

Two Novel Methods for BBA Approximation Based on Focal Element Redundancy

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Abstract—The theory of belief functions is a very appealing theory for uncertainty modeling and reasoning which has been widely used in information fusion. However, when the cardinality of the frame of discernment and the number of the focal elements are large the fusion of belief functions requires in general a high computational complexity. To circumvent this difficulty, many methods were proposed to implement more efficiently the combination rules and to approximate basic belief assignments (BBA's) into simplest ones to reduce the number of focal elements involved in the fusion process. In this paper, we present a novel principle for approximating a BBA by withdrawing more redundant focal elements of the original BBA. Two methods based on this principle are presented (using batch and recursive implementations). Numerical examples, simulations and related analyses are provided to illustrate and evaluate the performances of this new BBA approximation method.

Index Terms—Evidence theory; belief functions; basic belief assignment; approximation.

I. INTRODUCTION

The original theory of belief functions, also known as Dempster-Shafer Theory (DST) [1] has been widely used in information fusion, pattern recognition and decision making due to its advantages in representing uncertain information and partial knowledge. However, the computational complexity is one of its drawbacks [2], specially for combining sources of evidences expressing their BBA's with respect to large frames of discernment (FoD). The computational complexity of the evidence combination is strongly affected by the cardinality of the FoD and the number of focal elements of the BBA of the sources to combine.

To reduce the computational complexity of evidence combination, various approaches have been proposed, which generally fit within the following two categories:

- a) Efficient implementation for performing exact computations of the chosen rule of combination. For example, an optimal algorithm for Dempster's rule of combination was proposed by Kennes [3]. Barnett [4], Shafer and Logan's [5] works are also representatives of this aspect.
- b) Approximation of simplification of BBA's. For example, $k-l-x$ approach [6], summarization approach [7], the D1 approximation [8], inner and outer approximations [9], Monte-Carlo based approximation [10], etc., remove

focal elements and redistribute the corresponding mass assignments. In our previous works, we also had proposed hierarchical proportional redistribution approach [11], and the optimization-based BBA approximations [12].

The work presented in this paper focuses on the reduction of evidence combination's computational cost thanks to BBA approximations. In the aforementioned works of category b), the different methods propose to remove some focal elements according to some criteria, typically based either on their mass values or on their cardinalities. We think that only mass values or focal element cardinality are not enough for selecting the focal elements to remove for making good BBA approximation. We propose a novel approach using the notion of focal element redundancy. Those relatively redundant focal elements should be removed and those relatively non-redundant ones should be remained. To quantify this notion of redundancy, we use the average distance between a given focal element and all the other focal elements. Smaller average distance means that the given focal element carries similar information when compared with others, i.e., it is more redundant and should be removed at first. User can preset the desired number of remaining focal elements (also the number of removed focal elements). Two removing procedures (including a batch mode and a iterative mode) are proposed in the sequel, followed by the re-normalization or redistribution. Numerical examples, simulations and related analyses are provided to show the rationality and interest of these novel BBA approximation approaches.

II. BASICS OF BELIEF FUNCTIONS

The theory of belief functions has been developed by Shafer [1] in 1976 from early works of Dempster. In DST, the elements in frame of discernment (FoD) Θ are mutually exclusive and exhaustive. A basic belief assignment (BBA), also called a mass function, is a mapping $m(\cdot) : 2^\Theta \rightarrow [0, 1]$ satisfying $m(\emptyset) = 0$ and

$$\sum_{A \in 2^\Theta} m(A) = 1 \quad (1)$$

If $m(A) > 0$, A is called a focal element of the BBA $m(\cdot)$. In DST, the combination of two distinct bodies of evidence

