A New Belief Function Based Approach for Multi-Criteria Decision-Making Support

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***Abstract*—In this paper, we propose a new approach based on belief functions for multi-criteria decision-making (MCDM) support which is inspired by the technique for order preference by similarity to ideal solution (TOPSIS). This new approach, called BF-TOPSIS (Belief Function based TOPSIS), includes four distinct methods with different computational complexities. BF-TOPSIS offers the advantage of avoiding the problem of the choice of data normalization, of dealing with some missing scores, and of taking into account the reliability of each source (or criterion) that provides the scores of alternatives. We present results of BF-TOPSIS for different MCDM examples and discuss it robustness to rank reversal phenomena.**

Keywords: Information fusion, multi-criteria, decision-making, belief functions, TOPSIS, MCDM, DSmT.

I. Introduction

Classical Multi-Criteria Decision-Making (MCDM) consists to choose an alternative among a known set of alternatives based on their quantitative evaluations (numerical scores) obtained with respect to different criteria. The MCDM problem, although easily formulated, is not easy to solve because the scores are usually expressed in different (physical) units[1] and different scales which generally necessitates a choice of a normalization step that yields many problems, e.g. rank reversal[2] [1], [2].

Many methods have been proposed in the literature to try to solve the MCDM problem [3]–[12]. Among them, the following ones have attracted great interests in the operational research community and are widely used: AHP[3] [13], ELECTRE[4] [14], TOPSIS[5] [15], [16]. These methods however are not exempt of problems and none of them makes consensus in the MCDM community, see discussions in [14], [19]–[23]. More recently, COWA-ER[6] and Fuzzy-COWA-ER techniques based on belief functions [17], [18] have been proposed by the authors. In 2013, a new interesting MCDM method called estimator ranking vector (ERV) obtained from a Multiple-Attribute Competition Measure Matrix (MACMM) has been proposed by Yin et al. [24]. The ERV uses the "joint" information just like Pitman's closeness measure (PCM) to rank the performance of different estimators[7]. As shown by the authors, the ERV method performs better than PCM and it offers the advantage to avoid data normalization, but ERV method is not exempt from rank reversal problem.

In this work, we propose a new method based on belief functions inspired by the ERV and TOPSIS methods called BF-TOPSIS which does not involve direct data normalization, and which evaluates more precisely how much better or worse to some extent an alternative is with respect to the others. The main idea is to build basic belief assignments (BBAs) directly from the available scores values that reflect the evidences supporting each alternative on one hand, and the evidences supporting its complement on the other hand. Once all BBAs of rankings are determined for each pair *(alternative,criterion)*, then different techniques can be used to make the final ranking for MCDM support. This new approach offers the advantage to deal directly with negative, zero and positive score values, and eventually with missing score values (if any), and unreliable sources of information related to each criterion as well.

The paper is organized as follows. After a brief recall of the notations of the classical MCDM problem in II, and basics of belief functions in Section III, we present our new method of construction of BBA for the MCDM context in section IV. Four new BF-TOPSIS methods are then detailed in section V, with some examples in section VI. Conclusions, perspectives and open challenging questions are discussed in section VII.

II. Classical MCDM problem

We consider a classical[8] MCDM problem with a given set of alternatives $\mathbf{A} \triangleq \{A_1, A_2, \dots, A_M\}$ ($M > 2$), and a given set of criteria $\mathbf{C} \triangleq \{C_1, C_2, \dots, C_N\}$ ($N \geq 1$). Each alternative A_i represents a possible choice (a possible

[1] For example, in a car selection problem, fuel economy can be measured in miles per gallon (or in Km/L), and price can be expressed in different currencies (e.g. pound sterling, US dollars, or Euros), etc.

[2] A rank reversal is a change in the rank ordering if we change the structure of the MCDM problem by adding (or deleting) some alternatives. The rank reversal phenomenon's appearing in most of MCDM methods is partially due to the choice of direct data normalization as explained in [1], [2], [7], [14], [19], [20], [22], [23], [31], [33].

[3] Analytic Hierarchy Process

[4] elimination and choice translating reality

[5] technique for order preference by similarity to ideal solution

[6] Cautious Ordered Weighted Averaging with Evidential Reasoning

[7] In our general context, we speak about alternatives instead of estimators.

[8] A MCDM problem is said *classical* if all criteria C_j and all alternatives A_i are known as well as all their related scores values S_{ij}. Unclassical MCDM problems refer to problems involving incomplete or qualitative information.

decision to make). In a general context, each criterion is also characterized by a relative importance weighting factor $w_j \in [0,1]$, $j=1,\dots,N$ which are normalized by imposing the condition $\sum_j w_j = 1$. The set of normalized weighting factors is denoted by $\mathbf{w} \triangleq \{w_1, w_2, \dots, w_N\}$. The score[9] of each alternative A_i with respect to each criteria C_j is expressed by a real number S_{ij} called the score value of A_i based on C_j. We denote $\mathbf{S}$ the score[10] $M \times N$ matrix which is defined as $\mathbf{S} \triangleq [S_{ij}]$. The MCDM problem aims to select the best alternative $A^\star \in \mathbf{A}$ given $\mathbf{S}$ and the weighting factors $\mathbf{w}$ of criteria.

III. Basics of the theory of belief functions

To follow classical notations of Dempster-Shafer Theory, also called the theory of belief functions [25], we assume that the answer[11] of the problem under concern belongs to a known (or given) finite discrete frame of discernement (FoD) $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$, with $n > 1$, and where all elements of Θ are exclusive[12]. The set of all subsets of Θ (including empty set $\emptyset$ and Θ) is the power-set of Θ denoted by 2^Θ. A basic belief assignment (BBA) associated with a given source of evidence is defined [25] as the mapping $m(\cdot): 2^\Theta \to [0,1]$ satisfying $m(\emptyset) = 0$ and $\sum_{A \in 2^\Theta} m(A) = 1$. The quantity $m(A)$ is called the mass of A committed by the source of evidence. Belief and plausibility functions are respectively defined by

$$Bel(A) = \sum_{\substack{B \subseteq A \\ B \in 2^\Theta}} m(B), \quad \text{and} \quad Pl(A) = 1 - \text{Bel}(\bar{A}). \tag{1}$$

If $m(A) > 0$, A is called a focal element of $m(\cdot)$. When all focal elements are singletons then $m(\cdot)$ is called a *Bayesian BBA* [25] and its corresponding $Bel(\cdot)$ function is homogeneous to a (subjective) probability measure. The vacuous BBA, or VBBA for short, representing a totally ignorant source is defined as[13] $m_v(\Theta) = 1$.

