Credibilistic Independence of Two Propositions

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***Abstract*—In this paper the notion of (probabilistic) independence of two events defined classically in the theory of probability is extended in the theory of belief functions as the credibilistic independence of two propositions. This new notion of independence which is compatible with the probabilistic independence as soon as the belief function is Bayesian, is defined from Fagin-Halpern belief conditioning formulas drawn from Total Belief Theorem (TBT) when working in the framework of belief functions to model epistemic uncertainties. We give some illustrative examples of this notion at the end of the paper.**

Keywords: credibilistic independence, belief functions, belief conditioning, total belief theorem.

I. Introduction

In this paper the notion of (probabilistic) independence of two events defined classically in the theory of probability [1] is extended in the theory of belief functions [2]. We call it the *credibilistic independence* of two propositions to make a clear distinction between the origin of uncertainty related to events (i.e. the random or stochastic uncertainty), and in a more general context the origin of uncertainty of propositions (i.e. the epistemic uncertainty due to lack of knowledge). The epithet *credibilistic* refers to a credal system chosen for the codification of belief. In this work our credal system is the mathematical framework of belief functions.

Several works have been proposed in the past to define different notions of independencies in imprecise probability framework and in the theory of belief functions. For example, Couso et al. [3] did propose several notions of independencies illustrated by different combined Ellsberg's urns experiments. In 2000's Ben Yaghlane, Smets and Mellouli [4], [5] did explore the notion of independence and they define the *doxastic independence*. Their proposal is however essentially based on Dempster's rule of combination which is known problematic and incompatible with imprecise conditional probabilistic calculus as shown in [6]–[8]. More recently Jirousek and Vejnarova in [9] did propose a definition of conditional independence which is based on some complicate factorization principles of the joint basic belief assignment (BBA) into separate marginal spaces of the variables. All aforementioned works share two same basic principles for attempting to define the notion(s) of independence: 1) work on joint (Cartesian) product space, and 2) work with BBAs. These two fundamental principles yield to quite complicate definitions of independence(s) difficult to use by most engineers or researchers for their own applications or developments.

In this research work we adopt a radically different standpoint. We work with a BBA defined with respect to a single frame of discernment[1] (FoD), and we work directly with belief intervals induced by Fagin-Halpern conditioning rule [6], [7], rather than some factorization principles of joint BBA or extension principles of marginal BBAs. Our approach is constructive, easier than previous attempts to define independence, and consistent with the notion of probabilistic (or stochastic) independence of two events defined in the theory of probability. Our notion of credibilistic independence can be used easily to check if two propositions are credibilistically independent, or not, given a BBA. This new approach could be helpful for practitioners of belief functions. We do not have yet made more investigations for showing its usefulness for applications, but we expect it will generate some interest because this problem has been already explored by several researchers in the past based on different standpoints.

This paper is organized as follows. After a brief recall of basics of probability theory and belief functions in Sections II and III, we characterize mathematically the notion of *credibilistic independence* of two propositions in Section IV. Some basic illustrative examples are shown in Section V, with conclusions in Section VI.

II. Basics of probability theory

In probability theory [1], the elements θ_i of the space Θ are experimental outcomes. The subsets of Θ are called events and the event $\{\theta_i\}$ consisting of the single element θ_i is an elementary event. The space Θ is called the *sure event* and the empty set $\emptyset$ is the *impossible event*. We assign to each event A a number $P(A)$ in $[0,1]$, called the probability of A, which satisfies the three Kolmogorov's axioms: 1) $P(\emptyset)=0$; 2) $P(\Theta)=1$; and 3) if $A\cap B=\{\emptyset\}$, then $P(A\cup B)=P(A)+P(B)$. The fundamental Total Probability Theorem (TPT), also called the law of total probability, see [1] states that for any event B and any partition $\{A_1,A_2,\ldots,A_k\}$ of the space Θ, the following equality holds

$$P(B)=P(B\cap A_1)+P(B\cap A_2)+\ldots+P(B\cap A_k) \quad (1)$$

[1]It can be any Cartesian product space in fact. The main point is that the (joint) BBA we work with is defined with respect to this space.