(BOEs) $m_1(\cdot)$ and $m_2(\cdot)$ is done using Dempster's rule as follows. $\forall A \in 2^\Theta$:

$$m(A) = \begin{cases} 0, & \text{if } A = \emptyset \\ \frac{1}{1-K} \sum_{A_i \cap B_j = A} m_1(A_i) m_2(B_j), & \text{if } A \neq \emptyset \end{cases} \quad (2)$$

where $K = \sum_{A_i \cap B_j = \emptyset} m_1(A_i) m_2(B_j)$ is the total conflicting mass assignments, which is discarded by normalization in Dempster's rule. It can be found from Eq. (2) that Dempster's rule is both commutative and associative. Dempster's rule has been seriously criticized for its counter-intuitive behaviors both in high conflicting and low conflicting situations [13], and other rules of combination have been developed in the literature – see [14] for details. These modified or refined combination rules focus on suppressing the counter-intuitive behaviors of Dempster's rule. However, like Dempster's rule, they all have to face the problem of high computational complexity with the increase of the FoD's cardinality and the quantity of the focal elements.

To reduce the computational cost of combination of BBA's and make the fusion process tractable, we can as a first strategy switch to more simple rules of combination or try to develop efficient implementations of sophisticate rules, or as a second strategy simplify (approximate) original BBA to combine by simplest BBA with less focal elements of smaller cardinalities, or we can mix both strategies as well. In this paper, we focus on the second strategy devoted to BBA approximation, which is more intuitive for human to catch the meaning [15].

III. EXISTING BBA APPROXIMATION APPROACHES

Some existing BBA approximation approaches are briefly recalled in this section for the purpose of comparisons with the novel methods proposed in this paper.

1) *k-l-x method* [6]: This approach has been proposed by Tessem in 1993. The simplified BBA is obtained by

- keeping no less than k focal elements;
- keeping no more than l focal elements;
- by deleting the masses which are no greater than x .

In $k-l-x$, all original focal elements are sorted according to their mass values in a decreasing order. Then, the first p focal elements are chosen such that $k \leq p \leq l$ and such that the sum of the mass assignments of these first p focal elements is no less than $1-x$. The removed mass values are redistributed to remaining focal elements by a classical normalization procedure.

2) *Summarization method* [7]: This method is similar to the $k-l-x$ and it also keeps focal elements having highest mass values. The mass values of focal elements to remove are accumulated and assigned to the their union set. Suppose k is the desired number of focal elements in the approximated BBA $m_S(\cdot)$ of a given BBA $m(\cdot)$. Let M be the set of $k-1$ focal elements with the highest mass values in $m(\cdot)$. Then $m_S(\cdot)$ is obtained from $m(\cdot)$ by

$$m_S(A) = \begin{cases} m(A), & \text{if } A \in M \\ \sum_{A' \subseteq A, A' \notin M} m(A'), & \text{if } A = A_0 \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

where A_0 is determined by

$$A_0 \triangleq \bigcup_{A' \notin M, m(A') > 0} A' \quad (4)$$

3) *D1 method* [8]: Let $m(\cdot)$ be the original BBA to approximate. $m_S(\cdot)$ denotes the approximated BBA and the desired number of focal elements is k . Let M be the set of $k-1$ focal elements with the highest mass values in $m(\cdot)$ and M^- be the set including all the other focal elements of $m(\cdot)$. The basic idea of the D1 method is to keep all the members of M as the focal elements of $m_S(\cdot)$ and to assign the mass values of the focal elements in M^- among the focal elements in M according to the following procedure.

Given a focal element $A \in M^-$, in M , find all the supersets of A to form the collection M_A . If M_A is not empty, the mass value of A is uniformly assigned among the focal elements with smallest cardinality in M_A . When M_A is empty, then construct M'_A as

$$M'_A = \{B \in M \mid |B| \geq |A|, B \cap A \neq \emptyset\} \quad (5)$$

Then, if M'_A is not empty, $m(A)$ is assigned among the focal elements with smallest cardinality in M'_A . The value assigned to a focal element B depends on the value of $|B \cap A|$. Such a procedure is executed iteratively until all $m(A)$ have been assigned to the focal elements in M .

If M'_A is empty, there are two possible cases:

- 1) If the total set $\Theta \in M$, the sum of mass values of the focal elements in M^- will be added to Θ ;
- 2) If $\Theta \notin M$, then set Θ as a focal element of $m_S(\cdot)$ and assign the sum of mass values of the focal elements in M^- to $m_S(\Theta)$.