The main challenge of the decision-maker consists at first to combine efficiently the possible multiple BBAs $m_s(\cdot)$ given by $s > 1$ distinct sources of evidence to obtain a global (combined) BBA representing the pooling of all evidences available, and then to make a final decision from it. Of course, the difficulty is reinforced if the sources are possibly unreliable. Many rules of combination have been proposed in the literature over the decades[14] to combine BBAs $m_1(\cdot)$, …, $m_s(\cdot)$ ($s \geq 2$) defined on same fusion space 2^Θ. Historically, the combination is accomplished with Dempster's rule in DST [25], which is both associative and commutative and preserves the neutrality of the VBBA. s sources of evidence are said in total conflict if $m_{1,\dots,s}(\emptyset) = 1$. In this case the combination of the sources by Dempster's rule cannot be done because of mathematical indetermination in Dempster's rule formula (due to a division by zero). While appealing, the justification and behavior of this rule have been seriously casted in doubt over the years (from both theoretical and practical standpoints [27]–[30]), and that is why a new proportional conflict redistribution principle has been proposed to combine more efficiently BBAs, see [26], Vol. 3 for justifications and examples.

IV. Construction of BBAs for MCDM problems

Let us consider an $M \times N$ score matrix[15] $\mathbf{S} = [S_{ij}]$ of a MCDM problem with $M > 1$ alternatives A_i and $N \geq 1$ criteria C_j. If $N = 1$ we are concerned with mono-criterion problem. The j-th column[16] $\mathbf{s}_j \triangleq [S_{1j}\ S_{2j}\ \dots\ S_{Mj}]^T$ of the matrix $\mathbf{S}$ corresponds to the evaluation of all alternatives A_i's based on the criterion C_j only. For each column $\mathbf{s}_j$ of $\mathbf{S}$, the values of the elements S_{ij} can take any real value expressed in same unit. The units are usually different from one criterion (i.e column) to another. For simplicity, we assume that each criterion C_j expresses a benefit so that the ranking is done according to the preference *the greater is better*[17]. For a mono-criteria problem and if one has no same multiple score values, it is easy to rank alternatives A_i directly from the score values by descending order. In MCDM problems, the direct rankings associated with different criteria can be inconsistent (different). Therefore, efficient fusion techniques must be developed in order to provide the global ranking solution to solve the MCDM problem.

To diminish rank reversal phenomenon, we want to avoid direct data normalization [23], and of course we also want to estimate the global ranking vector drawn from the score values. For this aim, we propose to estimate the ranking vector (ERV) from all evidences that support[18] or refute[19] each alternative thanks to BBAs [25]. In our approach, the FoD is the set of alternatives, that is $\Theta \triangleq \{A_1, A_2, \dots, A_M\}$. The construction of BBAs is based on the following theorem.

Theorem: We consider a criterion C_j and a score vector $\mathbf{s}^j = [S_{1j}\, S_{2j} \dots S_{Mj}]^T$ with $S_{ij} \in \mathbb{R}$ associated to the FoD $\Theta = \{A_1, A_2, \dots, A_M\}$. For any proposition A_i of Θ and its positive and negative evidence supports defined by

$$Sup_j(A_i) \triangleq \sum_{k \in \{1,\dots M\} | S_{kj} \leq S_{ij}} |S_{ij} - S_{kj}| \tag{2}$$

$$Inf_j(A_i) \triangleq - \sum_{k \in \{1,\dots M\} | S_{kj} \geq S_{ij}} |S_{ij} - S_{kj}| \tag{3}$$

If $A^j_{\text{max}} \triangleq \max_i Sup_j(A_i)$ and $A^j_{\text{min}} \triangleq \min_i Inf_j(A_i)$ are different from zero, the following inequality holds

$$\frac{Sup_j(A_i)}{A^j_{\text{max}}} \leq 1 - \frac{Inf_j(A_i)}{A^j_{\text{min}}}. \tag{4}$$

[9]Depending on the context of the MCDM problem, the score can be interpreted either as a cost/expense or as a reward/benefit. In the sequel, by convention and without loss of generality we will interpret the score as a reward having monotonically increasing preference. Thus, the best alternative w.r.t. a given criteria will be the one providing the highest reward/benefit.

[10]also called benefit or payoff matrix.

[11]i.e. the solution, or the decision to take.

[12]This is so-called Shafer's model of FoD [26].

[13]The complete ignorance is denoted Θ in Shafer's book [25].

[14]see [26], Vol. 2 for a detailed list of fusion rules.

[15]In some references, the score matrix is also called the decision matrix because it is the matrix from which the decision must be taken.

[16]By convention, $\mathbf{x}^T$ denotes the transpose of $\mathbf{x}$.

[17]Of course, a similar presentation can be done with the preference *the lower is better* if the criterion corresponds to a loss.

[18]We call it *positive* evidence.

[19]We call it *negative* evidence.

Proof: Given in Appendix 1.

$Sup_j(A_i)$ is called the *positive support* of A_i because it measures how much A_i is better than other alternatives according to criterion C_j, and $Inf_j(A_i)$ is called the *negative support* of A_i because it measures how much A_i is worse than other alternatives according to criterion C_j. The length of $[0, Sup_j(A_i)]$ measures the support in favor of A_i as being the best alternative with respect to all other ones, and the length of $[Inf_j(A_i), 0]$ measures the support against A_i based on the criterion C_j.

