Multi-Criteria Decision-Making with Imprecise Scores and BF-TOPSIS

Jean Dezert
ONERA
The French Aerospace Lab
F-91761 Palaiseau, France.
E-mail: jean.dezert@onera.fr

Deqiang Han
CIESR
Xi'an Jiaotong University
Xi'an, China 710049.
E-mail: deqhan@gmail.com

Jean-Marc Tacnet
UGA, Irstea, UR ETGR
2 rue de la Papeterie-BP 76
F-38402 St-Martin-d'Hères, France.
E-mails: tacnet@irstea.fr

***Abstract*—In 2016 we developed a new approach for Multi-Criteria Decision-Making (MCDM) inspired by the technique for order preference by similarity to ideal solution (TOPSIS) and based on belief functions (BF). Our BF-TOPSIS (Belief Function based TOPSIS) approach assumes that the input score of each hypothesis for each criterion was a real precise number which is a quite restrictive assumption. In this paper we extend our BF-TOPSIS to deal with imprecise score values (intervals of real numbers) and we call it Imp-BF-TOPSIS. This new approach follows main ideas of BF-TOPSIS but extends its applicability for more realistic MCDM problems where the scores are given with a finite precision. Imp-BF-TOPSIS is based on Interval Arithmetic (IA), new probabilistic order relations between intervals and belief functions. We also present results of Imp-BF-TOPSIS for simple examples for illustrating its effectiveness.**

Keywords: Information fusion, multi-criteria decision-making, MCDM, belief functions, TOPSIS.

I. Introduction

The Multi-Criteria Decision-Making (MCDM) aims to choose an alternative among a known set of alternatives based on their quantitative or qualitative evaluations (scores) obtained with respect to different criteria. MCDM can be considered as a decision-level information fusion, and it has been widely used in many decision-making applications. In classical MCDM problem, all the criteria and all alternatives are known, and the score values are usually real numbers (precisely known). Depending on the context of the MCDM problem, the score can be interpreted either as a cost/expense or as a reward/benefit. In the sequel, by convention and without loss of generality we will interpret the score as a reward having monotonically increasing preference. Thus, the best alternative with respect to a given criteria will be the one providing the highest reward/benefit.The set of score values is represented by a quantitative benefit or payoff matrix. Each criterion can also have a relative importance weight. Many methods have been proposed in the literature to solve the classical MCDM [1]. When the score values are incomplete or imprecise (quantitative or qualitative), traditional approaches for classical MCDM problems do not work. In this paper, we focus on these unclassical MCDM problems. We propose to extend the BF-TOPSIS approach to deal with imprecise score values to cover a broader spectrum of real MCDM applications. We use the theory of belief functions and the interval arithmetic. This extension of BF-TOPSIS method is referred as Imp-BF-TOPSIS method in the sequel, where *Imp* is an abbreviation standing for *Imprecise* to specify that the BF-TOPSIS will work with imprecise score values (or more generally with imprecise basic belief assignments (BBAs)). The rest of this paper is organized as follows. In section II, the formulation of classical MCDM problem is provided. In Section III, we introduce Interval Arithmetic and propose new (probabilistic) order relations for intervals as well as distances between intervals. In section IV, basics of belief functions are recalled. In section V we recall the principle of BF-TOPSIS for classical (precise scores) MCDM. The Imp-BF-TOPSIS for imprecise score values is presented in section VI, with simple examples in section VII. SectionVIII concludes this paper.

II. Formulation of Classical MCDM

A classical MCDM problem has a given set of alternatives $\mathbf{A} \triangleq \{A_1, A_2, \ldots, A_M\}$ ($M > 2$), and a given set of criteria $\mathbf{C} \triangleq \{C_1, C_2, \ldots, C_N\}$ ($N \geq 1$). Each alternative A_i represents a possible choice (a possible decision to make). In a general context, each criterion is also characterized by a relative importance weighting factor $w_j \in [0,1]$, $j = 1, \ldots, N$ which are normalized by imposing the condition $\sum_j w_j = 1$. The set of normalized weighting factors is denoted by $\mathbf{w} \triangleq \{w_1, w_2, \ldots, w_N\}$. The score of each alternative A_i with respect to each criteria C_j is expressed by a real number S_{ij} called the score value of A_i based on C_j. We denote $\mathbf{S}$ the score $M \times N$ matrix which is defined as $\mathbf{S} \triangleq [S_{ij}]$. The MCDM problem aims to select the best alternative $A^\star \in \mathbf{A}$ given $\mathbf{S}$ and the weighting factors $\mathbf{w}$ of criteria.

III. Calculus with intervals

A closed interval $\mathbf{x}$ is denoted by $\mathbf{x} = [\underline{x}, \bar{x}] = \{x | \underline{x} \leq x \leq \bar{x}\}$. $\underline{x} = \inf(\mathbf{x})$ is the infimum (lower endpoint) of $\mathbf{x}$ and $\bar{x} = \sup(\mathbf{x})$ is the supremum (upper endpoint) of $\mathbf{x}$ taken values in $\mathbb{R}$. The set of intervals over $\mathbb{R}$ is denoted by $\mathbb{IR}$. An interval in which one endpoint is included and the other is excluded is called a half-closed interval (or half-open interval) and it is called an open interval if its endpoints are excluded. Any precise number x can be expressed with *imprecise number notation* as the degenerate interval $\mathbf{x} = [x, x]$. A non-degenerate interval is called a *proper interval*. The numbers

$wid(\mathbf{x}) \triangleq \bar{x}-\underline{x}$, $rad(\mathbf{x}) = wid(\mathbf{x})/2$ and $mid(\mathbf{x}) \triangleq \frac{1}{2}(\underline{x}+\bar{x})$ are respectively the width, the radius and the midpoint of $\mathbf{x}$. If $\mathbf{x}$ is a precise number (i.e. a degenerate interval), then $wid(\mathbf{x}) = 0$ and $\mathbf{x} = [mid(\mathbf{x}), mid(\mathbf{x})]$. The number $mag(\mathbf{x}) \triangleq \max\{|x| \mid x \in \mathbf{x}\} = \max\{|\underline{x}|, |\bar{x}|\}$ is the magnitude of $\mathbf{x}$, and $mig(\mathbf{x}) \triangleq \min\{|x| \mid x \in \mathbf{x}\} = \min\{|\underline{x}|, |\bar{x}|\}$ is the mignitude of $\mathbf{x}$. If $\mathbf{x}$ and $\mathbf{y}$ are overlapped intervals then $\mathbf{x} \cap \mathbf{y}$ and $\mathbf{x} \cup \mathbf{y}$ are also intervals defined by $\mathbf{x} \cap \mathbf{y} = [\max\{\underline{x}, \underline{y}\}, \min\{\bar{x}, \bar{y}\}]$ and $\mathbf{x} \cup \mathbf{y} = [\min\{\underline{x}, \underline{y}\}, \max\{\bar{x}, \bar{y}\}]$. If $\mathbf{x}$ and $\mathbf{y}$ do not overlap, then $\mathbf{x} \cap \mathbf{y}$ is empty, and $\mathbf{x} \cup \mathbf{y}$ is not a proper interval but the union of two disjoint[1] intervals. In this case, the interval $[\min\{\underline{x}, \underline{y}\}, \max\{\bar{x}, \bar{y}\}]$ is the tightest interval that includes $\mathbf{x} \cup \mathbf{y}$ and it is called the *interval hull* of $\mathbf{x}$ and $\mathbf{y}$. The interval $\mathbf{x}$ is a subset of $\mathbf{y}$ if $(\underline{y} \leq \underline{x}) \wedge (\bar{x} \leq \bar{y})$. The interval $\mathbf{x}$ is equal to $\mathbf{y}$ if $(\underline{x} = \underline{y}) \wedge (\bar{x} = \bar{y})$.

