian BBAs that are equi-reliable and fit with Zadeh's BBAs inputs and let the Nature combine them physically, and we just observe what happens. Even if the two Zadeh's Bayesian BBAs appear as highly conflicting (in Shafer's sense), we have shown that it is however possible to make a physical experiment in which each source provides a BBA as chosen by Zadeh. This is possible because each source has only a partial knowledge of the state of the world.

In this work, we have just designed a simple physical experiment in which the fusion procedure is just governed by the physical law of Nature. All the fusion rules aim to obtain good and reasonable fusion results. We do think that to use such a physical experiment for testing DS rule (a type of fusion rule) makes sense and is rational, and our results indicate that DS rule does not agree with the physical (natural) fusion process. To certify that DS fusion rule is undoubtedly valid and really useful in practical applications, it should be

[1]Although the rule has been proposed originally by Dempster, we call it Dempster-Shafer rule because it has been widely promoted by Shafer in DST.
[2]otherwise DS rule is mathematically not defined because of 0/0 indeterminacy.

proved valid through an undisputed experimental protocol and tested on real experiments, and not claimed valid from specific justifications conditioned by particular choices of semantics that have been disputed since more than four decades in the scientific community. The choice of a semantic interpretation of fusion, although interesting, is not our major concern here. So far, and to authors knowledge, there is no undisputed experimental protocol proving that Dempster's rule is valid, even if Shafer proposed an interpretation based on a *random-code* interpretation of belief functions (BF) in [13]. It is also worth recalling that DS rule is not a generalization of Bayesian inference because even when BBAs are Bayesian, DS and Bayes rules become incompatible as soon as the a priori is truly informative (i.e. it is not vacuous, nor uniform) – as it is in the vast majority of practical cases in fact, see [12] and references inside for justifications with examples. That is why, it is vain (in our opinion) to search for a real valid and general physical experiment validating DS rule in the general context of belief functions.

After a brief recall of the basics of DST and Zadeh's example, we will present in details our Z-box experiment and discuss its results in the next sections.

II. Basics of DST

Let $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$ be a frame of discernment of a problem under consideration containing n distinct exclusive and exhaustive elements θ_i, $i = 1, \dots, n$. A basic belief assignment[3] (BBA), $m(.) : 2^\Theta \to [0, 1]$ is a mapping from the power set of Θ (i.e. the set of subsets of Θ), denoted 2^Θ, to $[0, 1]$, that must satisfy the following conditions: 1) $m(\emptyset) = 0$, i.e. the mass of empty set (impossible event) is zero; 2) $\sum_{X \in 2^\Theta} m(X) = 1$, i.e. the mass of belief is normalized to one. The quantity $m(X)$ represents the mass of belief exactly committed to X. An element $X \in 2^\Theta$ is called a focal element if and only if $m(X) > 0$. The set $\mathcal{F}(m) \triangleq \{X \in 2^\Theta | m(X) > 0\}$ of all focal elements of a BBA $m(.)$ is called the core of the BBA. The vacuous BBA characterizing the full ignorance is defined by $m_v(.) : 2^\Theta \to [0; 1]$ such that $m_v(X) = 0$ if $X \neq \Theta$, and $m_v(\Theta) = 1$.

From any BBA $m(.)$, the belief function $Bel(.)$ and the plausibility function $Pl(.)$ are defined for $\forall X \in 2^\Theta$ as: $Bel(X) = \sum_{Y|Y \subseteq X} m(Y)$ and $Pl(X) = \sum_{Y|X \cap Y \neq \emptyset} m(Y)$. $Bel(X)$ and $Pl(X)$ are classically interpreted as lower and upper bounds of an unknown subjective probability $P(.)$ and one has the following inequality satisfied $Bel(X) \leq P(X) \leq Pl(X)$, $\forall X \in 2^\Theta$. In DST, the combination (fusion) of several distinct sources of evidences is done with DS rule of combination, which corresponds to the normalized conjunctive consensus operator [1], assuming that the sources are not in total conflict[4]. DS combination of two independent BBAs $m^1_\Theta(.)$ and $m^2_\Theta(.)$ is defined by $m_\Theta(\emptyset) = 0$, and for all $X \in 2^\Theta \setminus \{\emptyset\}$ by