Thanks to the previous theorem, the construction of BBAs for each alternative A_i based on the score vector $\mathbf{s}_j$ relative to a criteria C_j can be done as follows:

- **Step 1**: For each A_i, evaluate the evidential supports $Sup_j(A_i)$ and $Inf_j(A_i)$ by (2) and (3).
- **Step 2**: Compute the min value A^j_{min} of $Inf_j(A_i)$, and the max value A^j_{max} of $Sup_j(A_i)$, $\forall i = 1, \ldots, M$.
- **Step 3**: We define the belief of A_i as the evidential support of hypothesis "A_i is better than its competitors $\bar{A}_i$" by taking

$$Bel_{ij}(A_i) \triangleq Sup_j(A_i)/A^j_{\text{max}} \tag{5}$$

Similarly, we define the belief of $\bar{A}_i$ (i.e. the complement of A_i in Θ) by taking

$$Bel_{ij}(\bar{A}_i) \triangleq Inf_j(A_i)/A^j_{\text{min}} \tag{6}$$

By construction, $Bel_{ij}(A_i)$ and $Bel_{ij}(\bar{A}_i)$ belong to $[0, 1]$, and thanks to the theorem this belief function construction is perfectly consistent. More specificly, the inequality $Bel_{ij}(A_i) \leq Pl_{ij}(A_i)$ is satisfied with the plausibility of A_i defined by [25]

$$Pl_{ij}(A_i) = 1 - Bel_{ij}(\bar{A}_i) = 1 - \frac{Inf_j(A_i)}{A^j_{\text{min}}} \tag{7}$$

If we do not have evidential support of A_i (i.e. $A^j_{\text{max}} = 0$) then we take naturally $Bel_{ij}(A_i) = 0$, and if we do not have evidential support of $\bar{A}_i$ (i.e. $A^j_{\text{min}} = 0$) then we take $Bel_{ij}(\bar{A}_i) = 0$. The belief interval of choice A_i based on the criterion C_j and based on the local comparisons with respect to its alternatives is given by[20]

$$[Bel_{ij}(A_i); Pl_{ij}(A_i)] \triangleq [\frac{Sup_j(A_i)}{A^j_{\text{max}}}, 1 - \frac{Inf_j(A_i)}{A^j_{\text{min}}}] \tag{8}$$

From this belief interval, one computes it corresponding BBA $m_{ij}(\cdot)$ defined on the power set of Θ_i by taking for $i = 1, \ldots, M$

$$m_{ij}(A_i) = Bel_{ij}(A_i) \tag{9}$$

$$m_{ij}(\bar{A}_i) = Bel_{ij}(\bar{A}_i) = 1 - Pl_{ij}(A_i) \tag{10}$$

$$m_{ij}(A_i \cup \bar{A}_i) = Pl_{ij}(A_i) - Bel_{ij}(A_i) \tag{11}$$

At the output of step 3 of our BBA construction, one has an $M \times N$ BBA matrix $\mathbf{M} = [m_{ij}(\cdot)]$, where each element $m_{ij}(\cdot)$ of the BBA matrix corresponds in fact to a triplet $(m_{ij}(A_i), m_{ij}(\bar{A}_i), m_{ij}(A_i \cup \bar{A}_i))$ defined by the formulae (9)–(11).

For a given criterion C_j, if all score values S_{ij} are the same, we get for all A_i, $Sup_j(A_i) = Inf_j(A_i) = 0$. Hence, no evidence support can be drawn for or against A_i and we take $Bel_{ij}(A_i) = Bel_{ij}(\bar{A}_i) = 0$, so that $Pl(A_i) = 1$ and consequently $m_{ij}(A_i) = m_{ij}(\bar{A}_i) = 0$ and $m_{ij}(A_i \cup \bar{A}_i) = 1$. In this very particular case, the fusion of the M BBA $m_{ij}(\cdot)$ provides a combined BBA $m_j(\cdot)$ equals to the vacuous belief assignment over the refined frame Θ, that is $m_j(A_1 \cup \ldots \cup A_M) = 1$ from which naturally no specific ranking can be drawn, which makes a perfect sense.

This approach of BBA construction is very interesting for applications because it is invariant to the bias and scaling effects of score values[21]. Also, it allows us to model our lack of evidence (if any) with respect to an (or several) alternative(s) when their corresponding score values are missing for any reason. For example, if a numerical value S_{ij} is missing or indeterminate, then we will use the vacuous belief assignment $m_{ij}(A_i \cup \bar{A}_i) = 1$ because we have no evidential support for A_i and for $\bar{A}_i$. In our BF-TOPSIS methods presented in the next section, we will use only the score matrix and the importance weighting factor related to each criteria, but it is worth noting that one can also (as a pre-processing step) discount the BBA $m_{ij}(\cdot)$ by a reliability factor using the classical Shafer's discounting method if one wants to express some doubts on the reliability of $m_{ij}(\cdot)$, see [25] for reliability discounting formulae.

V. New BF-TOPSIS methods

In the previous Section, we have presented an appealing method (invariant to bias and scale effects on score values) to build the BBA matrix from any general score matrix. The major concern of MCDM problem is now how to deal with these elementary BBAs $m_{ij}(\cdot)$, and how to establish a final ranking from them? Also if possible, how to avoid the rank reversal problem?

Our first idea to solve this MCDM problem was to use directly some rules of combination, mainly Dempster's rule [25], or PCR6 rule [26] to combine all the BBAs $m_{ij}(\cdot)$ together to obtain a global basic belief assignment $m(\cdot)$ from which a final ranking of alternatives would be drawn. Unfortunately, such a very simple idea fails to provide correct ranking result, even in the simplest mono-criterion case (see the example 1 in the next section), because all the BBAs $m_{ij}(\cdot)$ for a given criterion C_j are not independent since they are built on same set of score values $\{S_{ij}, i = 1, ..., M\}$, so that DS and PCR6 rules theoretically should not be used. Of course, such a global fusion approach would also require too high computational complexity and resources to solve high dimension MCDM problems.

[20] assuming that $A^j_{\text{max}} \neq 0$ and $A^j_{\text{min}} \neq 0$. If $A^j_{\text{max}} = 0$ then $Bel_{ij}(A_i) = 0$, and if $A^j_{\text{min}} = 0$ then $Pl_{ij}(A_i) = 1$.

[21] More specificly, if for a given criterion C_j, the score values S_{ij} of alternatives are replaced by $S'_{ij} = a \cdot S_{ij} + b$, with a scale factor $a > 0$ and a bias $b \in \mathbb{R}$, then the corresponding BBAs $m_{ij}(\cdot)$ and $m'_{ij}(\cdot)$ built by our method are equal.