More details on D1 method with examples can be found in [8].

The basic principle of these three previous approaches of BBA approximation is to remove the focal elements having smaller mass values because they are deemed as unimportant. Besides these methods, there exist other works on BBA approximations. For example, Denœux inner and outer approximations [9], Grabisch's k -additive BBA approximation [16], and our previous works based on hierarchical proportional distribution (HPR) [11] and optimization-based BBA approximations [12]. In these methods, the aim is to remove the focal elements with larger cardinalities because they bring more computational cost in the fusion process in general (see related references for details).

IV. NEW BBA APPROXIMATIONS USING THE PRINCIPLE OF FOCAL ELEMENT REDUNDANCY

As briefly shown in the previous section, the existing BBA approximation approaches propose to remove some focal elements by eliminating those with smaller mass values, or with larger cardinalities. Although these methods have some rational justification, only mass values or cardinalities are not enough in our opinion for judging which focal elements should be removed for making BBA approximation. We consider that

it is quite hazardous (risky) to deem focal elements having small mass values as unimportant. It may also be dangerous to remove the focal elements with large cardinality justified only by the possible high computational cost they may cause in the fusion process. So, we should be cautious when adopting a BBA approximation technique. We agree with the fact that focal elements that are considered unimportant must be removed at first in an approximation method. However, focal elements' mass values are not enough for judging their importance. A more solid index (criterion) should be found to estimate the importance of a focal element to keep. Because the very redundant focal elements can reasonably be considered as unimportant and the relatively non-redundant focal elements can reasonably be considered as important, we define the degree of non-redundancy for a focal element at first. From this degree of non-redundancy, we can then develop new BBA approximation methods as it will be shown.

A. Degree of non-redundancy of focal elements

Suppose a BBA $m(\cdot)$ has l focal elements. A distance between focal elements A_i and A_j proposed by Denœux [9] is defined as

$$\delta_{\cap}(A_i, A_j) = m(A_i) \cdot |A_i| + m(A_j) \cdot |A_j| - [m(A_i) + m(A_j)] \cdot |A_i \cap A_j| \quad (6)$$

If a focal element A_i has the smallest average distance with other focal elements $A_j \subseteq \Theta, j \neq i$, then A_i shares most common information with other focal elements, i.e., A_i is the most redundant. Therefore, we can define the degree of non-redundancy based on the average distance between a focal element and others. First, we calculate the distance matrix for all the focal elements of $m(\cdot)$ as

$$Mat_{FE} \triangleq \begin{bmatrix} \delta_{\cap}(A_1, A_1) & \delta_{\cap}(A_1, A_2) & \cdots & \delta_{\cap}(A_1, A_l) \\ \delta_{\cap}(A_2, A_1) & \delta_{\cap}(A_2, A_2) & \cdots & \delta_{\cap}(A_2, A_l) \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{\cap}(A_l, A_1) & \delta_{\cap}(A_l, A_2) & \cdots & \delta_{\cap}(A_l, A_l) \end{bmatrix}$$

It should be noted that $\delta_{\cap}(A_i, A_i) = 0$ and $\delta_{\cap}(A_i, A_j) = \delta_{\cap}(A_j, A_i)$ where $i = 1, \dots, l$. Hence, it is not necessary to calculate all the elements in Mat_{FE} because the matrix is symmetric.

We define the degree of non-redundancy of the focal element A_i by

$$\text{nRd}(A_i) \triangleq \frac{1}{l-1} \sum_{j=1}^{l-1} \delta_{\cap}(A_i, A_j) \quad (7)$$

The larger $\text{nRd}(A_i)$ value, the larger non-redundancy (less redundancy) for A_i . The less $\text{nRd}(A_i)$ value, the less non-redundancy (larger redundancy) for A_i .

Based on the focal element redundancy, i.e., to use the degree of non-redundancy in (7), we propose two new BBA approximation methods described in the next subsections, where the more non-redundant focal elements will be remained and the more redundant ones will be removed.