$$m_\Theta(X) = \frac{1}{1 - K_{12}} \times \sum_{\substack{X_1, X_2 \in 2^\Theta \\ X_1 \cap X_2 = X}} m^1_\Theta(X_1) m^2_\Theta(X_2) \quad (1)$$

where

$$K_{12} \triangleq \sum_{\substack{X_1, X_2 \in 2^\Theta \\ X_1 \cap X_2 = \emptyset}} m^1_\Theta(X_1) m^2_\Theta(X_2) \quad (2)$$

defines the so-called *conflict* between the two sources of evidence characterized by the BBAs $m^1_\Theta(.)$ and $m^2_\Theta(.)$.

III. Zadeh's Example

The famous Zadeh's example considers two doctors examining a patient who suffers from either meningitis (A), concussion (B) or brain tumor (C). The frame of discernment is chosen as $\Theta = \{A, B, C\}$ and it is assumed as exhaustive and exclusive. Both doctors agree in their low expectation of a tumor, but disagree in likely cause and provide the following diagnosis, described by the following BBAs $m_1(.)$ and $m_2(.)$ satisfying

$$m_1(A) = 0.90, \; m_1(B) = 0.00, \; m_1(C) = 0.10 \quad (3)$$

$$m_2(A) = 0.00, \; m_2(B) = 0.90, \; m_2(C) = 0.10 \quad (4)$$

If one combines the two BBAs using DS rule of combination, the following counter-intuitive final conclusion is obtained

$$m_{DS}(A) = 0.0, \; m_{DS}(B) = 0.0, \; m_{DS}(C) = 1.0 \quad (5)$$

The conclusion made on the base of DS rule is that the patient has for sure a brain tumor because it is the only diagnose that both doctors agree on even if the two experts (doctors) agree that tumor is unlikely but are in almost full contradiction for the other causes of the disease. What is even more questionable is that the same conclusion (the brain tumor is unlikely) would be obtained regardless of the probabilities associated with the other possible diagnoses. This very simple but interesting example shows the limitations of practical use of the DST for automated reasoning and has widely been discussed in the literature [2]–[12].

A more emblematic and interesting example, involving possibly low conflicting sources, has been discovered recently and discussed in [10]–[12]. It corresponds to the case where the two equi-reliable doctors' reports concern the following BBAs satisfying $m_1(A) = a$, $m_1(A \cup B) = 1 - a$ and $m_2(A \cup B) = b_1$, $m_2(C) = 1 - b_1 - b_2$, $m_2(A \cup B \cup C) = b_2$, with parameters $0 < a, b_1, b_2, < 1$. It is easy to verify that the conflict given by (2) is equal to $K_{12} = m_1(A)m_2(C) + m_1(A \cup B)m_2(C) = 1 - b_1 - b_2$. Surprisingly, this conflict does not impact (it can be very high, or very low) the DS fusion result because one always has in this new example $m_{DS}(.) = m_1(.)$. This result is also abnormal and counter-intuitive because the second source $m_2(.)$ (the 2nd doctor diagnosis) does not count at all in DS fusion process, even if $m_2(.)$ is not vacuous (it is informative) and truly conflicting with the first doctor's diagnosis $m_1(.)$.

[3] also called a belief mass function (BMF) by some authors, or a basic probability assignment (BPA) by Shafer.

[4] otherwise DS rule is mathematically not defined because of 0/0 indeterminacy.

IV. A Real Z-box Experiment

In this section, we propose an electronic circuit (called Z-box scheme) as shown in Fig. 1 to generate BBAs according to Zadeh's example and to test experimentally the physical fusion of these BBAs.