Therefore, in this paper, we propose four MCDM methods inspired by the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) that uses belief masses $m_{ij}(\cdot)$ as defined in the previous section. We call them the Belief-based TOPSIS methods, and we denote them as BF-TOPSIS1, BF-TOPSIS2, BF-TOPSIS3 and BF-TOPSIS4. Our four BF-TOPSIS methods are somehow inspired by ERV method [24] which has proved that data normalization can be avoided for MCDM, and also by the TOPSIS [15], [16] method for borrowing the idea of using the best and worst ideal solutions. Our BF-TOPSIS methods are however totally new in the way of processing information to obtain the final ranking of alternatives. Our BF-TOPSIS methods present different complexity of implementation and robustness to rank reversal. These methods are detailed below:

A. BF-TOPSIS1 method

Step 1: From the score matrix $\mathbf{S}$, compute BBAs $m_{ij}(A_i)$ $m_{ij}(\bar{A}_i)$, and $m_{ij}(A_i \cup \bar{A}_i)$ using (9)–(11).
Step 2: For each alternative A_i, compute the Belief Interval-based Euclidean distance[22] $d^E_{BI}(m_{ij}, m^{\text{best}}_{ij})$ (defined in [43]) between $m_{ij}(\cdot)$ and the best ideal BBA defined by $m^{\text{best}}_{ij}(A_i) \triangleq 1$, and the distances $d^E_{BI}(m_{ij}, m^{\text{worst}}_{ij})$ between $m_{ij}(\cdot)$ and the worst ideal BBA defined by $m^{\text{worst}}_{ij}(\bar{A}_i) \triangleq 1$.
Step 3: Compute the weighted average of $d^E_{BI}(m_{ij}, m^{\text{best}}_{ij})$ values with relative importance weighting factor w_j of criteria C_j. Similarly, compute the weighted average of $d^E_{BI}(m_{ij}, m^{\text{worst}}_{ij})$ values. More specificly, compute

$$d^{\text{best}}(A_i) \triangleq \sum_{j=1}^{N} w_j \cdot d^E_{BI}(m_{ij}, m^{\text{best}}_{ij}) \tag{12}$$

$$d^{\text{worst}}(A_i) \triangleq \sum_{j=1}^{N} w_j \cdot d^E_{BI}(m_{ij}, m^{\text{worst}}_{ij}) \tag{13}$$

Step 4: The relative closeness of the alternative A_i with respect to ideal best solution A^{best} is then defined by

$$C(A_i, A^{\text{best}}) \triangleq \frac{d^{\text{worst}}(A_i)}{d^{\text{worst}}(A_i) + d^{\text{best}}(A_i)} \tag{14}$$

Because $d^{\text{best}}(A_i) \geq 0$ and $d^{\text{worst}}(A_i) \geq 0$, then $C(A_i, A^{\text{best}}) \in [0, 1]$. If $d^{\text{best}}(A_i) = 0$, it means that alternative A_i coincides with the ideal best solution and thus $C(A_i, A^{\text{best}}) = 1$ (the relative closeness of A_i with respect to A^{best} is maximal). Contrariwise, if $d^{\text{worst}}(A_i) = 0$, it means that alternative A_i coincides with the ideal worst solution and thus $C(A_i, A^{\text{best}}) = 0$ (the relative closeness of A_i with respect to A^{best} is minimal).
Step 5: (Preference ranking) The set of alternatives is preference ranked according to the descending order of $C(A_i, A^{\text{best}}) \in [0, 1]$, where a larger $C(A_i, A^{\text{best}})$ value means a better alternative (or higher preference).

[22]The justification of this distance with respect to other existing ones (Jousselme's, Tessem's, etc.) has been given in [43].

B. BF-TOPSIS2 method

Steps 1, 2 and 5 are the same as in BF-TOPSIS1. Only steps 3 and 4 differ as follows:
Step 3: For each criteria C_j, compute the relative closeness of the alternative A_i w.r.t. ideal best solution A^{best} by

$$C_j(A_i, A^{\text{best}}) \triangleq \frac{d^E_{BI}(m_{ij}, m^{\text{worst}}_{ij})}{d^E_{BI}(m_{ij}, m^{\text{worst}}_{ij}) + d^E_{BI}(m_{ij}, m^{\text{best}}_{ij})} \tag{15}$$

Step 4: The relative closeness of the alternative A_i with respect to ideal best solution A^{best} is then defined by the weighted average of $C_j(A_i, A^{\text{best}})$ that is

$$C(A_i, A^{\text{best}}) \triangleq \sum_{j=1}^{N} w_j \cdot C_j(A_i, A^{\text{best}}) \tag{16}$$

C. BF-TOPSIS3 method

This third method is more complicate to implement because it requires the direct fusion of N BBAs $m_{ij}(\cdot)$ for a given alternative index i with the PCR6 rule of combination [26], [32]. The Step 1, 4 and 5 are the same as in BF-TOPSIS1. Only the Step 2 and 3 differ as follows:
Step 2: For each alternative A_i and for the set of BBAs $m_{ij}(\cdot)$ and criteria importance factors w_j, compute with the PCR6 combination rule[23] [32], the fused BBA $m^{\text{PCR6}}_i(\cdot)$.
Step 3: Compute the Belief Interval-based Euclidean distances (see [43]) $d^E_{BI}(m^{\text{PCR6}}_i, m^{\text{best}}_i)$ between $m^{\text{PCR6}}_i(\cdot)$ and the ideal best BBA defined by $m^{\text{best}}_i(A_i) \triangleq 1$, and the distances $d^E_{BI}(m^{\text{PCR6}}_i, m^{\text{worst}}_i)$ between $m^{\text{PCR6}}_i(\cdot)$ and the ideal worst BBA defined by $m^{\text{worst}}_i(\bar{A}_i) \triangleq 1$. More specificly, compute

$$d^{\text{best}}(A_i) \triangleq d^E_{BI}(m^{\text{PCR6}}_i, m^{\text{best}}_i) \tag{17}$$

$$d^{\text{worst}}(A_i) \triangleq d^E_{BI}(m^{\text{PCR6}}_i, m^{\text{worst}}_i) \tag{18}$$

D. BF-TOPSIS4 method

BF-TOPSIS4 method is similar to BF-TOPSIS3 except that we use the more complicate ZPCR6 fusion rule which is a modified version of PCR6 rule taking into account Zhang's degree of intersection of focal elements in the conjunctive consensus operator. ZPCR6 rule is explained in details with examples in [34].