B. Batch approximation method

Let $m(\cdot)$ denote the original BBA to approximate with l focal elements. In the approximation, we want to keep $k < l$ focal elements. First, we propose a BBA approximation with a batch processing, which means that the number of focal elements is reduced from l to k in one processing cycle as follows.

- **Step 1:** Calculate Mat_{FE} at first, and for each $A_i, i = 1, \dots, l$ compute its non-redundancy value $\text{nRd}(A_i)$;
- **Step 2:** Sort all the elements in descending order according to the values of $\text{nRd}(A_i)$;
- **Step 3:** Remove the $l - k$ bottom focal elements;
- **Step 4:** Normalize the mass values of the remaining k focal elements and output the approximated BBA $m_S^{BRd}(\cdot)$.

C. Iterative approximation method

In this method, we remove iteratively one most redundant focal element (with the least nRd value) in each cycle until k focal elements are remained. This method consists of the following steps:

- **Step 1:** Calculate Mat_{FE} and nRd for each $A_i, i = 1, \dots, l$;
- **Step 2:** Sort all the elements in descending order according to their values of $\text{nRd}(A_i)$;
- **Step 3:** Remove the bottom focal element A_r ;
- **Step 4:** If the number of remaining focal element is larger than k , recalculate $\text{nRd}(A_i)$ for $i = 1, \dots, l, i \neq r$ and go to Step 3. Otherwise, go to Step 5 ;
- **Step 5:** Normalize the mass values of the remaining k focal elements and output the approximated BBA $m_S^{IRd}(\cdot)$.

For this iterative method, the degrees of non-redundancy are recalculated in each cycle after removing a focal element in the previous cycle. That is to say, in each cycle, only the non-redundancy of the current remaining focal elements are concerned.

D. Illustrative examples

Here we provide a simple numerical example to illustrate the implementation procedures of some available BBA approximation approaches with respect to our two new methods.

Example 1: Let consider the BBA $m(\cdot)$ defined over the FoD $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5\}$ listed in Table I.

TABLE I
FOCAL ELEMENTS AND MASS VALUES OF $m(\cdot)$

Focal Elements	Mass values
$A_1 = \{\theta_1, \theta_2\}$	0.50
$A_2 = \{\theta_1, \theta_3, \theta_4\}$	0.30
$A_3 = \{\theta_3\}$	0.10
$A_4 = \{\theta_3, \theta_4\}$	0.05
$A_5 = \{\theta_4, \theta_5\}$	0.05

1) Using $k-l-x$ method [6]: Here k and l are set to 3. x is set to 0.1. The focal elements $A_4 = \{\theta_3, \theta_4\}$ and $A_5 = \{\theta_4, \theta_5\}$ are removed without violating the constraints in $k-l-x$. The remaining total mass value is $1 - 0.05 - 0.05 = 0.9$. Then, all the remaining focal elements' mass values are divided by 0.9 to accomplish the normalization. The approximated BBA $m_S^{klx}(\cdot)$ obtained by $k-l-x$ method is listed in Table II, where $A'_i, i = 1, 2, 3$ are the focal elements of $m_S^{klx}(\cdot)$.

TABLE II
 $m_S^{klx}(\cdot)$ OBTAINED USING $k-l-x$

Focal Elements	Mass values
$A'_1 = \{\theta_1, \theta_2\}$	0.5556
$A'_2 = \{\theta_1, \theta_3, \theta_4\}$	0.3333
$A'_3 = \{\theta_3\}$	0.1111

2) Using summarization method [7]: Here k is set to 3. According to the summarization method, the focal elements $A_3 = \{\theta_3\}$, $A_4 = \{\theta_3, \theta_4\}$ and $A_5 = \{\theta_4, \theta_5\}$ are removed, and their union $\{\theta_3, \theta_4, \theta_5\}$ is generated as a new focal element with mass value $m(\{\theta_3\}) + m(\{\theta_3, \theta_4\}) + m(\{\theta_4, \theta_5\}) = 0.2$. The approximated BBA m_S^{Sum} is listed in Table III below.