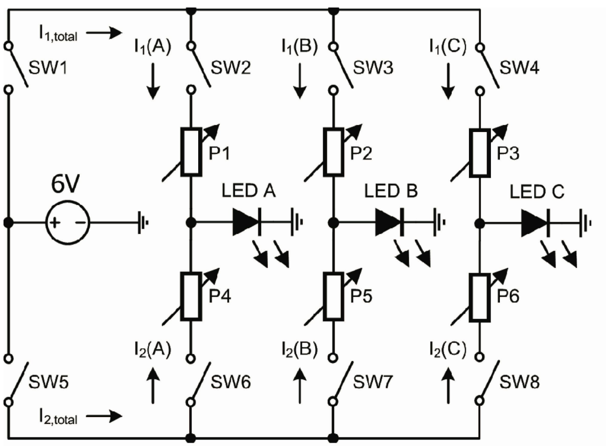

Figure 1. Z-box Scheme.

It is clear that this scheme can be easily extended to build and combine more than three Bayesian sources of evidence as well, which is out of the scope of this paper. This scheme utilizes a simple battery of 6 Volts as an only circuit's power supply. The switches SW1 and SW5 are used to obtain two independent sub-circuits, in order to realize two independent sources of information for the purpose of our task. Three simple linear potentiometers (P_1, P_2, P_3) and three switches (SW2, SW3, and SW4) are used to establish the first source (sub-circuit 1), respectively three potentiometers (P_4, P_5, P_6) and three switches (SW6, SW7, and SW8) for the second source (sub-circuit 2). Each of these two sources of information provides its relative truth, established on its own knowledge only, by setting the special tuning of corresponding sets of potentiometers. Three white Light Emitting Diodes (LED's - LED_A, LED_B, and LED_C) are put to be utilized as light indicators. The light intensity is proportional to the current values through the LED's. We are concerned with the answer of the question: which LED emits the light with strongest intensity? Our frame is $\Theta = \{A \triangleq LED_A, B \triangleq LED_B, C \triangleq LED_C\}$. The Z-box experiment consists in three main steps: 1) tuning the source no. 1 (Sub-circuit 1) to generate BBA $m_1(.)$; 2) tuning the source no. 2 (Sub-circuit 2) to generate BBA $m_2(.)$; and 3) the physical fusion of the two BBAs. The descriptions of these steps are given in the sequel and are illustrated in the figures 2-4.

Step 1: Tuning the first source (Sub-circuit 1) according to Fig.2. Only the upper branch of the circuit is active with the following settings:

- Switch SW1 is closed and switch SW5 is open.
- Switches SW2 and SW4 are closed. Switch SW3 is left open, providing a zero-current through LED_B: $I_1(LED_B) = 0.0$ mA.
- The potentiometers (P_1, P_3) are tuned to provide the following current values through the LED's: $I_1(LED_A) \approx$ 32.5 mA, $I_1(LED_B) = 0.0$ mA and $I_1(LED_C) \approx 3.6$ mA, where the index $\{1\}$ is used to denote the 1st source of information.

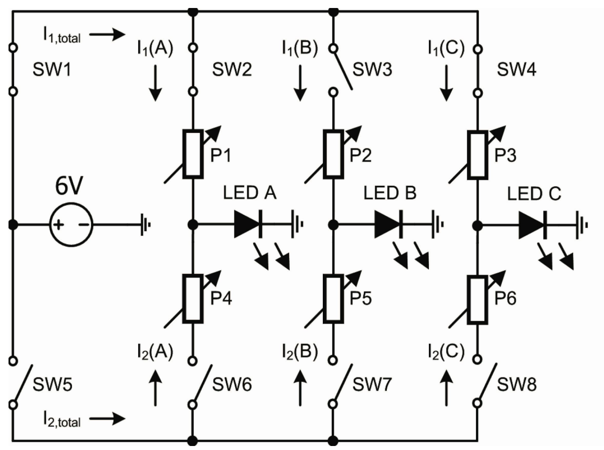

Figure 2. Step 1 of the experiment : setting the BBA $m_1(.)$.