VI. Examples

In this section, we provide the results of our new BF-TOPSIS methods when applied to different examples.

A. Example 1 (Mono-criterion)

Let's consider a criterion[24] C_j and seven alternatives A_i, $(i = 1, \ldots, 7)$ with the following corresponding score values $\mathbf{s}_j = [10, 20, -5, 0, 100, -11, 0]^T$. The direct ranking with the preference "greater is better" yields[25] $A_5 \succ A_2 \succ A_1 \succ (A_4 \sim A_7) \succ A_3 \succ A_6$. Because A_4 and A_7 have same

[23]The choice of PCR6 rule instead of DS rule for taking into account importance discounting has been justified in [32]. The weighted average fusion rule has been also tested but it can provide rank reversal.

[24]In this example $j = 1$ because we consider a mono-criterion example.

[25]where the symbol $\succ$ means *better than (or is preferred to).*

score values, then both ranking vectors $\mathbf{r}_j = [5, 2, 1, 4, 7, 3, 6]$ and $\mathbf{r}'_j = [5, 2, 1, 7, 4, 3, 6]$ are admissible ranking solutions. In applying formulas (9)–(11), we get the following set of BBAs:

Table I
BBAs CONSTRUCTED FROM SCORE VALUES.

	$m_{ij}(A_i)$	$m_{ij}(\bar{A}_i)$	$m_{ij}(A_i \cup \bar{A}_i)$
A_1	0.0955	0.5236	0.3809
A_2	0.1809	0.4188	0.4003
A_3	0.0102	0.8115	0.1783
A_4	0.0273	0.6806	0.2921
A_5	1.0000	0	0
A_6	0	1.0000	0
A_7	0.0273	0.6806	0.2921

It can be verified that the combination of these BBAs by Dempster's rule yields $m_{DS}(A_5) = 1$ from which no ranking of alternatives can be inferred, but A_5 is the best one. The combination of these BBAs by PCR6 yields $m_{PCR6}(A_1) = 0.0063$, $m_{PCR6}(A_2) = 0.0255$, $m_{PCR6}(A_4) = 0.0014$, $m_{PCR6}(A_5) = 0.9665$ and $m_{PCR6}(A_1 \cup A_2 \cup A_3 \cup A_5 \cup A_6 \cup A_7) = 0.0003$. If we sort the alternatives by decreasing order of belief or plausibility values computed from $m_{PCR6}(\cdot)$, we will get the following preferences order $A_5 \succ A_2 \succ A_1 \succ A_4 \succ (A_3 \sim A_6 \sim A_7)$, which is better than the *preferences order* result obtained with Dempster's rule. Unfortunately, it is still not fully consistent with the direct ranking.

This example shows that Dempster's and PCR6 rules cannot infer the correct ranking even in a simple mono-criterion example. The main reason is because the BBAs to combine are not built from independent sources of evidence, so that Dempster's and PCR6 rules should not apply. This simple example motivates the development of our new BF-TOPSIS methods based on BF and TOPSIS for MCDM.

Using BF-TOPSIS methods[26], we get the following distance values, and the relative closeness measures:

Table II
DISTANCES AND RELATIVE CLOSENESS MEASURES.

	$d^E_{BI}(m_{ij}, m^{\text{best}}_{ij})$	$d^E_{BI}(m_{ij}, m^{\text{worst}}_{ij})$	$C(A_i, A^{\text{best}})$
A_1	0.7380	0.0940	0.1130
A_2	0.6676	0.1615	0.1948
A_3	0.8112	0.0214	0.0257
A_4	0.7954	0.0405	0.0485
A_5	0	0.8229	1.0000
A_6	0.8229	0	0
A_7	0.7954	0.0405	0.0485

In sorting $C(A_i, A^{\text{best}})$ values by the descending order, we get the correct preferences order

$$A_5 \succ A_2 \succ A_1 \succ (A_4 \sim A_7) \succ A_3 \succ A_6.$$

Let's modify a little bit this example by changing the score of A_5 from 100 to 21 (21 is now very close to the score of A_2). So, if we start with modified score values $\mathbf{s}_j = [10, 20, -5, 0, 21, -11, 0]^T$, the result of the direct ranking method remains unchanged with respect to previous one because A_5 still has the highest score among all scores of alternatives. In applying BF-TOPSIS methods, we get now

Table III
DISTANCES AND RELATIVE CLOSENESS MEASURES.

	$d^E_{BI}(m_{ij}, m^{\text{best}}_{ij})$	$d^E_{BI}(m_{ij}, m^{\text{worst}}_{ij})$	$C(A_i, A^{\text{best}})$
A_1	0.4067	0.4185	0.5072
A_2	0.0435	0.7796	0.9472
A_3	0.7741	0.0560	0.0675
A_4	0.6988	0.1315	0.1584
A_5	0	0.8229	1.0000
A_6	0.8229	0	0
A_7	0.6988	0.1315	0.1584

In sorting $C(A_i, A^{\text{best}})$ values by the descending order, we see that we still get the correct same ranking

$$A_5 \succ A_2 \succ A_1 \succ (A_4 \sim A_7) \succ A_3 \succ A_6.$$

Of course A_5 is still the best alternative to select because $C(A_5, A^{\text{best}}) = 1$, but A_2 can now be considered also as very close to the best solution also (with good confidence because $C(A_2, A^{\text{best}}) = 0.9472$), which makes perfectly sense in this modified example. So these new BF-TOPSIS methods provide the correct preferences order in the mono-criterion case, and they are able to capture somehow how much we must be confident in the ranks of the ranking result.

B. Example 2 (Non informative case in mono-criterion)

If all score values are equal, then one gets from the BBA construction $m_{ij}(A_i \cup \bar{A}_i) = 1$ for each alternative index value i. Consequently with BF-TOPSIS methods, all relative closeness values $C(A_i, A^{\text{best}})$ are the same, and therefore no specific choice of a specific alternative with respect to the other ones can be drawn, which is perfectly normal in such degenerate (non informative) case.