A. Interval Arithmetic

Interval Arithmetic (IA) is an arithmetic defined on intervals of $\mathbb{IR}$. Its modern development started with Moore's works [4]–[7] and yielded recently to an IEEE Standard [8]. The INTLAB Matlab™ toolbox for IA has been developed and proposed by Rump in [9] with a tutorial in [10]. Other tools implementing IA are listed in [7] with more resources available on Kreinovich's interval computation web site [11]. The basic operations[2] on intervals are:

- **Addition**: $\mathbf{x} + \mathbf{y} = [\underline{x} + \underline{y}, \bar{x} + \bar{y}]$
- **Subtraction**: $\mathbf{x} - \mathbf{y} = [\underline{x} - \bar{y}, \bar{x} - \underline{y}]$. In particular, $-\mathbf{x} = [-\bar{x}, -\underline{x}]$, because $-\mathbf{x} = [0, 0] - [\underline{x}, \bar{x}]$.
- **Multiplication**: $\mathbf{x} \times \mathbf{y} = [\min\{S_\times(\mathbf{x}, \mathbf{y})\}, \max\{S_\times(\mathbf{x}, \mathbf{y})\}]$, where $S_\times(\mathbf{x}, \mathbf{y}) \triangleq \{\underline{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y}\}$ is the set of all possible products[3] of endpoints of $\mathbf{x}$ and $\mathbf{y}$. In particular, $-\mathbf{x} = [-\bar{x}, -\underline{x}]$ because $-\mathbf{x} = [-1, -1] \times [\underline{x}, \bar{x}] = [\underline{x}, \bar{x}] \times [-1, -1]$.
- **Division**: $\mathbf{x}/\mathbf{y} = [\min\{S_\div(\mathbf{x}, \mathbf{y})\}, \max\{S_\div(\mathbf{x}, \mathbf{y})\}]$, if $0 \notin \mathbf{y}$ and where $S_\div(\mathbf{x}, \mathbf{y}) \triangleq \{\underline{x}/\underline{y}, \underline{x}/\bar{y}, \bar{x}/\underline{y}, \bar{x}/\bar{y}\}$ is the set of all possible divisions of endpoints of $\mathbf{x}$ and $\mathbf{y}$. If $0 \in \mathbf{y}$ then the division by $\mathbf{y}$ can be handled with more effort using extended interval arithmetic [7], [12] not detailed in this paper.
- **Inverse**: if $\underline{x} > 0$ or $\bar{x} < 0$, $\frac{1}{\mathbf{x}} = [1/\bar{x}, 1/\underline{x}]$.

The following algebraic properties hold for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{IR}$:

- **Associativity**: $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$ and $(\mathbf{x}\mathbf{y})\mathbf{z} = \mathbf{x}(\mathbf{y}\mathbf{z})$.
- **Commutativity**: $(\mathbf{x} + \mathbf{y}) = (\mathbf{y} + \mathbf{x})$ and $(\mathbf{x}\mathbf{y}) = (\mathbf{y}\mathbf{x})$.
- **Neutral elements**: $\mathbf{0} + \mathbf{x} = \mathbf{x} + \mathbf{0} = \mathbf{x}$ where $\mathbf{0} \triangleq [0, 0]$, $\mathbf{0} \cdot \mathbf{x} = \mathbf{x} \cdot \mathbf{0} = \mathbf{0}$ and $\mathbf{1} \cdot \mathbf{x} = \mathbf{x} \cdot \mathbf{1} = \mathbf{x}$ where $\mathbf{1} \triangleq [1, 1]$.

Proper intervals do not have additive or multiplicative inverses and the distributivity law does not hold for intervals. Instead, the following *sub-distributivity low* (weaker version of distributivity) holds : $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{IR}$, $\mathbf{x}(\mathbf{y} + \mathbf{z}) \subseteq \mathbf{x}\mathbf{y} + \mathbf{x}\mathbf{z}$.

Although the interval arithmetic is appealing and looks simple for basic operations with intervals, the so-called dependency problem is a major obstacle to its application when complicate expressions have to be calculated to find tightest range enclosure. In fact, we must take care of dependencies of variables involved in formulas before applying IA in order to get tightest results. To reduce the dependency effect in the result, we need to replace (if possible) the original expression to compute by an equivalent simpler one having less (or none) redundant variables. For example, the derivation of $\mathbf{x}/[\mathbf{x} + \mathbf{y}]$ for $0 \notin \mathbf{x}$ must be computed with IA by $\mathbf{1}/[\mathbf{1} + \mathbf{y}/\mathbf{x}]$ to get tightest result. Also, the power 2 of $\mathbf{x}$ must not be computed by $[\underline{x}, \bar{x}] \times [\underline{x}, \bar{x}]$ because the unknown precise value of x in $[\underline{x}, \bar{x}]$ must be exactly the same (strong dependency) in the multiplication operation in the derivation of $\mathbf{x}^2$. Hence, $\mathbf{x}^2 = \{x^2 | -2 \leq x \leq 2\} = [0, 4]$ is different of $[-2, 2] \times [-2, 2] = [-4, 4]$.

[1] $\mathbf{x} = [\underline{x}, \bar{x}]$ and $\mathbf{y} = [\underline{y}, \bar{y}]$ are disjoint if $(\bar{x} < \underline{y}) \vee (\bar{y} < \underline{x})$.

[2] For simplicity, we use operations on closed intervals.

[3] The product of $\mathbf{x}$ and $\mathbf{y}$ will also be denoted $\mathbf{x} \cdot \mathbf{y}$, or $\mathbf{x}\mathbf{y}$ for simplicity.

B. Basic interval functions

Here several functions that are used in the sequel. More interval functions can be found in [7], [13].

- **Absolute value** [13]:

$$|\mathbf{x}| = \begin{cases} [|\bar{x}|, |\underline{x}|], & \text{if } \bar{x} \leq 0 \\ [|\underline{x}|, |\bar{x}|], & \text{if } \underline{x} \geq 0 \\ [0, \max\{|\underline{x}|, |\bar{x}|\}], & \text{if } \underline{x} < 0 \text{ and } \bar{x} > 0 \end{cases} \tag{1}$$

- **Power** [7]:
  - If $n > 0$ is an odd number: $\mathbf{x}^n = [\underline{x}^n, \bar{x}^n]$
  - If $n > 0$ is an even number

$$\mathbf{x}^n = \begin{cases} [\underline{x}^n, \bar{x}^n], & \text{if } \underline{x} > 0 \\ [\bar{x}^n, \underline{x}^n], & \text{if } \bar{x} < 0 \\ [0, \max\{\underline{x}^n, \bar{x}^n|\}], & \text{if } 0 \in \mathbf{x}. \end{cases}$$

  - If $z > 0$ and $\underline{x} > 0$, $\mathbf{x}^z = [\underline{x}^z, \bar{x}^z]$.
- **Square root** [7]:

$$\mathbf{x}^{\frac{1}{2}} = \sqrt{\mathbf{x}} = \begin{cases} [\sqrt{\underline{x}}, \sqrt{\bar{x}}], \text{for } \underline{x} > 0 \\ [0, \sqrt{\bar{x}}], \text{ if } 0 \in \mathbf{x}. \end{cases}$$