Starting from TPT formula (1) and assuming $P(B) > 0$, we get for any $i \in \{1, \ldots, k\}$ (after dividing each side of (1) by $P(B)$ and rearranging terms) the equality

$$\frac{P(A_i \cap B)}{P(B)} = 1 - \sum_{\substack{j=1,\ldots,k \\ j \neq i}} \frac{P(A_j \cap B)}{P(B)} = 1 - \frac{P(\bar{A}_i \cap B)}{P(B)} \quad (2)$$

This equality allows us to define the conditional probability $P(A_i|B)$ by[2]

$$P(A_i|B) \triangleq P(A_i \cap B)/P(B) \quad (3)$$

One can verify that the conditional probability (3) satisfies the three axioms of the Theory of Probability [1].

Similarly, by considering an event A_i of Θ and the partition $\{B, \bar{B}\}$ of Θ, the formula $P(A_i) = P(A_i \cap B) + P(A_i \cap \bar{B})$ applies, and by dividing it by $P(A_i)$ (assuming $P(A_i) > 0$), one gets

$$\frac{P(A_i \cap B)}{P(A_i)} = 1 - \frac{P(A_i \cap \bar{B})}{P(A_i)} \quad (4)$$

This allows to define the reverse conditional probability $P(B|A_i)$ by

$$P(B|A_i) \triangleq P(A_i \cap B)/P(A_i) \quad (5)$$

Probabilistic Independence: Two events A_i and B are said to be *probabilistically independent* (or P-independence for short) if and only if $P(A_i|B) = P(A_i)$ and $P(B|A_i) = P(B)$. From conditioning formulas (3) and (5) and because conditional probabilities formulas $P(A_i|B)$ and $P(B|A_i)$ are mathematically defined only if $P(B) > 0$ and $P(A_i) > 0$, one determines the condition of P-independence (which is well defined even if $P(A_i) = 0$, or $P(B) = 0$, or both) by the formula

$$P(A_i \cap B) = P(A_i)P(B) \quad (6)$$

III. Basics of belief functions

Based on Dempster's works [10], [11], Shafer did introduce Belief Functions (BF) to model the epistemic uncertainty and to reason under uncertainty in his Mathematical Theory of Evidence [2], also known as Dempster-Shafer Theory (DST). We consider a finite discrete frame of discernement (FoD) $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$, with $n > 1$, and where all exhaustive and exclusive elements of Θ represent the set of the potential solutions of the problem under concern. The set of all subsets of Θ is the power-set of Θ denoted by 2^Θ. The number of elements (i.e. the cardinality) of 2^Θ is $2^{|\Theta|}$. A basic belief assignment (BBA) associated with a given source of evidence is defined as the mapping $m(\cdot) : 2^\Theta \to [0, 1]$ satisfying the conditions $m(\emptyset) = 0$ and $\sum_{A \in 2^\Theta} m(A) = 1$. The quantity $m(A)$ is the mass of belief of subset A committed by the source of evidence (SoE). A focal element X of a BBA $m(\cdot)$ is an element of 2^Θ such that $m(X) > 0$. Note that the empty set $\emptyset$ is not a focal element of a BBA because $m(\emptyset) = 0$ (closed-world assumption of Shafer's model for the FoD). The set of all focal elements of $m(\cdot)$ is denoted[3] $\mathcal{F}_\Theta(m) \triangleq \{X \subseteq \Theta | m(X) > 0\} = \{X \in 2^\Theta | m(X) > 0\}$. Belief and plausibility functions are defined by[4]

$$Bel(A) = \sum_{\substack{X \in 2^\Theta \\ X \subseteq A}} m(X) = \sum_{X \in \mathcal{F}_A(m)} m(X) \quad (7)$$

$$Pl(A) = \sum_{\substack{X \in 2^\Theta \\ X \cap A \neq \emptyset}} m(X) = 1 - \text{Bel}(\bar{A}) \quad (8)$$