C. Example 3 (Multi-criteria and rank reversal)

This interesting example is drawn from [22] (Table 7, p. 1224). We consider the following set of alternatives (the FoD) $\Theta \triangleq \{A_1, A_2, A_3, A_4, A_5\}$, four criteria with importance weighting vector $\mathbf{w} = [1/6, 1/3, 1/3, 1/6]$, and the score matrix

$$\mathbf{S} = \begin{bmatrix} 36 & 42 & 43 & 70 \\ 25 & 50 & 45 & 80 \\ 28 & 45 & 50 & 75 \\ 24 & 40 & 47 & 100 \\ 30 & 30 & 45 & 80 \end{bmatrix}$$

As shown in [22] (Table 9, p. 1226) and in the following tables, the (classical) TOPSIS method suffers from rank reversal. As seen in next Tables, BF-TOPSIS1 and BF-TOPSIS2 give same preference order results and they also suffer from rank reversal in this example. BF-TOPSIS3 and BF-TOPSIS4 preserve the preference order (yielding no rank reversal in this example). It is worth noting that the simple weighted BBA averaging rule instead of PCR6 and ZPCR6 rules has also been tested, and it gives same result as with BF-TOPSIS1 (i.e., rank reversal).

[26]in mono-criterion case, all BF-TOPSIS methods are equivalent because there is no need of making fusion.

Table IV
TOPSIS RESULT FOR EXAMPLE 3.

Set of alternatives	Classical TOPSIS result
$\{A_1, A_2, A_3\}$	$A_3 \succ A_2 \succ A_1$
$\{A_1, A_2, A_3, A_4\}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$\{A_1, A_2, A_3, A_4, A_5\}$	$A_3 \succ A_2 \succ A_4 \succ A_1 \succ A_5$

Table V
BF-TOPSIS1 AND BF-TOPSIS2 RESULTS FOR EXAMPLE 3.

Set of alternatives	BF-TOPSIS1 & BF-TOPSIS2 results
$\{A_1, A_2, A_3\}$	$A_2 \succ A_3 \succ A_1$
$\{A_1, A_2, A_3, A_4\}$	$A_3 \succ A_2 \succ A_4 \succ A_1$
$\{A_1, A_2, A_3, A_4, A_5\}$	$A_3 \succ A_2 \succ A_4 \succ A_1 \succ A_5$

Table VI
BF-TOPSIS3 AND BF-TOPSIS4 RESULTS FOR EXAMPLE 3.

Set of alternatives	BF-TOPSIS3 & BF-TOPSIS4 results
$\{A_1, A_2, A_3\}$	$A_3 \succ A_2 \succ A_1$
$\{A_1, A_2, A_3, A_4\}$	$A_3 \succ A_2 \succ A_4 \succ A_1$
$\{A_1, A_2, A_3, A_4, A_5\}$	$A_3 \succ A_2 \succ A_4 \succ A_1 \succ A_5$

D. Example 4 (Multi-criteria for car selection)

Let's consider a more concrete car selection problem. We consider a set of four cars $\{A_1, A_2, A_3, A_4\}$ as follows:

- A_1 = TOYOTA YARIS 69 VVT-i Tendance;
- A_2 = SUZUKI SWIFT MY15 1.2 VVT So'City;
- A_3 = VOLKSWAGEN POLO 1.0 60 Confortline;
- A_4 = OPEL CORSA 1.4 Turbo 100 ch Start/Stop Edition;

We consider the following five criteria for making the choice of the best car to buy:

- C_1 is the price (in €);
- C_2 is fuel consumption (in L/km);
- C_3 is the CO2 emission (in g/km);
- C_4 is the fuel tank volume (in L);
- C_5 is the trunk volume (in L);

The score matrix $\mathbf{S} = [S_{ij}]$ is built from information extracted from car-makers technical characteristics available on the world wide web[27]. For the chosen cars, the corresponding score matrix is given by

$$\mathbf{S} = \begin{bmatrix} 15000 & 4.3 & 99 & 42 & 737 \\ 15290 & 5.0 & 116 & 42 & 892 \\ 15350 & 5.0 & 114 & 45 & 952 \\ 15490 & 5.3 & 123 & 45 & 1120 \end{bmatrix}$$

For criteria C_1, C_2 and C_3 smaller is better. For criteria C_4 and C_5 larger is better. To make the preference order homogeneous in the score matrix, we multiply values of columns C_1, C_2 and C_3 by -1 so that our MCDM problem is described by a modified score matrix with homogeneous preference order ("larger is better") for each column.

For simplicity, the importance $imp(C_j)$ of each criteria C_j takes a value in $\{1, 2, 3, 4, 5\}$, where 1 means *the least important*, and 5 means *the most important*. In this example we take $imp(C_1) = 5$, $imp(C_2) = 4$, $imp(C_3) = 4$, $imp(C_4) = 1$ and $imp(C_5) = 3$ which means that the price of a car (criteria C_1) is the most important criteria for us, and the volume of fuel tank (criteria C_4) is the least important one. From these importances values and after normalization, we get the following vector of relative weights of criteria

$$\mathbf{w} = [\frac{5}{17} \quad \frac{4}{17} \quad \frac{4}{17} \quad \frac{1}{17} \quad \frac{3}{17}]$$

Intuitively, based on the score matrix $\mathbf{S}$ and importances of criteria, the choice of car A_1 is anticipated to be the best choice because the three most important criteria meet clearly their best values for car A_1. If we apply the classical TOPSIS [15], [16], one gets $A_4 \succ A_1 \succ A_3 \succ A_2$, that is A_4 would be the best car to buy, whereas A_2 would be the worst one. This result is quite surprising and counter-intuitive because in this very simple and concrete example A_1 should have been selected as the best choice without ambiguity by any rational decision-maker. With BF-TOPSIS methods (1, 2, 3 and 4) we get a more satisfactory preference order $A_1 \succ A_3 \succ A_2 \succ A_4$, which is also what we get with AHP method [13], or with the SAW (Simple Additive Weighting) method [3], [22] in this example. The unexpected classical TOPSIS results is due to the choice of normalization of scores [15], and the problem can be (not always) circumvented by changing the normalization procedure, see discussion in [22]. The choice of a normalization procedure for TOPSIS is always an open challenging question. This problem of choice of normalization is avoided in BF-TOPSIS methods because no direct score normalization is necessary for its implementation.