C. Order relations for intervals

The real numbers are ordered by the relation $<$ (or $>$) and comparing two real numbers is in general not a difficult task. In the methods developed in this paper, we need to compare imprecise numbers represented by intervals. We interpret an interval to mean "there is a point that lies between the bounds" and the relation between two intervals is a relation between the two points belonging to intervals (i.e. a *possibly* relation). Comparing intervals is not obvious in the general case when the intervals have a non-empty intersection. For this, we propose a method for comparing intervals and we then explain how to find the $\min$ (or $\max$) element of a set of intervals. To make comparisons, we assume that the unknown precise value belonging to an imprecise number is uniformly distributed in the interval under concern. This assumption is motived by the principle of insufficient reason. The comparative test that we propose does not provide a true or false answer (boolean function), but only a probability value that the test is satisfied or not. To implement the comparison between to intervals $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{IR}$, we define $W \triangleq wid(\mathbf{x})wid(\mathbf{y})$ for notation convenience and we need to distinguish all possible situations as follows:

- Case 1: $\underline{x} < \bar{x} < \underline{y} < \bar{y}$. In this case, $\mathbf{x} < \mathbf{y}$ with probability $P(\mathbf{x} < \mathbf{y}) = 1$.
- Case 2: $\underline{y} < \bar{y} < \underline{x} < \bar{x}$. In this case, $\mathbf{x} < \mathbf{y}$ with probability $P(\mathbf{x} < \mathbf{y}) = 0$.
- Case 3: $\underline{x} < \underline{y} < \bar{x} < \bar{y}$. In this case, $\mathbf{x} < \mathbf{y}$ with

$$P(\mathbf{x} < \mathbf{y}) = \frac{1}{W}[wid(\mathbf{a})wid(\mathbf{b}) + wid(\mathbf{a})wid(\mathbf{c}) + (wid(\mathbf{b})^2/2) + wid(\mathbf{b})wid(\mathbf{c})] \quad (2)$$

where $\mathbf{a} \triangleq [\underline{x}, \underline{y}]$, $\mathbf{b} \triangleq [\underline{y}, \bar{x}]$ and $\mathbf{c} \triangleq [\bar{x}, \bar{y}]$.
- Case 4: $\underline{x} < \underline{y} < \bar{y} < \bar{x}$. In this case, $\mathbf{x} < \mathbf{y}$ with

$$P(\mathbf{x} < \mathbf{y}) = \frac{1}{W}[wid(\mathbf{a})wid(\mathbf{b}) + (wid(\mathbf{b})^2/2)] \quad (3)$$

where $\mathbf{a} \triangleq [\underline{x}, \underline{y}]$ and $\mathbf{b} \triangleq [\underline{y}, \bar{y}]$.
- Case 5: $\underline{y} < \underline{x} < \bar{y} < \bar{x}$. In this case, $\mathbf{x} < \mathbf{y}$ with

$$P(\mathbf{x} < \mathbf{y}) = \frac{1}{W}(wid(\mathbf{b})^2/2) \quad (4)$$

where $\mathbf{b} \triangleq [\underline{x}, \bar{y}]$.
- Case 6: $\underline{y} < \underline{x} < \bar{x} < \bar{y}$. In this case, $\mathbf{x} < \mathbf{y}$ with

$$P(\mathbf{x} < \mathbf{y}) = \frac{1}{W}[wid(\mathbf{b})wid(\mathbf{c}) + (wid(\mathbf{b})^2/2)] \quad (5)$$

where $\mathbf{b} \triangleq [\underline{x}, \bar{x}]$ and $\mathbf{c} \triangleq [\bar{x}, \bar{y}]$.

Formulae (2)-(5) are obtained by the probability calculus using uniform distributions over intervals and the total probability theorem. For case 3, one has $P(\mathbf{x} < \mathbf{y}) = P(\mathbf{x} < \mathbf{y}, x \in \mathbf{a}, y \in \mathbf{b}) + P(\mathbf{x} < \mathbf{y}, x \in \mathbf{a}, y \in \mathbf{c}) + P(\mathbf{x} < \mathbf{y}, x \in \mathbf{b}, y \in \mathbf{b}) + P(\mathbf{x} < \mathbf{y}, x \in \mathbf{b}, y \in \mathbf{c})$ with $P(\mathbf{x} < \mathbf{y}, x \in \mathbf{a}, y \in \mathbf{b}) = 1 \cdot \frac{wid(\mathbf{a})}{wid(\mathbf{x})} \frac{wid(\mathbf{b})}{wid(\mathbf{y})}$, $P(\mathbf{x} < \mathbf{y}, x \in \mathbf{a}, y \in \mathbf{c}) = 1 \cdot \frac{wid(\mathbf{a})}{wid(\mathbf{x})} \frac{wid(\mathbf{c})}{wid(\mathbf{y})}$, $P(\mathbf{x} < \mathbf{y}, x \in \mathbf{b}, y \in \mathbf{b}) = \frac{1}{2} \cdot \frac{wid(\mathbf{b})}{wid(\mathbf{x})} \frac{wid(\mathbf{b})}{wid(\mathbf{y})}$ and $P(\mathbf{x} < \mathbf{y}, x \in \mathbf{b}, y \in \mathbf{c}) = 1 \cdot \frac{wid(\mathbf{b})}{wid(\mathbf{x})} \frac{wid(\mathbf{c})}{wid(\mathbf{y})}$, which gives formula (2).

The value of $P(\mathbf{x} > \mathbf{y})$ can be computed by a similar approach. Of course, $P(\mathbf{x} \geq \mathbf{y}) = 1 - P(\mathbf{x} < \mathbf{y})$ and $P(\mathbf{x} \leq \mathbf{y}) = 1 - P(\mathbf{x} > \mathbf{y})$. Also, one has $P(\mathbf{x} \neq \mathbf{y}) = P(\mathbf{x} < \mathbf{y}) + P(\mathbf{x} > \mathbf{y}) = 1 - P(\mathbf{x} = \mathbf{y})$.

Example 1: $\mathbf{x} = [-3, 0]$, $\mathbf{y} = [-1, 4]$, then $P(\mathbf{x} < \mathbf{y}) = 0.9667$.

Because we know how to compute the probability $P(\mathbf{x} < \mathbf{y})$ for any two imprecise numbers $\mathbf{x}$ and $\mathbf{y}$, we are able to find the $\min$ (or $\max$) elements of a set of imprecise numbers $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_M\}$ with a given associated probability. For instance, for finding the $\min$ element of X we proceed as follows:

- Calculate the $M \times M$ square matrix[4]:

$$\mathbf{P} \triangleq [P_{ij} = P(\mathbf{x}_i < \mathbf{x}_j)] \quad (6)$$

- Calculate the likelihood $\lambda_i \triangleq \lambda(\mathbf{x}_i)$ of $\mathbf{x}_i$ to be the $\min$ of X as the sum of P_{ij} for $j \neq i$, that is

$$\lambda_i = \sum_{j=1,\ldots,M | j \neq i} P(\mathbf{x}_i < \mathbf{x}_j) = \sum_{j \neq i} P_{ij} \quad (7)$$

- The index of the most likely $\min$ element of X is

$$i_{\min} = arg \max_{i=1,\ldots,M} \lambda_i \quad (8)$$

The most likely $\min$ element of X is given by $\mathbf{x}_{i_{\min}}$ with the probability $P(\mathbf{x}_{i_{\min}} = \min\{X\}) = \lambda_i/(M-1)$.