Step 2: Tuning the second source (Sub-circuit 2) according to Fig. 3. Only the lower branch of the circuit is active with the following settings:

- Switch SW1 is open and switch SW5 is closed.
- Switches SW7 and SW8 are closed. Switch SW6 is left open, providing a zero-current through LED_A as $I_2(LED_A) = 0.0$ mA.
- The potentiometers (P_5, P_6) are tuned to provide the following current values through the LED's: $I_2(.) = \{I_2(LED_A) = 0.0$ mA, $I_2(LED_B) \approx 32.5$ mA, and $I_2(LED_C) \approx 3.6$ mA, where the index $\{2\}$ is used to denote the 2nd source of information.

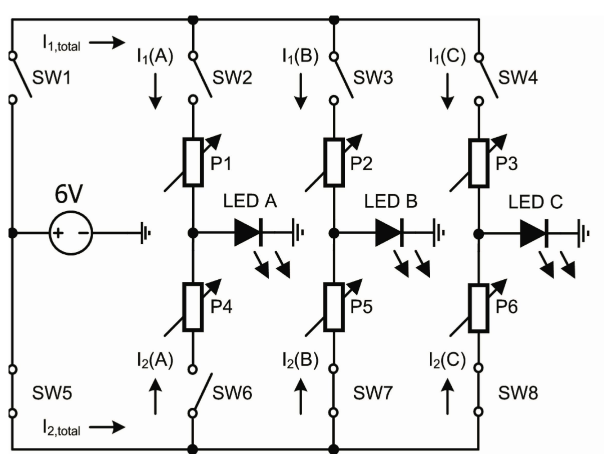

Figure 3. Step 2 of the experiment : setting the BBA $m_2(.)$.

Step 3: Both branches of the circuit are active at the same time for making the physical fusion. More precisely, we set the switches SW2, SW3 and SW4 and tune the potentiometers P_1, P_2 and P_3 according to Step 1, and we set the switches SW6, SW7 and SW8 and tune the potentiometers P_4, P_5 and P_6 according to Step 2. The switches SW1 and SW5 are closed to implement the fusion of the sources as shown in Fig. 4.

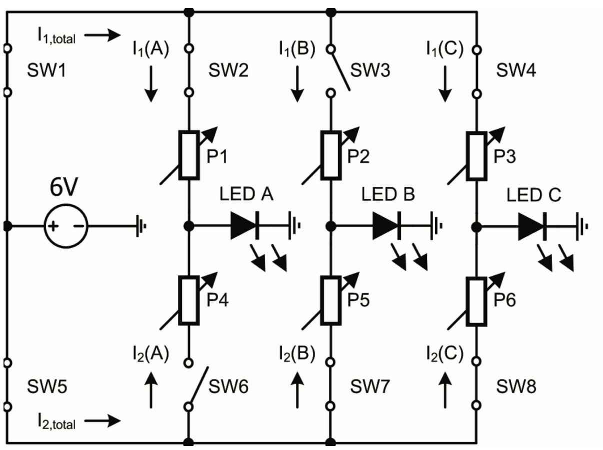

Figure 4. Step 3 of the experiment : the (physical) fusion of BBAs.