TABLE III
 $m_S^{Sum}(\cdot)$ OBTAINED USING SUMMARIZATION

Focal Elements	Mass values
$A'_1 = \{\theta_1, \theta_2\}$	0.50
$A'_2 = \{\theta_1, \theta_3, \theta_4\}$	0.30
$A'_3 = \{\theta_3, \theta_4, \theta_5\}$	0.20

3) Using DI method [8]: Here k is still 3. It can be obtained that A_1, A_2 belong to M , and A_3, A_4, A_5 belong to M^- . The focal element $A_1 = \{\theta_1, \theta_2\}$ has empty intersection with the focal elements in M^- , therefore its value will be unchanged. In M , A_2 is the unique superset of A_3 and A_4 , therefore, $m(A_3) + m(A_4) = 0.10 + 0.05$ is added to its original mass value. A_2 also covers half of A_5 , therefore, $m(A_5)/2 = 0.025$ is further added to the mass of A_2 . Finally, the rest mass value is assigned to the total set Θ . The approximated BBA m_S^{D1} is listed in Table IV.

TABLE IV
 $m_S^{D1}(\cdot)$ OBTAINED USING SUMMARIZATION

Focal Elements	Mass values
$A'_1 = \{\theta_1, \theta_2\}$	0.50
$A'_2 = \{\theta_1, \theta_3, \theta_4\}$	0.475
$A'_3 = \Theta$	0.025

4) Using Denœux inner approximation [9]: Because this method uses the focal element distance in Eq. (6), we also apply it in this example for comparison. With the inner approximation method, the focal elements pair with smallest distance are removed, and then their intersection is set as

the supplemented focal element whose mass value is the sum of the removed two focal elements' mass values. Such a procedure is repeated until the desired number of focal elements is reached. The results at each step are listed in Table V.

TABLE V
BBA'S OBTAINED USING INNER APPROXIMATION

Step 1		Step 2	
Focal elements	Mass values	Focal elements	Mass values
$A'_1 = \{\theta_1, \theta_2\}$	0.5	$A'_1 = \{\theta_1, \theta_2\}$	0.5
$A'_2 = \{\theta_1, \theta_3, \theta_4\}$	0.3	$A'_2 = \{\theta_1, \theta_3, \theta_4\}$	0.3
$A'_3 = \{\theta_3\}$	0.15	$A'_3 = \emptyset$	0.2
$A'_4 = \{\theta_4, \theta_5\}$	0.05		

As we can see in Table V, it generates the empty set as a focal element, which is not allowed in the classical Dempster-Shafer evidence theory under close-world assumption.

5) Using the redundancy-based batch approximation method: The desired remaining focal element is set to $k = 3$. We first calculate the distance matrix Mat_{FE} and we get

$$Mat_{FE} = \begin{matrix} & \begin{matrix} A_1 & A_2 & A_3 & A_4 & A_5 \end{matrix} \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{matrix} & \begin{bmatrix} 0 & 1.10 & 1.10 & 1.10 & 1.10 \\ 1.10 & 0 & 0.60 & 0.30 & 0.65 \\ 1.10 & 0.60 & 0 & 0.05 & 0.20 \\ 1.10 & 0.30 & 0.05 & 0 & 0.10 \\ 1.10 & 0.65 & 0.20 & 0.10 & 0 \end{bmatrix} \end{matrix}$$

Based on this matrix, the degree of non-redundancy for each focal elements of $m(\cdot)$ can be obtained. It is listed in Table VI.

TABLE VI
NON-REDUNDANCY FOR DIFFERENT FOCAL ELEMENTS

Focal Elements	Mass values	nRd(A_i)
$A_1 = \{\theta_1, \theta_2\}$	0.50	1.10
$A_2 = \{\theta_1, \theta_3, \theta_4\}$	0.30	0.6625
$A_3 = \{\theta_3\}$	0.10	0.4875
$A_4 = \{\theta_3, \theta_4\}$	0.05	0.3875
$A_5 = \{\theta_4, \theta_5\}$	0.05	0.5125

Since A_3 and A_4 at the bottom have the two least nRd values, they correspond the two focal elements with the lowest non-redundancy, i.e., the highest redundancy. Therefore, they are removed and their mass values are redistributed thanks to the classical normalization step. The approximated BBA m_S^{BRd} is listed in Table VII.