E. Example 5 (Multi-criteria for student evaluation)

We consider four students (i.e., alternatives A_1, A_2, A_3 and A_4) and ten attributes (i.e., criteria C_j, $j = 1, \dots, 10$) with equal attribute's weighting factor $w_j = 1/10$ ($j = 1, \dots, 10$), and the scores given in table below.

Table VII
ATTRIBUTES (SCORES) OF THE STUDENTS.

	A_1	A_2	A_3	A_4
$C_1 \triangleq$ Math	90	80	70	60
$C_2 \triangleq$ Arts	90	80	70	60
$C_3 \triangleq$ English	90	80	70	60
$C_4 \triangleq$ Geography	90	80	70	60
$C_5 \triangleq$ Physics	90	80	70	75
$C_6 \triangleq$ Music	90	80	70	95
$C_7 \triangleq$ History	80	90	70	85
$C_8 \triangleq$ Chemistry	80	90	70	85
$C_9 \triangleq$ Biology	80	90	70	85
$C_{10} \triangleq$ Long jump	3.5m	3.7m	4.0m	3.6m

At first, we only rank the comprehensive quality of the first three students A_1, A_2 and A_3. The tables VIII and IX present ranking vectors (with three digits approximation), and preference orders obtained with Multiple-Attribute Competition Measure matrix[28] of ERV method [24], and the BF-TOPSIS methods.

[27] http://www.choisir-sa-voiture.com

[28] We assume here that all attributes (criteria) have the same weighting factor 1/10, see [24] for details.

Table VIII
ERV & BF-TOPSIS RESULTS FOR EXAMPLE 5 (CASE WITH 3 STUDENTS).

Methods	Ranking vectors	Preferences orders
ERV	$[0.748, 0.636, 0.188]$	$A_1 \succ A_2 \succ A_3$
BF-TOPSIS1	$[0.729, 0.594, 0.100]$	$A_1 \succ A_2 \succ A_3$
BF-TOPSIS2	$[0.731, 0.597, 0.100]$	$A_1 \succ A_2 \succ A_3$
BF-TOPSIS3	$[0.803, 0.736, 0.100]$	$A_1 \succ A_2 \succ A_3$
BF-TOPSIS4	$[0.803, 0.736, 0.100]$	$A_1 \succ A_2 \succ A_3$

If we rank the comprehensive quality of the four students A_1, A_2, A_3 and A_4, we get the following results:

Table IX
ERV & BF-TOPSIS RESULTS FOR EXAMPLE 5 (CASE WITH 4 STUDENTS).

Methods	Ranking vectors	Preferences orders
ERV	$[0.620, 0.636, 0.248, 0.386]$	$A_2 \succ A_1 \succ A_4 \succ A_3$
BF-TOPSIS1	$[0.675, 0.620, 0.195, 0.320]$	$A_1 \succ A_2 \succ A_4 \succ A_3$
BF-TOPSIS2	$[0.677, 0.622, 0.194, 0.319]$	$A_1 \succ A_2 \succ A_4 \succ A_3$
BF-TOPSIS3	$[0.766, 0.775, 0.158, 0.288]$	$A_2 \succ A_1 \succ A_4 \succ A_3$
BF-TOPSIS4	$[0.766, 0.775, 0.158, 0.288]$	$A_2 \succ A_1 \succ A_4 \succ A_3$

In this example, one sees that ERV, BF-TOPSIS3 and BF-TOPSIS4 methods suffer from rank reversal, whereas BF-TOPSIS1 and BF-TOPSIS2 do not suffer from rank reversal.

VII. CONCLUSIONS

In this work, we have presented four new MCDM methods (BF-TOPSIS1–BF-TOPSIS4) inspired by TOPSIS and based on belief functions. We have shown that it is possible to establish basic belief assignments (BBAs) from any score values with respect to a criterion to get a mass for each alternative, its complement and uncertainty. With BF-TOPSIS methods, we can compute from these BBAs the relative closeness of each alternative to the ideal best and worst solutions for establishing the ranking to get the final preference order of alternatives. These new methods do not need direct score value normalization, and they can deal with any real score values at same time (negative, zero or positive). If necessary, these methods can also deal easily with missing score values and with the reliability of the sources as well. They are invariant to scales and bias effects in score values. As all other MCDM methods developed so far, these new BF-TOPSIS methods suffer from rank reversal in some cases, which are very difficult to anticipate. As shown in our examples, some BF-TOPSIS methods are robust to rank reversal in some cases, while others are robust in other cases. If rank reversal is really crucial in the MCDM problem under concern, we recommend to test the panel of BF-TOPSIS methods, and choose the least complex one which is robust to rank reversal. BF-TOPSIS1 and BF-TOPSIS2 are easy to implement, whereas BF-TOPSIS3 and BF-TOPSIS4 are more complicate and time consuming. In future research work, we would like to establish the theoretical conditions that any MCDM method must satisfy to not suffer from rank reversal. This remains a fundamental open challenging question for operational research and information fusion communities.