At this step, one gets:

$$\begin{cases} I_1(LED_A) \approx 32.5 \text{ mA}, \\ I_1(LED_B) = 0.00 \text{ mA}, \\ I_1(LED_C) \approx 3.6 \text{ mA} \end{cases} \quad (6)$$

and

$$\begin{cases} I_2(LED_A) = 0.00 \text{ mA}, \\ I_2(LED_B) \approx 32.5 \text{ mA}, \\ I_2(LED_C) \approx 3.6 \text{ mA} \end{cases} \quad (7)$$

The total current intensities are respectively equal to

$$\begin{cases} I_{1,total} = \sum_{i\in\{A,B,C\}} I_1(LED_i) \approx 36.1 \text{ mA} \\ I_{2,total} = \sum_{i\in\{A,B,C\}} I_2(LED_i) \approx 36.1 \text{ mA} \end{cases}$$

Fig. 5 shows the different LED's current values obtained in each step during the experiment's time duration of 5 sec. In the left subplots of Fig. 5 (result of step 1), one sees that the current through LED_A is 9 times higher than the current through LED_C, while the current through LED_B is almost zero, whereas in the middle subplots of Fig. 5 (result of step 2), one sees that the current through LED_B is 9 times higher than the current through LED_C, while the current through LED_A is almost zero. The observed results make perfect sense. Because the light intensity is proportional to current values through the LEDs, the same proportions are valid for the intensity of the light emitted from the LEDs. One sees that these settings fit with the input BBAs of Zadeh's example because after the normalization of current values one has the following masses of belief in the origin of the strongest light emission:

$$\begin{cases} m_1(A) \triangleq \frac{I_1(LED_A)}{I_{1,total}} \approx 0.9 \\ m_1(B) \triangleq \frac{I_1(LED_B)}{I_{1,total}} = 0.0 \\ m_1(C) \triangleq \frac{I_1(LED_C)}{I_{1,total}} \approx 0.1 \end{cases} \quad (8)$$

and

$$\begin{cases} m_2(A) \triangleq \frac{I_2(LED_A)}{I_{2,total}} = 0 \\ m_2(B) \triangleq \frac{I_2(LED_B)}{I_{2,total}} \approx 0.9 \\ m_2(C) \triangleq \frac{I_2(LED_C)}{I_{2,total}} \approx 0.1 \end{cases} \quad (9)$$

The results of steps 1 and 2 show that both of the sources (corresponding to 1st and 2nd sub-circuits), taken independently, are able to make a correct physical assessment of the real physical situation. The right subplots of Fig. 5 (result of step 3) show the real physical fusion results simulated from MicroSim DesignLab 8 [18], as shown through the screen copy given in Fig. 6. Here we use the index $\{12\}$ to denote that both sources (sub-circuits) are active. The observed current intensities are $I_{12}(LED_A) \approx 32.5$ mA, $I_{12}(LED_B) \approx 32.5$ mA, and $I_{12}(LED_C) \approx 6.9$ mA. After the normalization of $I_{12}(.)$, we get finally the combined BBA $m_{12}(.)$ over the frame of discernment $\Theta \triangleq \{A, B, C\}$ that is given by $m_{12}(A) \triangleq I_{12}(LED_A)/I_{12,total} \approx 0.45$, $m_{12}(B) \triangleq I_{12}(LED_B)/I_{12,total} \approx 0.45$, and $m_{12}(C) \triangleq I_{12}(LED_C)/I_{12,total} \approx 0.10$, where $I_{12,total} = I_{12}(LED_A) + I_{12}(LED_B) + I_{12}(LED_C) \approx 71.9$ mA.

Clearly, the observed fact is that after the real physical fusion, the current through LED_A is just equal to the current through LED_B, and both are approximately 5 times higher than the current through LED_C. The experimental fusion result does not fit with the predicted result based on DS rule (5), nevertheless in this experiment both BBA inputs match the medical experts' opinions as in Zadeh's example, and they are considered to be in high "conflict" according to the classical interpretation in DST. This result brings to light the fact that DS rule result (5) is not consistent in this experiment with what the physical fusion system provides. This real Z-box experiment supports Zadeh's intuition about the non-adequate behavior of DS rule, and the counter-intuitive decisions that can be drawn from it. Stated otherwise, the natural physical fusion does not follow DS rule of combination. In fact, the notion of "conflict", which plays an important role when manipulating belief functions, is questionable, since it appears quite artificial in physics (in natural phenomenon). The conflict plays however a main role in decision-making in human reasoning. The way in which the total or partial conflicts are managed by Shafer's evidential reasoning is incompatible with this simple physical experiment.