where $\bar{A} \triangleq \Theta - \{A\} = \{X | X \in \Theta \text{ and } X \notin A\}$, is the complement of A in Θ and the minus symbol denotes the set difference operator. The width $U(A^*) = Pl(A) - Bel(A) = \sum_{X \in \mathcal{F}_{A^*}(m)} m(X)$ of the belief interval $[Bel(A), Pl(A)]$ is called the *uncertainty on* A committed by the SoE. $\mathcal{F}_{A^*}(m)$ is the set of focal elements of $m(\cdot)$ not included in A and not included in $\bar{A}$, that is $\mathcal{F}_{A^*}(m) \triangleq \mathcal{F}_\Theta(m) - \mathcal{F}_A(m) - \mathcal{F}_{\bar{A}}(m)$ and $U(A^*)$ represents the imprecision on the (subjective) probability of A granted by the SoE which provides the BBA $m(\cdot)$. When all elements of $\mathcal{F}_\Theta(m)$ are only singletons, $m(\cdot)$ is called a *Bayesian BBA* [2] and its corresponding $Bel(\cdot)$ and $Pl(\cdot)$ functions are homogeneous to a same (subjective) probability measure $P(\cdot)$, and in this case $\mathcal{F}_{A^*}(m) = \emptyset$. According to Shafer's Theorem 2.9, see [2] page 39 with its proof on page 51, the belief functions can be characterized without referencing to a BBA. The quantities $m(\cdot)$ and $Bel(\cdot)$ are one-to-one, and for any $A \subseteq \Theta$ the BBA $m(\cdot)$ is obtained from $Bel(\cdot)$ by Möbius inverse formula (see [2], p.39)

$$m(A) = \sum_{B \subseteq A \subseteq \Theta} (-1)^{|A-B|} Bel(B) \quad (9)$$

Because for any partition $\{A_1, \ldots, A_k\}$ of the FoD Θ, the equality $\mathcal{F}_\Theta(m) = \mathcal{F}_{A_1}(m) \cup \ldots \cup \mathcal{F}_{A_k}(m) \cup \mathcal{F}_{A^*}(m)$ with $\mathcal{F}_{A^*}(m) \triangleq \mathcal{F}_\Theta(m) - \mathcal{F}_{A_1}(m) - \ldots - \mathcal{F}_{A_k}(m)$ is valid, the following Total Belief Theorem (TBT) holds – see proof in the companion paper [7].

Total Belief Theorem (TBT): Let's consider a FoD Θ with $|\Theta| \geq 2$ elements and a BBA $m(\cdot)$ defined on 2^Θ with the set of focal elements $\mathcal{F}_\Theta(m)$. For any chosen partition $\{A_1, \ldots, A_k\}$ of Θ and for any $B \subseteq \Theta$, one has

$$Bel(B) = \sum_{i=1,\ldots,k} Bel(A_i \cap B) + U(A^* \cap B) \quad (10)$$

where $U(A^* \cap B) \triangleq \sum_{X \in \mathcal{F}_{A^*}(m) | X \in \mathcal{F}_B(m)} m(X) \in [0, 1]$.

By expressing $Bel(\bar{B})$ using TBT and noting that $Pl(B) = 1 - Bel(\bar{B})$, one get the Total Plausibility Theorem (TPlT) [7], which states that for any partition $\{A_1, \ldots, A_k\}$ of Θ and any $B \subseteq \Theta$, one has

$$Pl(B) = \sum_{i=1,\ldots,k} Pl(\bar{A}_i \cup B) + 1 - k - U(A^* \cap \bar{B}) \quad (11)$$

[2]the notation $\triangleq$ means *equal by definition*

[3]More generally, the set of all focal elements of $m(\cdot)$ included in a subset $A \subseteq \Theta$ is denoted $\mathcal{F}_A(m)$.

[4]By convention, a *sum of non existing terms* (if it occurs in formulas depending on the given BBA) is always set to zero.

where $U(A^* \cap \bar{B}) \triangleq \sum_{X \in \mathcal{F}_{A^*}(m) | X \in \mathcal{F}_{\bar{B}}(m)} m(X) \ \in [0,1]$.