An approach similar is applied to find the $\max$ element of X using the likelihood $\lambda_i = \sum_{j \neq i} P(\mathbf{x}_i > \mathbf{x}_j)$ and $i_{\max} = arg \max_i \lambda_i$. The $\max$ element of X will be given by $\mathbf{x}_{i_{\max}}$ with the associated probability $P(\mathbf{x}_{i_{\max}} = \max\{X\}) = \lambda_i/(M-1)$. Moreover and if needed, we can also sort (probabilistically) all the elements of X by decreasing (or increasing) order based on the likelihood values λ_i.

Example 2: Let's consider the set of intervals $X = \{\mathbf{x}_1 = [-2, 2], \mathbf{x}_2 = [-3, 0], \mathbf{x}_3 = [0, 5], \mathbf{x}_4 = [-1, 3], \}$. From formulas of $P(\mathbf{x} < \mathbf{y})$ given for aforementioned cases 1–6, one obtains

$$\mathbf{P} \triangleq [P(\mathbf{x}_i < \mathbf{x}_j)] = \begin{bmatrix} 0 & 0.1667 & 0.9000 & 0.7188 \\ 0.8333 & 0 & 1.0000 & 0.9583 \\ 0.1000 & 0 & 0 & 0.2250 \\ 0.2812 & 0.0417 & 0.7750 & 0 \end{bmatrix}$$

with the likelihood values

$$\begin{bmatrix} \lambda_1 = \lambda(\mathbf{x}_1) \\ \lambda_2 = \lambda(\mathbf{x}_2) \\ \lambda_3 = \lambda(\mathbf{x}_3) \\ \lambda_4 = \lambda(\mathbf{x}_4) \end{bmatrix} = \begin{bmatrix} 1.7854 \\ 2.7917 \\ 0.3250 \\ 1.0979 \end{bmatrix}$$

The maximum likelihood is $\lambda_2 = 2.7917$ and the corresponding index is $i_{\min} = 2$. This means that $\mathbf{x}_2 = [-3, 0]$ is most likely the $\min$ element of X with the probability $P(\mathbf{x}_2 = \min\{X\}) = 2.7917/3 = 0.9306$. Using a similar approach, one will find that the $\max$ of X is $\mathbf{x}_3 = [0, 5]$ with the probability $P(\mathbf{x}_3 = \max\{X\}) = 2.6750/3 = 0.8917$. Based on the likelihood values of the min element of X sorted in decreasing order, we obtain $\mathbf{x}_2 < \mathbf{x}_1 < \mathbf{x}_4 < \mathbf{x}_3$, which corresponds to what we intuitively expect in such example.

D. Distances between intervals

There are many ways to define strict distance metrics between two intervals. The simplest one is the Hausdorff distance between two intervals $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{IR}$ which is the maximum distance $d(x, y)$ of $x \in \mathbf{x}$ to its nearest point $y \in \mathbf{y}$, where $d(x, y)$ is any chosen metric [14], more precisely $d_H(\mathbf{x}, \mathbf{y}) = \max_{x \in \mathbf{x}}\{\min_{y \in \mathbf{y}} d(x, y)\}$. For simplicity, if we choose the L_1 distance metric $d_{L_1}(x, y) \triangleq |x - y|$, then Hausdorff's distance is given by

$$\begin{aligned} d_H(\mathbf{x}, \mathbf{y}) &= \max\{|\underline{x} - \underline{y}|, |\bar{x} - \bar{y}|\} \\ &= |mid(\mathbf{x}) - mid(\mathbf{y})| + |rad(\mathbf{x}) - rad(\mathbf{y})| \end{aligned} \quad (9)$$

Another interesting distance successfully used for decision-making under uncertainty in the belief functions framework [1], [15], [16], is Wassertein's distance metric [17], [18] $d_W(\mathbf{x}, \mathbf{y})$ defined as

$$d_W(\mathbf{x}, \mathbf{y}) \triangleq \sqrt{[mid(\mathbf{x}) - mid(\mathbf{y})]^2 + \frac{1}{3}[rad(\mathbf{x}) - rad(\mathbf{y})]^2} \quad (10)$$

[4]By construction all diagonal elements P_{ii} equal zero.

which corresponds to Mallows' distance [19] between two probability distributions when we assume that each interval is the support of a uniform distribution.

Example 3: If $\mathbf{x} = [-3, 0]$ and $\mathbf{y} = [-1, 4]$, then $d_H(\mathbf{x}, \mathbf{y}) = 4$ whereas $d_W(\mathbf{x}, \mathbf{y}) \approx 3.0551$.

IV. Basics of belief functions

Belief functions have been introduced by Shafer in [20] to model epistemic uncertainty. We assume that the answer[5] of the problem under concern belongs to a known (or given) finite discrete frame of discernement (FoD) $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$, with $n > 1$, and where all elements of Θ are exclusive[6]. The set of all subsets of Θ (including empty set $\emptyset$ and Θ) is the power-set of Θ denoted by 2^Θ. A basic belief assignment (BBA) associated with a given source of evidence is defined [20] as the mapping $m(\cdot) : 2^\Theta \rightarrow [0, 1]$ satisfying $m(\emptyset) = 0$ and $\sum_{A \in 2^\Theta} m(A) = 1$. The quantity $m(A)$ is called the mass of A committed by the source of evidence. Belief and plausibility functions are respectively defined by

$$Bel(A) = \sum_{\substack{B \subseteq A \\ B \in 2^\Theta}} m(B), \quad \text{and} \quad Pl(A) = 1 - \text{Bel}(\bar{A}). \qquad (11)$$

If $m(A) > 0$, A is called a focal element of $m(\cdot)$. When all focal elements are singletons then $m(\cdot)$ is called a *Bayesian BBA* [20] and its corresponding $Bel(\cdot)$ function is homogeneous to a (subjective) probability measure. The vacuous BBA, or VBBA for short, representing a totally ignorant source is defined as[7] $m_v(\Theta) = 1$.

Shafer [20] proposed to combine $s \geq 2$ distinct sources of evidence represented by BBAs $m_1(.), \ldots, m_s(.)$ over the same FoD with Dempster's rule (i.e. the normalized conjunctive rule). The justification and behavior of Dempster's rule have been disputed over the years from many counter-examples involving high or low conflicting sources (from both theoretical and practical standpoints) as reported in [22]–[25]. Many rules of combination exist[8], and we recommend the new interesting rules based on the proportional conflict redistribution (PCR) principle, see [21], Vol. 3 for details.

A true distance metric between two BBAs $m_1(.)$ and $m_2(.)$ defined on the same FoD, has been defined in [15] as follows[9]

$$d_{BI}(m_1, m_2) \triangleq \sqrt{N_c \cdot \sum_{X \in 2^\Theta} d_W^2(BI_1(X), BI_2(X))} \qquad (12)$$

where the *Belief-Intervals* are defined by $BI_1(X) \triangleq [Bel_1(X), Pl_1(X)]$ and $BI_2(X) \triangleq [Bel_2(X), Pl_2(X)]$, and where $d_W(BI_1(X), BI_2(X))$ is Wassertein's distance between intervals calculated by (10). $N_c = 1/2^{|\Theta|-1}$ is a normalization factor to get $d_{BI}(m_1, m_2) \in [0, 1]$.

[5] i.e. the solution, or the decision to take.

[6] This is so-called Shafer's model of FoD [21].

[7] The complete ignorance is denoted Θ in Shafer's book [20].

[8] see [21], Vol. 2 for a detailed list of fusion rules.