TABLE VII
 $m_S^{BRd}(\cdot)$ OBTAINED USING THE BATCH APPROXIMATION BASED ON REDUNDANCY

Focal Elements	Mass values
$A'_1 = \{\theta_1, \theta_2\}$	0.5882
$A'_2 = \{\theta_1, \theta_3, \theta_4\}$	0.3530
$A'_3 = \{\theta_4, \theta_5\}$	0.0588

6) *Using the redundancy-based iterative approximation method:* The number of remaining focal elements is still set to $k = 3$, so that two focal elements have to be removed. In the iterative mode, only one focal element is removed in each cycle, thus two cycles are needed.

In cycle I, the degree of non-redundancy is the same as listed in Table V. Then, the focal element A_4 is removed in first cycle.

In cycle II, recalculate nRd for A_i , $i = 1, \dots, 5, i \neq 4$ according to $\text{nRd}(A_i) = \sum_{j=1, j \neq 4, j \neq i}^5 \delta(A_i, A_j)$. The results are

$$\begin{aligned} \text{nRd}(A_1) &= 1.1000, \text{nRd}(A_2) = 0.7833, \\ \text{nRd}(A_3) &= 0.6333, \text{nRd}(A_5) = 0.6500 \end{aligned}$$

Then, A_3 is removed in this cycle due to its the lowest nRd value (the highest redundancy among the remaining focal elements). The approximated BBA obtained using iterative way is the same as the one listed in Table VII. It should be noted that the batch approximation and the iterative approximation will not always output the same results as shown in the next example.

Example 2: Suppose that FoD is $\Theta = \{\theta_1, \theta_2, \theta_3\}$. The BBA to approximate is listed in Table VIII, and the desired number of remaining focal elements is $k = 3$.

TABLE VIII
FOCAL ELEMENTS AND MASS VALUES OF $m(\cdot)$

Focal Elements	Mass values
$A_1 = \{\theta_1, \theta_2\}$	0.1780
$A_2 = \{\theta_2, \theta_3\}$	0.2477
$A_3 = \{\theta_2\}$	0.2322
$A_4 = \{\theta_3\}$	0.1758
$A_5 = \Theta$	0.1662

The distance matrix Mat_{FE} is

$$\begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \left[\begin{array}{ccccc} 0 & 0.4258 & 0.1780 & 0.5319 & 0.1662 \\ 0.4258 & 0 & 0.2477 & 0.2477 & 0.1662 \\ 0.1780 & 0.2477 & 0 & 0.4080 & 0.3325 \\ 0.5319 & 0.2477 & 0.4080 & 0 & 0.3325 \\ 0.1662 & 0.1662 & 0.3325 & 0.3325 & 0 \end{array} \right]$$

The degree of non-redundancy of focal elements are

$$\begin{aligned} \text{nRd}(A_1) &= 0.3255, \text{nRd}(A_2) = 0.2719, \\ \text{nRd}(A_3) &= 0.2916, \text{nRd}(A_4) = 0.3800, \text{nRd}(A_5) = 0.2494 \end{aligned}$$

With the batch approximation, the focal elements A_2 and A_5 are removed. After normalization, we get the approximated BBA listed in Table IX.