It is worth noting that the physical fusion of sources of Zadeh's example is consistent with the simple averaging rule, and (relatively) consistent with PCR6 fusion rule [17] (Vol. 2) which will provide in this example $m_{PCR6}(A) = 0.486$, $m_{PCR6}(B) = 0.486$, and $m_{PCR6}(C) = 0.028$. Contrarily to DS rule, PCR6 is fully consistent with the averaging

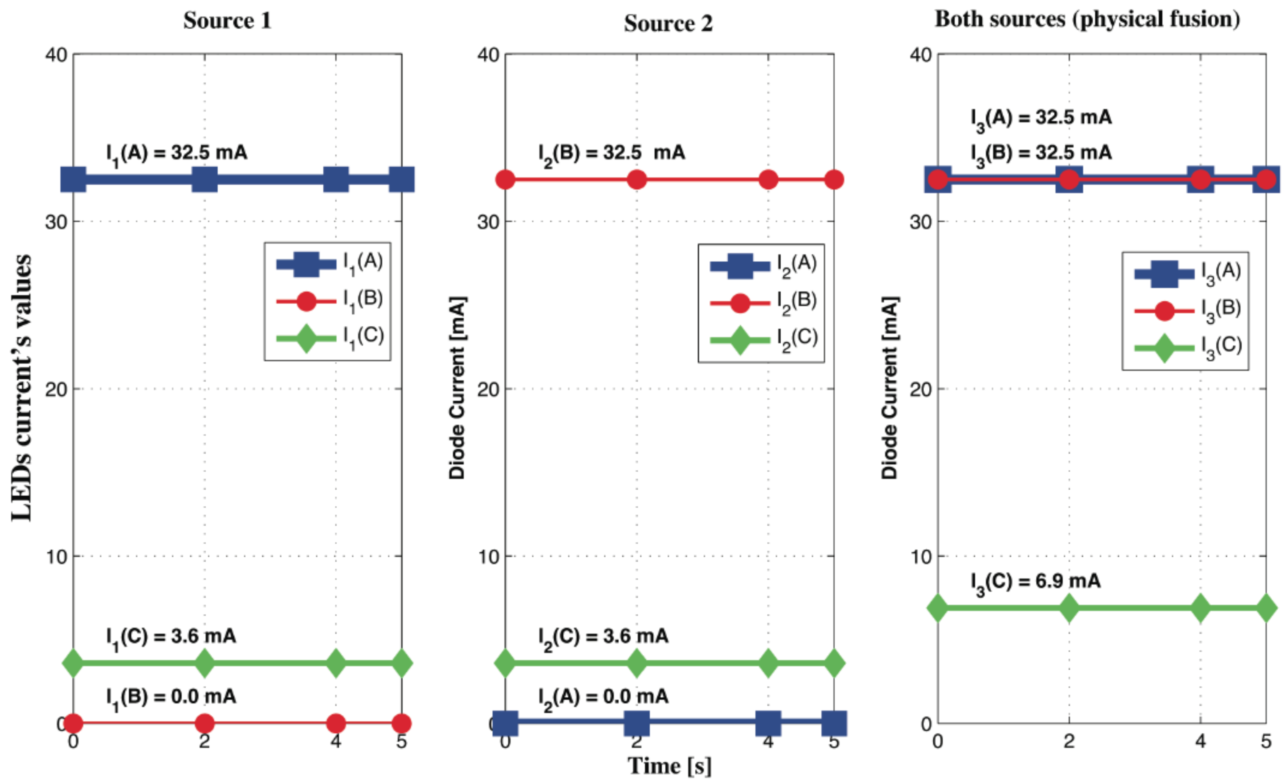

Figure 5. LEDs current values for source 1, source 2, both sources (by physical fusion).