[9] Another well-known real distance metric $d_J(m_1, m_2)$ had been proposed before by Jousselme et al. in [26] which could also be used but we prefer to work with $d_{BI}(m_1, m_2)$ distance for reasons explained in [27].

Making decision on an element of FoD from a given BF ($Bel(.)$, $Pl(.)$, or $m(.)$) can be done in many manners. For instance,

- in taking the argument of max of $\{Bel(\theta_i), i = 1, \ldots, n\}$. This is a pessimistic decisional attitude.
- in taking the argument of max of $\{Pl(\theta_i), i = 1, \ldots, n\}$. This is an optimistic decisional attitude.
- in approximating the BBA $m(.)$ by a subjective probability measure $P(.)$ and taking the argument of max of $\{P(\theta_i), i = 1, \ldots, n\}$. This is a compromise decisional attitude.
- in taking the argument of min of $\{d_{BI}(m(.), m_{\theta_i}), i = 1, \ldots, n\}$, where m_{θ_i} is the BBA entirely focused on θ_i defined by $m_{\theta_i}(X) = 1$, if $X = \theta_i$ and $m_{\theta_i}(X) = 0$, if $X \neq \theta_i$.

In the sequel, we will use the latter method which has been proved very effective in [16], [27].

V. BF-TOPSIS with precise scores

Four BF-TOPSIS methods have been proposed in [1] with an increasing complexity and robustness to rank reversal phenomenon for MDCM support. In this section we briefly recall the main ideas of BF-TOPSIS. For further mathematical details, please refer to [1]. All these methods start with constructing BBAs from the precise score values of the score matrix $\mathbf{S}$ as briefly explained. Only the way those BBAs are processed differs from one BF-TOPSIS method to another one.

A. From precise scores to precise BBAs

In [1], one has proved that BBAs can be consistently built from the precise score matrix $\mathbf{S}$ as follows:

$$Bel_{ij}(A_i) \triangleq \begin{cases} \frac{Sup_j(A_i)}{A^j_{\max}} & \text{if } A^j_{\max} \neq 0 \\ 0 & \text{if } A^j_{\max} = 0 \end{cases} \qquad (13)$$

$$Bel_{ij}(\bar{A}_i) \triangleq \begin{cases} \frac{Inf_j(A_i)}{A^j_{\min}} & \text{if } A^j_{\min} \neq 0 \\ 0 & \text{if } A^j_{\min} = 0 \end{cases} \qquad (14)$$

where $\bar{A}_i$ is the complement of A_i in the FoD $\Theta \triangleq \{A_1, A_2, \ldots, A_M\}$ ($M > 2$), and

$$Sup_j(A_i) \triangleq \sum_{k \in \{1,\ldots,M\} | S_{kj} \leq S_{ij}} |S_{ij} - S_{kj}| \qquad (15)$$

$$Inf_j(A_i) \triangleq - \sum_{k \in \{1,\ldots,M\} | S_{kj} \geq S_{ij}} |S_{ij} - S_{kj}| \qquad (16)$$

The denominators involved in Eqs. (13)-(14), are defined by $A^j_{\max} \triangleq \max_i Sup_j(A_i)$ and $A^j_{\min} \triangleq \min_i Inf_j(A_i)$, and they are supposed different from zero[10]. Therefore, the belief interval of choosing hypothesis A_i considering criterion C_j is given by:

$$[Bel_{ij}(A_i); Pl_{ij}(A_i)] \triangleq [\frac{Sup_j(A_i)}{A^j_{\max}}, 1 - \frac{Inf_j(A_i)}{A^j_{\min}}] \qquad (17)$$

[10] If $A^j_{\max} = 0$ then $Bel_{ij}(A_i) = 0$, and if $A^j_{\min} = 0$ then $Pl_{ij}(A_i) = 1$, so that $Bel_{ij}(\bar{A}_i) = 0$.

From this belief interval, we deduce the BBA $m_{ij}(\cdot)$ which is the triplet $(m_{ij}(A_i), m_{ij}(\bar{A}_i), m_{ij}(A_i \cup \bar{A}_i))$ defined by:

$$m_{ij}(A_i) \triangleq Bel_{ij}(A_i) \quad (18)$$

$$m_{ij}(\bar{A}_i) \triangleq Bel_{ij}(\bar{A}_i) = 1 - Pl_{ij}(A_i) \quad (19)$$

$$m_{ij}(A_i \cup \bar{A}_i) \triangleq m_{ij}(\Theta) = Pl_{ij}(A_i) - Bel_{ij}(A_i) \quad (20)$$

If a numerical value S_{ij} is missing in $\mathbf{S}$, one uses $m_{ij}(\cdot) \triangleq (0,0,1)$, i.e. one takes the vacuous belief assignment.

Using the formulae (13)-(20), we obtain from any $M \times N$ precise score matrix[11] $\mathbf{S}$ the general $M \times N$ matrix $\mathbf{M} \triangleq [m_{ij}(.)]$ of BBAs that are involved in BF-TOPSIS methods. This construction of BBAs is very interesting for applications because it is invariant to the bias and scaling effects of score values [1]. Also, it allows us to model our lack of evidence (if any) with respect to an (or several) alternative(s) when their corresponding score values are missing for any reason.

B. BF-TOPSIS1 method

From the BBA matrix $\mathbf{M}$ and for each alternative A_i, one computes distances $d_{BI}(m_{ij}, m_{ij}^{\text{best}})$ between $m_{ij}(\cdot)$ and the ideal best BBA defined by $m_{ij}^{\text{best}}(A_i) \triangleq 1$, and the distances $d_{BI}(m_{ij}, m_{ij}^{\text{worst}})$ between $m_{ij}(\cdot)$ and the ideal worst BBA defined by $m_{ij}^{\text{worst}}(\bar{A}_i) \triangleq 1$. Then, one computes the weighted average distances with relative importance weighting factor w_j of criteria C_j as follows:

$$d^{\text{best}}(A_i) \triangleq \sum_{j=1}^{N} w_j \cdot d_{BI}(m_{ij}, m_{ij}^{\text{best}}) \quad (21)$$

$$d^{\text{worst}}(A_i) \triangleq \sum_{j=1}^{N} w_j \cdot d_{BI}(m_{ij}, m_{ij}^{\text{worst}}) \quad (22)$$

The relative closeness of the alternative A_i with respect to the ideal best solution A^{best} defined by

$$C(A_i, A^{\text{best}}) \triangleq \frac{d^{\text{worst}}(A_i)}{d^{\text{worst}}(A_i) + d^{\text{best}}(A_i)} \quad (23)$$

is used to make the preference ordering according to the descending order of $C(A_i, A^{\text{best}}) \in [0,1]$, where a larger $C(A_i, A^{\text{best}})$ value means a better alternative (higher preference).