APPENDIX 1

Proof of Theorem: Proving the following inequality

$$\frac{Sup_j(A_i)}{A^j_{\max}} \leq 1 - \frac{Inf_j(A_i)}{A^j_{\min}} \tag{19}$$

for any A_i, is equivalent to prove

$$\frac{Inf_j(A_i)}{A^j_{\min}} + \frac{Sup_j(A_i)}{A^j_{\max}} \leq 1. \tag{20}$$

For proving (20), we sort the score values S_{ij} is descending order. For notation convenience, we denote them B_1, B_2, $\ldots B_M$, and we have $B_1 \geq B_2 \geq \ldots \geq B_M$, so that $(B_k - B_s) \geq 0$ for $k < s$, and $(B_s - B_k) \geq 0$ for $k > s$ for any particular value B_s, where $1 \leq s \leq M$. Moreover, one has also $Inf_j(B_s) = -\sum_{k=1}^{s}(B_k - B_s)$ and $A^j_{\min} = -\sum_{k=1}^{M}(B_k - B_M)$, and $Sup_j(B_s) = \sum_{k=s+1}^{M}(B_s - B_k)$ and $A^j_{\max} = \sum_{k=1}^{M}(B_1 - B_k)$. Proving (20), is then equivalent to prove the following inequality for any B_s

$$\frac{-\sum_{k=1}^{s}(B_k - B_s)}{-\sum_{k=1}^{M}(B_k - B_M)} + \frac{\sum_{k=s+1}^{M}(B_s - B_k)}{\sum_{k=1}^{M}(B_1 - B_k)} \leq 1. \tag{21}$$

Because $\sum_{k=1}^{s}(B_k - B_M) \leq \sum_{k=1}^{M}(B_k - B_M)$, then the first fraction[29] $F_1 \triangleq \sum_{k=1}^{s}(B_k - B_s)/\sum_{k=1}^{M}(B_k - B_M)$ of l.h.s of (21) is bounded by

$$0 \leq F_1 \leq \frac{\sum_{k=1}^{s}(B_k - B_s)}{\sum_{k=1}^{s}(B_k - B_M)} \tag{22}$$

The upper bound of (22) can be rewritten as

$$\begin{aligned}\frac{\sum_{k=1}^{s}(B_k - B_s)}{\sum_{k=1}^{s}(B_k - B_M)} &= \frac{\sum_{k=1}^{s}(B_1 - B_s - B_1 + B_k)}{\sum_{k=1}^{s}(B_1 - B_M - B_1 + B_k)} \\ &= \frac{\sum_{k=1}^{s}(B_1 - B_s) - \sum_{k=1}^{s}(B_1 - B_k)}{\sum_{k=1}^{s}(B_1 - B_M) - \sum_{k=1}^{s}(B_1 - B_k)} \\ &= \frac{s \cdot c - a}{s \cdot d - a}\end{aligned} \tag{23}$$

where $a \triangleq \sum_{k=1}^{s}(B_1 - B_k)$, $c \triangleq B_1 - B_s$, and $d \triangleq B_1 - B_M$.

Because $s \cdot c - a \geq 0$, $s \cdot d - a > 0$, $d > c$, $a \geq 0$ and $s > 0$, then one has the following inequalities satisfied

$$\begin{aligned}a \cdot d \geq a \cdot c &\Rightarrow -a \cdot d \leq -a \cdot c \\ &\Rightarrow s \cdot d \cdot c - a \cdot d \leq s \cdot d \cdot c - a \cdot c \\ &\Rightarrow (s \cdot c - a) \cdot d \leq (s \cdot d - a) \cdot c \\ &\Rightarrow \frac{s \cdot c - a}{s \cdot d - a} \leq \frac{c}{d}\end{aligned} \tag{24}$$

Therefore, $F_1 \leq \frac{c}{d}$, or equivalently

$$\frac{\sum\limits_{k=1}^{s}(B_k - B_s)}{\sum\limits_{k=1}^{M}(B_k - B_M)} \leq \frac{B_1 - B_s}{B_1 - B_M} \tag{25}$$

Because $\sum_{k=s+1}^{M}(B_1 - B_k) \leq \sum_{k=1}^{M}(B_1 - B_k)$, then the second fraction $F_2 \triangleq \sum_{k=s+1}^{M}(B_s - B_k)/\sum_{k=1}^{M}(B_1 - B_k)$ of l.h.s of (21) is bounded by

$$0 \leq F_2 \leq \frac{\sum\limits_{k=s+1}^{M}(B_s - B_k)}{\sum\limits_{k=s+1}^{M}(B_1 - B_k)} \tag{26}$$

[29]after the simplification by -1 in the numerator and denominator of the fraction.

The upper bound of (26) can be rewritten as

$$\frac{\sum_{k=s+1}^{M}(B_s - B_k)}{\sum_{k=s+1}^{M}(B_1 - B_k)} = \frac{\sum_{k=s+1}^{M}(B_s - B_M - B_k + B_M)}{\sum_{k=s+1}^{M}(B_1 - B_M - B_k + B_M)}$$
$$= \frac{\sum_{k=s+1}^{M}(B_s - B_M) - \sum_{k=s+1}^{M}(B_k - B_M)}{\sum_{k=s+1}^{M}(B_1 - B_M) - \sum_{k=s+1}^{M}(B_k - B_M)}$$
$$= \frac{(M-s)\cdot c' - a'}{(M-s)\cdot d - a'} \quad (27)$$

where $a' \triangleq \sum_{k=s+1}^{M}(B_k - B_M)$, and $c' \triangleq B_s - B_M$.

Because $(M-s)\cdot c' - a' \geq 0$, $(M-s)\cdot d - a' > 0$, $d > c'$, $a' \geq 0$ and $s > 0$, then one has the following inequalities satisfied

$$\begin{aligned}
& a'\cdot d \geq a'\cdot c' \Rightarrow -a'\cdot d \leq -a'\cdot c' \\
& \Rightarrow (M-s)\cdot d\cdot c' - a'\cdot d \leq (M-s)\cdot d\cdot c' - a'\cdot c' \\
& \Rightarrow ((M-s)\cdot c' - a')\cdot d \leq ((M-s)\cdot d - a')\cdot c' \\
& \Rightarrow \frac{(M-s)\cdot c' - a'}{(M-s)\cdot d - a'} \leq \frac{c'}{d}
\end{aligned} \quad (28)$$

Therefore, $F_2 \leq \frac{c'}{d}$, or equivalently

$$\frac{\sum_{k=s+1}^{M}(B_s - B_k)}{\sum_{k=1}^{M}(B_1 - B_k)} \leq \frac{B_s - B_M}{B_1 - B_M} \quad (29)$$

From inequalities (25) and (29), we finally get

$$F_1 + F_2 \leq (\frac{B_1 - B_s}{B_1 - B_M} + \frac{B_s - B_M}{B_1 - B_M} = 1) \quad (30)$$

Therefore, the inequality (21) is satisfied, and consequently inequality (19) is also satisfied, which completes the proof of the theorem.

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