In DST framework, Shafer [2] did propose to combine $s \geq 2$ distinct sources of evidence represented by BBAs $m_1(.), \ldots, m_s(.)$ over the same FoD with Dempster's rule of combination (i.e. the normalized conjunctive rule). The justification and behavior of Dempster's rule have however been strongly disputed from both theoretical and practical standpoints as reported in [12]–[15]. Furthermore, Shafer did use also Dempster's rule to establish formulas for conditional belief and plausibility functions [2]. Unfortunately, Shafer's conditioning formulas are inconsistent with lower and upper bounds of imprecise conditional probability values as discussed in [6], [16], [18] – see also Ellsberg's urn example in [7]. That is why we do not recommend Shafer's conditioning and Dempster' rule in applications involving belief functions. This standpoint has been already shared by several authors before us, see by example [6], [8], [16], [19]–[21].

Recently in [7], we have proved that Fagin-Halpern conditional belief and plausibility formulas [6], [16], [17] can be directly obtained from TBT to define the conditional belief as the lower envelope (i.e. the infimum) of a family of conditional probability functions to make belief conditioning consistent with imprecise conditional probability calculus. In this paper we do not enter in details on the justification of Fagin-Halpern conditioning formulas but we just need to recall their expressions because they will be used in the next section to define the notion of credibilistic independence (or C-independence for short). Assuming $Bel(B) > 0$, Fagin and Halpern proposed the following conditional formulas (FH formulas for short)

$$Bel(A|B) = Bel(A \cap B)/(Bel(A \cap B) + Pl(\bar{A} \cap B)) \quad (12)$$

$$Pl(A|B) = Pl(A \cap B)/(Pl(A \cap B) + Bel(\bar{A} \cap B)) \quad (13)$$

Fagin and Halpern proved in [6] that $Bel(A|B)$ given by (12) is a true belief function[5]. Later, Sundberg and Wagner in [20] (p. 268) did give a clearer proof also (not very easy to follow though). By switching notations and assuming $Bel(A) > 0$, the previous FH formulas yield

$$Bel(B|A) = Bel(A \cap B)/(Bel(A \cap B) + Pl(\bar{B} \cap A)) \quad (14)$$

$$Pl(B|A) = Pl(A \cap B)/(Pl(A \cap B) + Bel(\bar{B} \cap A)) \quad (15)$$

In [7], we did also generalize Bayes' Theorem for working in the framework of belief functions as follows.

Generalized Bayes' Theorem (GBT): For any partition $\{A_1, \ldots, A_k\}$ of a FoD Θ, any belief function $Bel(\cdot) : 2^\Theta \mapsto [0,1]$, and any subset B of Θ with $Bel(B) > 0$, one has for $i \in \{1, \ldots, k\}$

$$Bel(A_i|B) = \frac{Bel(B|A_i)q(A_i, B)}{\sum_{i=1}^{k} Bel(B|A_i)q(A_i, B) + U((\bar{A}_i \cap B)^*)} \quad (16)$$

[5]satisfying the three conditions of Shafer's Theorem 2.9, see [2] page 39.

where

$$\begin{aligned} q(A_i, B) &\triangleq Bel(A_i) + U((\bar{B} \cap A_i)^*) - U(B^* \cap A_i) \\ U((\bar{B} \cap A_i)^*) &\triangleq Pl(\bar{B} \cap A_i) - Bel(\bar{B} \cap A_i) \\ U((\bar{A}_i \cap B)^*) &\triangleq Pl(\bar{A}_i \cap B) - Bel(\bar{A}_i \cap B) \\ U(B^* \cap A_i) &\triangleq \sum_{X \in \mathcal{F}_{B^*}(m) | X \in \mathcal{F}_{A_i}(m)} m(X) \end{aligned}$$

Note that FH formulas are consistent with Bayes formula (i.e. conditional probability formula) when the underlying BBA $m(\cdot)$ is Bayesian. Indeed if $m(\cdot)$ is Bayesian, then $Pl(A \cap B) = Bel(A \cap B) = P(A \cap B)$, $Pl(\bar{A} \cap B) = Bel(\bar{A} \cap B) = P(\bar{A} \cap B)$ and $Pl(\bar{B} \cap A) = Bel(\bar{B} \cap A) = P(\bar{B} \cap A)$ and FH formulas become

$$Bel(A|B) = Pl(A|B) = P(A \cap B)/P(B) = P(A|B) \quad (17)$$

$$Bel(B|A) = Pl(B|A) = P(A \cap B)/P(A) = P(B|A) \quad (18)$$

The advantage of FH formulas is their complete compatibility with the bounds of conditional probability calculus [20] and their theoretical constructive justification drawn from TBT.