C. BF-TOPSIS2 method

For each criteria C_j, one computes at first the relative closeness of each alternative A_i w.r.t. its ideal best solution A^{best} by

$$C_j(A_i, A^{\text{best}}) \triangleq \frac{d_{BI}(m_{ij}, m_{ij}^{\text{worst}})}{d_{BI}(m_{ij}, m_{ij}^{\text{worst}}) + d_{BI}(m_{ij}, m_{ij}^{\text{best}})} \quad (24)$$

The global relative closeness $C(A_i, A^{\text{best}})$ of each alternative A_i with respect to its ideal best solution A^{best} used to make the final preference ordering is then obtained by the weighted average of $C_j(A_i, A^{\text{best}})$, that is

$$C(A_i, A^{\text{best}}) \triangleq \sum_{j=1}^{N} w_j \cdot C_j(A_i, A^{\text{best}}) \quad (25)$$

D. BF-TOPSIS3 method

For each alternative A_i, one fuses the N precise BBAs $m_{ij}(\cdot)$ discounted with importance factor w_j (see [28]) with PCR6 rule of combination [21] (Vol. 3) to get the precise fused BBA m_i^{PCR6}, from which one computes the distance $d^{best}(A_i) = d_{BI}(m_i^{PCR6}, m_i^{best})$ between $m_i^{PCR6}(\cdot)$ and its ideal best BBA $m_i^{best}(A_i) \triangleq 1$. Similarly, one computes the distance $d^{worst}(A_i) = d_{BI}(m_i^{PCR6}, m_i^{worst})$ between $m_i^{PCR6}(\cdot)$ and $m_i^{worst}(\bar{A}_i) \triangleq 1$. The relative closeness of each A_i with respect to ideal best solution $C(A_i, A^{best})$ is computed by (23), and is used to make the preference ordering according to the descending order of $C(A_i, A^{best})$.

E. BF-TOPSIS4 method

This method is similar to BF-TOPSIS3 except that we use the more complicate ZPCR6 fusion rule taking into account Zhang's degree of intersection of focal elements in the conjunctive consensus operator, see [29] for details.

VI. BF-TOPSIS WITH IMPRECISE SCORES

In this section we present the extension of BF-TOPSIS methods to deal with imprecise score values $S_{ij} = [\underline{S}_{ij}, \bar{S}_{ij}]$, $i = 1,\dots,M$ and $j = 1,\dots,N$. These extensions will be referred as Imp-BF-TOPSIS in the sequel. The basic idea is to follow principles of BF-TOPSIS using Interval Arithmetic (IA) instead of classical arithmetic on reals.

A. From imprecise scores to imprecise BBAs

The application of formulae (13)–(20) using IA operations does not work directly because of potential division by intervals including zero, and because of comparison tests involving boolean $\leq$ and $\geq$ functions. To circumvent these problems, we need to avoid intervals including zero, and replace boolean $\leq$ and $\geq$ functions by their probabilistic counterpart presented in section III-C. This is done as follows:

- Step 1 (Offset correction): To work only with positive intervals, we apply at first an offset correction of imprecise score values $S_{ij} = [\underline{S}_{ij}, \bar{S}_{ij}]$ for each column j of imprecise score matrix $\mathbf{S} = [S_{ij}]$. This is a preprocessing step. We are allowed to do this because, by construction, the BBAs based on formulae (13)–(20) are invariant to bias and scaling effects [1]. Therefore, we can always replace the original imprecise score $[\underline{S}_{ij}, \bar{S}_{ij}]$ by

$$[\underline{S}'_{ij}, \bar{S}'_{ij}] = [\underline{S}_{ij}, \bar{S}_{ij}] + [\delta_j + \epsilon, \delta_j + \epsilon] \quad (26)$$

where $\epsilon > 0$ is an arbitrary positive number to ensure the strict positivity of intervals, and the offset correction value δ_j is given for $j = 1,\dots,N$ by

$$\delta_j = - \min_{i=1,\dots,M} \{\underline{S}_{ij}\} \quad (27)$$

[11] Note that each element $m_{ij}(.)$ is in fact a 3-uple of masses given by (18)–(20).

Example 4: Let's consider the FoD $\Theta \triangleq \{A_1, A_2, A_3, A_4\}$ with four alternatives, a criterion C_1 and the following associated imprecise scores $S_{11} = [-2, 2]$, $S_{21} = [-3, 0]$, $S_{31} = [0, 5]$ and $S_{41} = [-1, 3]$. The offset correction is then $\delta_1 = -\min\{-2, -3, 0, -1\} = 3$. If we take, $\epsilon = 1$ then we will get the corrected (positive) imprecise scores

$$\begin{aligned} S'_{11} &= [-2, 2] + [3+1, 3+1] = [2, 6] \\ S'_{21} &= [-3, 0] + [3+1, 3+1] = [1, 4] \\ S'_{31} &= [0, 5] + [3+1, 3+1] = [4, 9] \\ S'_{41} &= [-1, 3] + [3+1, 3+1] = [3, 7] \end{aligned}$$

- Step 2: Replace the $S_{kj} \leq S_{ij}$ and $S_{kj} \geq S_{ij}$ tests involved in (15) and (16), by their probabilistic counterparts $P(S_{kj} \leq S_{ij}) \geq 0.5$ and $P(S_{kj} \geq S_{ij}) \geq 0.5$ because S_{kj} and S_{ij} are imprecise numbers (i.e. intervals), where $P(S_{kj} \leq S_{ij})$ and $P(S_{kj} \geq S_{ij})$ are computed as in section III-C.

In the sequel, we assume that the offset correction of score as been applied (step 1 done) and for notation simplicity we denote these (corrected) strictly positive imprecise scores S_{ij}. The imprecise BBAs can now be computed from the (offset-corrected) imprecise scores values as follows[12]:

$$[\underline{Bel}_{ij}(A_i), \overline{Bel}_{ij}(A_i)] \triangleq \begin{cases} \frac{Sup_j(A_i)}{A^j_{\max}} & \text{if } A^j_{\max} \neq [0,0] \\ [0,0] & \text{if } A^j_{\max} = [0,0] \end{cases} \quad (28)$$

$$[\underline{Bel}_{ij}(\bar{A}_i), \overline{Bel}_{ij}(\bar{A}_i)] \triangleq \begin{cases} \frac{Inf_j(A_i)}{A^j_{\min}} & \text{if } A^j_{\min} \neq [0,0] \\ [0,0] & \text{if } A^j_{\min} = [0,0] \end{cases} \quad (29)$$

where

$$\begin{aligned} Sup_j(A_i) &= [\underline{Sup}_j(A_i), \overline{Sup}_j(A_i)] \\ &\triangleq \sum_{k\in\{1,\ldots,M\}|P(S_{kj}\leq S_{ij})\geq 0.5} |S_{ij} - S_{kj}| \end{aligned} \quad (30)$$

$$\begin{aligned} Inf_j(A_i) &= [\underline{Inf}_j(A_i), \overline{Inf}_j(A_i)] \\ &\triangleq -\sum_{k\in\{1,\ldots,M\}|P(S_{kj}\geq S_{ij})\geq 0.5} |S_{ij} - S_{kj}| \end{aligned} \quad (31)$$

The denominators involved in Eqs. (28)-(29), are defined by $A^j_{\max} = [\underline{A}^j_{\max}, \bar{A}^j_{\max}] \triangleq \max_i Sup_j(A_i)$ and $A^j_{\min} = [\underline{A}^j_{\min}, \bar{A}^j_{\min}] \triangleq \min_i Inf_j(A_i)$, and they are supposed different from $[0, 0]$[13]. Therefore, in non-degenerate case (when $A^j_{\max} \neq [0, 0]$ and $A^j_{\min} \neq [0, 0]$) the belief interval of hypothesis A_i considering criterion C_j has now imprecise bounds given by

$$Bel_{ij}(A_i) = [\underline{Bel}_{ij}(A_i), \overline{Bel}_{ij}(A_i)] = \frac{Sup_j(A_i)}{A^j_{\max}} \quad (32)$$

$$Pl_{ij}(A_i) = [\underline{Pl}_{ij}(A_i), \overline{Pl}_{ij}(A_i)] = [1, 1] - \frac{Inf_j(A_i)}{A^j_{\min}} \quad (33)$$

[12]Remember that operations involved in the formulas of this section are IA operations defined in section III.