With the iterative approximation method, the degree of non-redundancy obtained at Cycle I are also

$$\begin{aligned} \text{nRd}^I(A_1) &= 0.3255, \text{nRd}^I(A_2) = 0.2719, \\ \text{nRd}^I(A_3) &= 0.2916, \text{nRd}^I(A_4) = 0.3800, \\ \text{nRd}^I(A_5) &= 0.2494 \end{aligned}$$

TABLE IX
 $m_S^{BRd}(\cdot)$ OBTAINED USING THE BATCH APPROXIMATION BASED ON REDUNDANCY

Focal Elements	Mass values
$A'_1 = \{\theta_1, \theta_2\}$	0.3038
$A'_2 = \{\theta_2\}$	0.3962
$A'_3 = \{\theta_3\}$	0.3000

The iterative approximation first removes the focal element A_5 because it has the least nRd value. Then we recalculate the nRd values for A_1, A_2, A_3 , and A_4 which gives us

$$\begin{aligned} \text{nRd}^{II}(A_1) &= 0.3786, \text{nRd}^{II}(A_2) = 0.3071, \\ \text{nRd}^{II}(A_3) &= 0.2779, \text{nRd}^{II}(A_4) = 0.3959 \end{aligned}$$

At Cycle II, the focal element A_3 having the least nRd value is removed. After normalization, we get the approximated BBA $m_S^{SRd}(\cdot)$ using iterative approximation as listed in Table X.

TABLE X
 $m_S^{SRd}(\cdot)$ OBTAINED USING THE BATCH APPROXIMATION BASED ON REDUNDANCY

Focal Elements	Mass values
$A'_1 = \{\theta_1, \theta_2\}$	0.2960
$A'_2 = \{\theta_2, \theta_3\}$	0.4118
$A'_3 = \{\theta_3\}$	0.2922

which is different of the result of Table IX using the batch approximation.

V. COMPARATIVE ANALYSIS

In this section, we present simulation results to compare the different BBA approximation approaches in terms of the computational cost and the closeness to the original one in average meaning. A BBA transformation with less computational cost and more closeness is preferred. To measure the closeness or the dissimilarity between different BBAs, a distance measure between BBA is used. In this work, we use Jousselme's distance [17] because it remains one of the most widely used distance of evidence. This distance is defined as

$$d_J(m_1, m_2) \triangleq \sqrt{\frac{1}{2} \cdot (m_1 - m_2)^T \mathbf{Jac} (m_1 - m_2)} \quad (8)$$

where \mathbf{Jac} is the so-called Jaccard's weighting matrix whose elements $J_{ij} = \mathbf{Jac}(A_i, B_j)$ are defined by

$$\mathbf{Jac}(A_i, B_j) = \frac{|A_i \cap B_j|}{|A_i \cup B_j|} \quad (9)$$

A BBA $m(\cdot)$ here can be considered as a column vector according to the geometric interpretation of the theory of belief functions [18]. There are also other types of distance of evidence [18]. We choose to use Jousselme's distance of evidence in this paper, because it has been proved to be a strict distance metric [19].

Our comparative analysis is based on a Monte Carlo simulation using $M = 200$ random runs. In j -th simulation

run, the BBA to approximate $m^j(\cdot)$ is randomly generated and the different approximation results $\{m_{S_i}^j(\cdot)\}$ are obtained using the different approximation approaches, where i denotes the i -th BBA approximation approach. We calculate the computational time of the original evidence combination of $m^j(\cdot) \oplus m^j(\cdot)$ with Dempster's rule, and the computation time of Dempster's combination of each approximated BBA $m_{S_i}^j(\cdot) \oplus m_{S_i}^j(\cdot)$. As stated before, there are many available BBA approximation approaches. Here we only compare our proposed approaches with $k-l-x$ method, D1 method, Summarization method because with these methods the number of the remaining focal elements and the empty set is never considered as a valid focal element (contrarily to inner approximation method which will bring troubles for making the comparisons because Jousselme distance cannot be computed if one allows to put mass on empty set because $|\emptyset| = 0$).

In our simulations, the cardinality of the FoD Θ is chosen to 3. In each random generation, there are 7 focal elements in the original BBA to approximate. The remaining number of focal elements for all the approaches used here are set to 6, 5, 4, 3, and 2. Random generation of BBA is based on Algorithm 1 [18] below.

TABLE XI
ALGORITHM 1: RANDOM GENERATION OF BBA.