IV. Notion of credibilistic independence

In this section we generalize in the belief functions framework the notion of probabilistic independence of two events A and B expressed by the condition $P(A \cap B) = P(A)P(B)$.

A. Definition of credibilistic independence

To define the credibilistic independence of two propositions A and B, we start from the FH belief conditioning formulas (12)–(15) and we impose the Credibilistic Independence Constraints (CIC) by analogy of what has been done in the framework of probabilistic framework. So, we require the conditions

$$Bel(A|B) = Bel(A) \quad (19)$$

$$Bel(B|A) = Bel(B) \quad (20)$$

$$Pl(A|B) = Pl(A) \quad (21)$$

$$Pl(B|A) = Pl(B) \quad (22)$$

which reflect the notion of independence of propositions A and B.

Working with conditional belief expressions, the formula (12) and the condition (19) yield

$$Bel(A)[Bel(A \cap B) + Pl(\bar{A} \cap B)] = Bel(A \cap B)$$

or equivalently

$$Bel(A)Pl(\bar{A} \cap B) = Bel(A \cap B)[1 - Bel(A)]$$

By noting that $1 - Bel(A) = Pl(\bar{A})$ and dividing both sides of the previous equality by $Pl(\bar{A})$ (assumed strictly positive), we get

$$Bel(A \cap B) = \frac{Bel(A)}{Pl(\bar{A})} Pl(\bar{A} \cap B) \quad (23)$$

Similarly, the formula (14) and the condition (20) yield

$$Bel(B)[Bel(A \cap B) + Pl(A \cap \bar{B})] = Bel(A \cap B)$$

or equivalently

$$Bel(B)Pl(A \cap \bar{B}) = Bel(A \cap B)[1 - Bel(B)]$$

By noting that $1 - Bel(B) = Pl(\bar{B})$ and dividing both sides of the previous equality by $Pl(\bar{B})$ (assumed strictly positive), we get

$$Bel(A \cap B) = \frac{Bel(B)}{Pl(\bar{B})} Pl(A \cap \bar{B}) \tag{24}$$

If CIC (19) and (20) are satisfied, then because of (23) and (24), one must have also the following equality satisfied

$$Bel(A \cap B) = \frac{Bel(A)}{Pl(\bar{A})} Pl(\bar{A} \cap B) = \frac{Bel(B)}{Pl(\bar{B})} Pl(A \cap \bar{B})$$

This equality imposes the following condition to be satisfied

$$Bel(A)Pl(\bar{B})Pl(\bar{A} \cap B) = Pl(\bar{A})Bel(B)Pl(A \cap \bar{B}) \tag{25}$$

One sees that this equality is always satisfied if one has

$$Pl(A \cap \bar{B}) = Bel(A)Pl(\bar{B}) \tag{26}$$

$$Pl(\bar{A} \cap B) = Pl(\bar{A})Bel(B) \tag{27}$$

Working with conditional plausibility expressions, the formula (13) and the condition (21) yield

$$Pl(A)[Pl(A \cap B) + Bel(\bar{A} \cap B)] = Pl(A \cap B)$$

or equivalently

$$Pl(A \cap B) = \frac{Pl(A)}{Bel(\bar{A})} Bel(\bar{A} \cap B) \tag{28}$$

The formula (15) and the condition (22) yield

$$Pl(B)[Pl(A \cap B) + Bel(A \cap \bar{B})] = Pl(A \cap B)$$

or equivalently

$$Pl(A \cap B) = \frac{Pl(B)}{Bel(\bar{B})} Bel(A \cap \bar{B}) \tag{29}$$

If CIC (21) and (22) are satisfied, then because of (28) and (29), one must have also the following equality satisfied

$$Pl(A \cap B) = \frac{Pl(A)}{Bel(\bar{A})} Bel(\bar{A} \cap B) = \frac{Pl(B)}{Bel(\bar{B})} Bel(A \cap \bar{B})$$