[13]If $A^j_{\max} = [0, 0]$ then $Bel_{ij}(A_i) = [0, 0]$, and if $A^j_{\min} = [0, 0]$ then $Pl_{ij}(A_i) = [1, 1]$, so that $Bel_{ij}(\bar{A}_i) = [0, 0]$.

From these imprecise bounds, we calculate the imprecise BBAs $m_{ij}(\cdot) = [\underline{m}_{ij}(\cdot), \bar{m}_{ij}(\cdot)]$ which is the triplet of intervals $(m_{ij}(A_i) = [\underline{m}_{ij}(A_i), \bar{m}_{ij}(A_i)], m_{ij}(\bar{A}_i) = [\underline{m}_{ij}(\bar{A}_i), \bar{m}_{ij}(\bar{A}_i)], m_{ij}(A_i \cup \bar{A}_i) = [\underline{m}_{ij}(A_i \cup \bar{A}_i), \bar{m}_{ij}(A_i \cup \bar{A}_i)])$ defined by:

$$m_{ij}(A_i) \triangleq Bel_{ij}(A_i) \quad (34)$$

$$m_{ij}(\bar{A}_i) \triangleq Bel_{ij}(\bar{A}_i) = [1, 1] - Pl_{ij}(A_i) \quad (35)$$

$$m_{ij}(A_i \cup \bar{A}_i) \triangleq m_{ij}(\Theta) = Pl_{ij}(A_i) - Bel_{ij}(A_i) \quad (36)$$

If a numerical (imprecise) value S_{ij} is missing in $\mathbf{S}$, one uses $m_{ij}(\cdot) \triangleq ([0, 0], [0, 0], [1, 1])$, i.e. one takes the vacuous belief assignment expressed in its degenerate interval form.

Using the formulae (28)-(36), we obtain from any $M \times N$ imprecise score matrix $\mathbf{S}$ the general $M \times N$ matrix $\mathbf{M} \triangleq \left[m_{ij}(.) = [\underline{m}_{ij}(.), \bar{m}_{ij}(.)]\right]$ of imprecise BBAs that are necessary in Imp-BF-TOPSIS methods.

It is worth to note that formulae (28)–(36) are fully consistent with (13)–(20) when all the elements S_{ij} of imprecise score matrix are degenerate (are precise numbers), that is when $\underline{S}_{ij} = \bar{S}_{ij}$. By choosing the midpoints of imprecise score values, we can always build a precise BBA that satisfies Shafer's BBA definition [1]. This midpoint-based BBA is always included in imprecise BBA bounds because of IA. Therefore, imprecise BBAs are always admissible and they can be combined by Dempster's or PCR6 rules thanks to IA operations. This has to be implemented with caution to avoid dependency effect [7].

B. Imp-BF-TOPSIS1 and Imp-BF-TOPSIS2 methods

These methods are similar to BF-TOPSIS1 and BF-TOPSIS2 except that we use IA operations. The distances $d_{BI}(m_{ij}, m^{\text{best}}_{ij})$ and $d_{BI}(m_{ij}, m^{\text{worst}}_{ij})$ become imprecise numbers computed with formula (12) adapted for interval calculus, where $m^{\text{best}}_{ij}(A_i) \triangleq [1, 1]$ and $m^{\text{worst}}_{ij}(\bar{A}_i) \triangleq [1, 1]$. Of course, all scalars involved in the formulae (12), (21), (22), and (25) must be expressed in their degenerate interval form in order to apply IA operations, for instance N_c is replaced by $[N_c, N_c]$, w_j by $[w_j, w_j]$, etc. The final preference ordering is found according to the descending order of imprecise $C(A_i, A^{\text{best}})$ obtained by the method explained at the end of section II-C, where a larger $C(A_i, A^{\text{best}})$ means a better alternative (higher preference).

C. Imp-BF-TOPSIS3 and Imp-BF-TOPSIS4 methods

These methods are similar to BF-TOPSIS3 and BF-TOPSIS4[14] but with special adaptation of PCR6 and ZPCR6 formulae to reduce dependency effects with IA operations. For example, the expression $\frac{m_1^2(X)m_2(X)}{m_1(X)+m_2(Y)}$ involved in PCR6 formula [21] (Vol. 3) for the fusion of two BBAs must be computed as $[\frac{1}{m_1(X)m_2(Y)} + \frac{1}{m_1^2(X)}]^{-1}$ with IA to get the tightest range enclosure. The implementation of conjunctive rule must also be done with precaution when using IA to reduce the dependency effect in the derivation.

[14]See their mathematical derivations in [1].

VII. EXAMPLES

A. Example 5 (mono-criterion)

• **Precise scores case** [1]: Let's consider a criterion C_1 and seven alternatives A_i, $(i = 1, \dots, 7)$ with the precise score values $\mathbf{S}_{11} = 10$, $\mathbf{S}_{21} = 20$, $\mathbf{S}_{31} = -5$, $\mathbf{S}_{41} = 0$, $\mathbf{S}_{51} = 100$, $\mathbf{S}_{61} = -11$, and $\mathbf{S}_{71} = 0$. The direct ranking with the preference "greater is better" yields[15] $A_5 \succ A_2 \succ A_1 \succ (A_4 \sim A_7) \succ A_3 \succ A_6$. In applying formulas (13)–(20), we get the BBAs listed in Table I. Using BF-TOPSIS methods[16], we get the distances, and the relative closeness measures of Table II. In sorting $C(A_i, A^{\text{best}})$ by the descending order, we get the correct preferences order $A_5 \succ A_2 \succ A_1 \succ (A_4 \sim A_7) \succ A_3 \succ A_6$ which is consistent with the direct ranking result.

Table I
BBAS CONSTRUCTED FROM PRECISE SCORE VALUES.

	$m_{ij}(A_i)$	$m_{ij}(\bar{A}_i)$	$m_{ij}(A_i \cup \bar{A}_i)$
A_1	0.0955	0.5236	0.3809
A_2	0.1809	0.4188	0.4003
A_3	0.0102	0.8115	0.1783
A_4	0.0273	0.6806	0.2921
A_5	1.0000	0	0
A_6	0	1.0000	0
A_7	0.0273	0.6806	0.2921

Table II
DISTANCES AND RELATIVE CLOSENESS MEASURES.

	$d_{BI}(m_{ij}, m_{ij}^{\text{best}})$	$d_{BI}(m_{ij}, m_{ij}^{\text{worst}})$	$C(A_i, A^{\text{best}})$
A_1	0.7380	0.0940	0.1130
A_2	0.6676	0.1615	0.1948
A_3	0.8112	0.0214	0.0257
A_4	0.7954	0.0405	0.0485
A_5	0	0.8229	1.0000
A_6	0.8229	0	0
A_7	0.7954	0.0405	0.0485

• **Imprecise scores case**: For simplicity, consider now the imprecise score values with midpoints consistent with previous example. For instance suppose $\mathbf{S}_{11} = [8, 12]$, $\mathbf{S}_{21} = [18, 22]$, $\mathbf{S}_{31} = [-7, -3]$, $\mathbf{S}_{41} = [-1, 1]$, $\mathbf{S}_{51} = [97, 103]$, $\mathbf{S}_{61} = [-12, -10]$, and $\mathbf{S}_{71} = [-1, 1]$. The offset factor is equal to $\delta_1 = 12$. After offset corrections with $\epsilon = 1$, we get the corrected positive imprecise scores $\mathbf{S}_{11} = [21, 25]$, $\mathbf{S}_{21} = [31, 35]$, $\mathbf{S}_{31} = [6, 10]$, $\mathbf{S}_{41} = [12, 14]$, $\mathbf{S}_{51} = [110, 116]$, $\mathbf{S}_{61} = [1, 3]$, and $\mathbf{S}_{71} = [12, 14]$. In applying formulas (28)–(36), we get the imprecise BBAs listed in Table III.