Input: Θ : Frame of discernment;
 N_{max} : Maximum number of focal elements
Output: m : BBA
Generate $\mathcal{P}(\Theta)$, which is the power set of Θ ;
Generate a random permutation of $\mathcal{P}(\Theta) \rightarrow \mathcal{R}(\Theta)$;
Generate an integer between 1 and $N_{max} \rightarrow l$;
FOReach First k elements of $\mathcal{R}(\Theta)$ do
Generate a value within $[0, 1] \rightarrow m_i, i = 1, \dots, l$;
END
Normalize the vector $m = [m_1, \dots, m_l] \rightarrow m'$;
 $m(A_i) = m'_i$;

The average distance values over 200 runs between the original BBA and the approximated BBA's obtained using different approaches given different remaining focal elements' numbers are shown in Fig. 1. The average (over all runs and all numbers of remaining focal elements) computation time and distance are shown in Table XII.

TABLE XII
COMPARISONS BETWEEN DIFFERENT BBA APPROXIMATIONS IN TERMS OF TIME AND CLOSENESS

Approaches	Distance	Time (ms)
Batch-redundancy	0.1162	0.1026
Iterative-redundancy	0.1147	0.1059
$k-l-x$	0.1181	0.1073
D1	0.1718	0.1039
Summarization	0.1624	0.1034

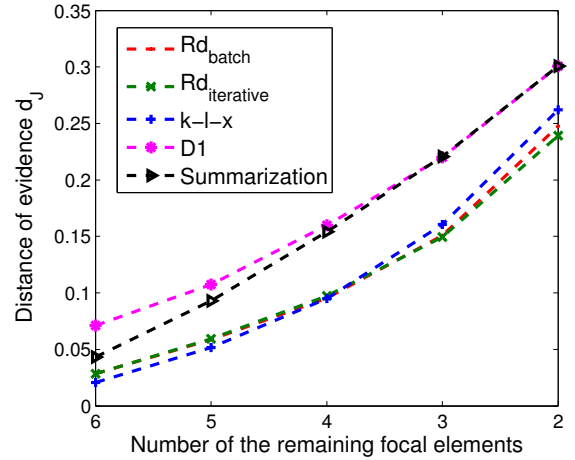


Fig. 1. Comparisons between different approximations in terms of the distance of evidence.

As we can see in Fig. 1 and in Table XII, all the method have the average computation time around 0.1 ms, which is reduced when compared with the original average computation time which is 0.2011 ms. It means that all the methods can well reduce the computational cost. Our new BBA approximation approaches based on focal element redundancy outputs BBA's which are closer to the original one when compared with other approaches. This means that our proposed approximation approaches output BBA's which are most faithful and with the least loss of information when compared with other approaches. So based on this comprehensive evaluation using two criteria including computation time and the closeness to the original BBA, our comparative analysis shows that our new methods perform better. The iterative version (having the smallest average distance) performs better than the batch version.

VI. CONCLUSION

The degree of non-redundancy of focal elements is defined, based on which, two novel BBA approximation methods have been proposed in this paper including a batch version and an iterative version. Our Monte Carlo simulation results show that these new methods can well reduce the computational cost when compared with other available approaches; at the same time, the approximated BBA's obtained using our new approaches are closer the original BBA in average, which represents the less loss of information in the approximation procedure.

In our future work, further theoretical analyses on the definition of the focal element non-redundancy or redundancy are needed, based on which, we will also attempt to design some new types of the focal element redundancy and to make additional comparison with the one used in this paper. Besides the computation time and the distance of evidence used in this paper, we will explore more comprehensive evaluation criteria of the BBA approximation approaches, and test other distance

measures of evidence [20] in our proposed approaches. This is crucial for the design of more effective approximations.

ACKNOWLEDGMENT

This work was supported by the Grant for State Key Program for Basic Research of China (973) (No. 2013CB329405), National Natural Science Foundation (No. 61203222, No. 61104214), Foundation for Innovative Research Groups of the National Natural Science Foundation of China (No. 61221063), Science and technology project of Shaanxi Province (No. 2013KJXX-46), Specialized Research Fund for the Doctoral Program of Higher Education (20120201120036), and Fundamental Research Funds for the Central Universities (No. xjj2012104, No. xjj2014122).

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