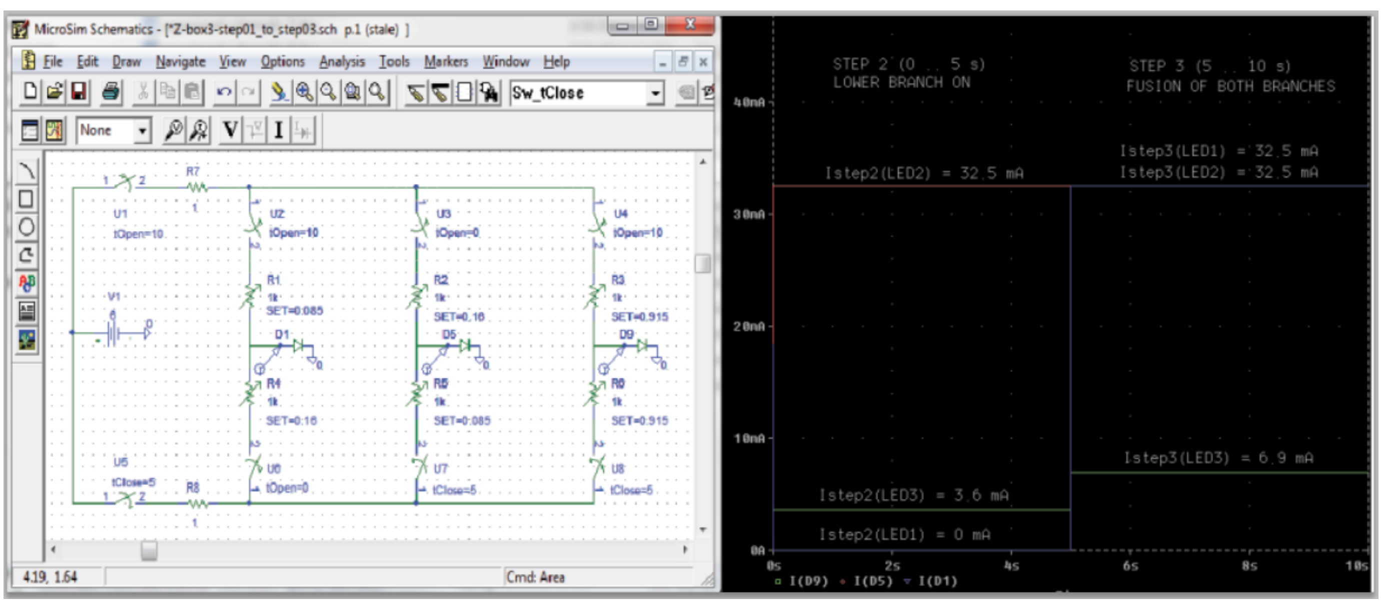

Figure 6. Screen copies of MicroSim schematics and its physical fusion result.

rule for estimating frequentist probabilities in binary random experiments, see [19] for details with examples.

V. Conclusions

In this paper a real experimental method for building basic belief assignments associated with two independent, informative, and equireliable sources of information, following the emblematic Zadeh's example has been presented. It is based on a particular electronic circuit box (called Z-box), enabling to observe and to check the fusion result experimentally. Zadeh's intuition about the non-adequate behavior of DS rule and the counter-intuitive decisions obtained on its base is perfectly defended by Nature through this experiment. A similar experiment, called Z-aquarium experiment can also be done with fluids (with a container filled of water) instead of an electronic circuit, but it is more complex to set up and it has not been reported in this paper. Our conclusion is that Dempster-Shafer Theory does not agree with the physical fusion process at least for a situation that fits with Zadeh's example. The more general question on the validity of DST (especially, when subjective beliefs are considered) was not the purpose of this paper because this question has already been addressed in details in our previous research works put in references.

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