As we see all imprecise BBAs values of Table III include precise BBAs values of Table I. Note that all negative bounds encountered in derivations (if any) are set to zero, and all bounds greater than one in derivations (if any) are set to one because masses values must belong to $[0, 1]$. Each imprecise

Table III
IMPRECISE BBAS CONSTRUCTED FROM IMPRECISE SCORE VALUES.

	$m_{ij}(A_i)$	$m_{ij}(\bar{A}_i)$	$m_{ij}(A_i \cup \bar{A}_i)$
A_1	[0.0701,0.1234]	[0.4375,0.6264]	[0.2501,0.4924]
A_2	[0.1452,0.2200]	[0.3606,0.4885]	[0.2915,0.4942]
A_3	[0.0049,0.0161]	[0.6538,1.0000]	[0,0.3413]
A_4	[0.0179,0.0376]	[0.5769,0.8046]	[0.1578,0.4051]
A_5	[0.9119,1.0000]	[0,0]	[0,0.0881]
A_6	[0,0]	[0.8365,1.0000]	[0,0.1635]
A_7	[0.0179,0.0376]	[0.5769,0.8046]	[0.1578,0.4051]

BBA represented by a row of Table III is said *admissible* because for a given hypothesis A_i one can find at least a point (a precise mass value) in each interval $m_{ij}(A_i)$, $m_{ij}(\bar{A}_i)$ and $m_{ij}(A_i \cup \bar{A}_i)$ such that the sum of the masses equals one. If all imprecisions of scores reduce to zero, the results of Table III will coincide with results of Table I. Using Imp-BF-TOPSIS methods, we get the imprecise distances, and the imprecise relative closeness measures listed in Table IV.

Table IV
IMPRECISE DISTANCES AND RELATIVE CLOSENESS MEASURES.

	$d_{BI}(m_{ij}, m_{ij}^{\text{best}})$	$d_{BI}(m_{ij}, m_{ij}^{\text{worst}})$	$C(A_i, A^{\text{best}})$
A_1	[0.5421,0.9338]	[0.0301,0.2864]	[0.0312,0.3457]
A_2	[0.5034,0.8313]	[0.0511,0.3253]	[0.0579,0.3926]
A_3	[0.5324,1.0000]	[0.0006,0.2977]	[0.0006,0.3586]
A_4	[0.5960,0.9956]	[0.0131,0.2317]	[0.0130,0.2800]
A_5	[0,0.1135]	[0.7176,0.8591]	[0.8634,1.0000]
A_6	[0.6900,0.9533]	[0,0.1360]	[0,0.1647]
A_7	[0.5960,0.9956]	[0.0131,0.2317]	[0.0130,0.2800]

For each element of $\mathcal{C} = \{C(A_i, A^{\text{best}}), i = 1, \dots, M\}$, we compute its likelihood $\lambda_i \triangleq \lambda(C(A_i, A^{\text{best}}))$ to be the max of $\mathcal{C}$ by the method explained in section III-C. Here, one gets

$$[\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7] \approx [3.00, 3.57, 2.80, 2.29, 6.00, 1.02, 2.29]$$

In sorting λ_i by the descending order, we get $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_6 \succ (A_4 \sim A_7)$. This result is of course a bit different of what we obtain with precise midpoints of scores because of imprecision degree in the input scores, which is normal. However when the imprecision degree (i.e. the width of each score interval) of the input scores reduces to zero, we always obtain the same result as with (precise) midpoint of score intervals because of the consistency of interval arithmetic operators with arithmetic on real numbers.

B. Example 6 (Multi-criteria)

An investor wants to invest some money in a company to get the highest profit. He considers four companies $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ and must take a decision according to the following two criteria: C_1 is the risk analysis (the min is better) with weight $w_1 = 0.6$ and C_2 is the growth analysis (the max

[15] where the symbol $\succ$ means *better than (or is preferred to).*

[16] in mono-criterion case, all BF-TOPSIS methods are equivalent because there is no need of making fusion.

is better) with $w_2 = 0.4$. Assume the imprecise scores are

$$\mathbf{S} = \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \overset{\begin{array}{cc} C_1(\text{in } \%) & C_2(\text{in } \%) \end{array}}{\begin{bmatrix} [8,12] & [8,10] \\ [17,19] & [15,19] \\ [3,5] & [8,12] \\ [4,8] & [5,7] \end{bmatrix}}$$

We get the final preference orderings with:

- Imp-BF-TOPSIS1: we get $A_3 \succ A_4 \succ A_1 \succ A_2$ because
$$[\lambda_1, \lambda_2, \lambda_3, \lambda_4] \approx [1.4939, 1.0013, 1.9396, 1.5652]$$
- Imp-BF-TOPSIS2: we get $A_3 \succ A_4 \succ A_1 \succ A_2$ because
$$[\lambda_1, \lambda_2, \lambda_3, \lambda_4] \approx [1.4968, 0.9323, 2.0535, 1.5175]$$
- Imp-BF-TOPSIS3: we get $A_3 \succ A_4 \succ A_1 \succ A_2$ because
$$[\lambda_1, \lambda_2, \lambda_3, \lambda_4] \approx [1.5300, 1.3487, 1.5639, 1.5574]$$
- Imp-BF-TOPSIS4: we get $A_4 \succ A_3 \succ A_1 \succ A_2$ because
$$[\lambda_1, \lambda_2, \lambda_3, \lambda_4] \approx [1.5093, 1.4686, 1.5101, 1.5120]$$

Imp-BF-TOPSIS1–3 methods provide here the same preference order $A_3 \succ A_4 \succ A_2 \succ A_1$, hence A_3 is the preferred choice. One sees that Imp-BF-TOPSIS3 and Imp-BF-TOPSIS4 have difficulties to provide very distinct likelihood values because unsurmountable dependency effects arise in IA operations when applying PCR6 and ZPCR6 rules which degrade substantially the final precision of the result. Based on this analysis, we recommend to use either Imp-BF-TOPSIS1or Imp-BF-TOPSIS2 because they provide tightest enclosure results and they are much simpler to implement than Imp-BF-TOPSIS3 and Imp-BF-TOPSIS4.

VIII. Conclusions

Four new methods (Imp-BF-TOPSIS1–Imp-BF-TOPSIS4) for MCDM have been proposed. We have shown how to calculate imprecise BBAs from imprecise scores, and how to evaluate the relative imprecise closeness of each alternative to the ideal best and worst solutions for making the preference ordering. These methods avoid scores normalization, and they can deal with imprecise scores, with missing scores, with the reliability of the sources as well, and they could also work with imprecise weightings of criteria. They are more complicate to implement (and slower) that their precise counterparts because of IA. They are consistent with BF-TOPSIS1–4 when the imprecision of scores reduces to zero. However because IA suffers of dependency effects, IA is not the *universal panacea* to work with imprecise values to get best results, specially for combining imprecise BBAs. More research efforts need to be done to circumvent these problems (if possible) by better implementations (or by Monte-Carlo approach) in order to improve the performance of Imp-BF-TOPSIS3 and Imp-BF-TOPSIS4 methods. Application of these methods for natural risk assessment in mountains is under development and will be reported in future publications.

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