This equality imposes the following condition to be satisfied

$$Pl(A)Bel(\bar{B})Bel(\bar{A} \cap B) = Bel(\bar{A})Pl(B)Bel(A \cap \bar{B}) \tag{30}$$

One sees that this equality is always satisfied if one has

$$Bel(A \cap \bar{B}) = Pl(A)Bel(\bar{B}) \tag{31}$$

$$Bel(\bar{A} \cap B) = Bel(\bar{A})Pl(B) \tag{32}$$

In summary, the four CIC are satisfied whenever the two following conditions are satisfied for the two belief intervals $[Bel(A \cap \bar{B}), Pl(A \cap \bar{B})]$ and $[Bel(\bar{A} \cap B), Pl(\bar{A} \cap B)]$.

- **Condition** C_1:

$$[Bel(A \cap \bar{B}), Pl(A \cap \bar{B})] = [Pl(A)Bel(\bar{B}), Bel(A)Pl(\bar{B})] \tag{33}$$

- **Condition** C_2:

$$[Bel(\bar{A} \cap B), Pl(\bar{A} \cap B)] = [Bel(\bar{A})Pl(B), Pl(\bar{A})Bel(B)] \tag{34}$$

The conditions C_1 and C_2 are in fact just necessary conditions but not sufficient conditions because one needs also to impose the coherence conditions C_3 and C_4 stating that right bound of any belief interval must always be greater (or equal) than its left bound. Hence the following inequalities (35) and (37) must also be satisfied.

- **Condition** C_3: The constraint $Bel(A \cap \bar{B}) \leq Pl(A \cap \bar{B})$ and (33) impose to have

$$Pl(A)Bel(\bar{B}) \leq Bel(A)Pl(\bar{B}) \tag{35}$$

which is equivalent to the condition[6]

$$Pl(A) - Bel(A) \leq Pl(A)Pl(B) - Bel(A)Bel(B) \tag{36}$$

- **Condition** C_4: The constraint $Bel(\bar{A} \cap B) \leq Pl(\bar{A} \cap B)$ and (34) impose to have

$$Bel(\bar{A})Pl(B) \leq Pl(\bar{A})Bel(B) \tag{37}$$

which is equivalent to the condition[7]

$$Pl(B) - Bel(B) \leq Pl(A)Pl(B) - Bel(A)Bel(B) \tag{38}$$

Thus, the conditions C_1, C_2, C_3 and C_4 characterize mathematically the notion of credibilistic independence (C-Indep) between two propositions A and B according to a given BBA. This allows us to establish the following theorem.

C-Indep Theorem: Consider a FoD Θ and a BBA $m(\cdot) : 2^\Theta \mapsto [0,1]$ and A and B two subsets of Θ. The two propositions A and B are said credibilistically independent if and only if

$$[Bel(A \cap \bar{B}), Pl(A \cap \bar{B})] = [Pl(A)Bel(\bar{B}), Bel(A)Pl(\bar{B})]$$
$$[Bel(\bar{A} \cap B), Pl(\bar{A} \cap B)] = [Bel(\bar{A})Pl(B), Pl(\bar{A})Bel(B)]$$

and

$$Pl(A) - Bel(A) \leq Pl(A)Pl(B) - Bel(A)Bel(B)$$
$$Pl(B) - Bel(B) \leq Pl(A)Pl(B) - Bel(A)Bel(B)$$

where $Bel(\cdot)$ and $Pl(\cdot)$ are respectively the belief and plausibility functions related to the BBA $m(\cdot)$.

Remark: Fagin-Halpern formulas (12)–(15) are defined only if $Bel(B) > 0$ and if $Bel(A) > 0$. This means that $[Bel(A), Pl(A)] =]a_1, a_2]$ is a left open interval (excluding $a_1 = 0$ and with $a_1 \leq a_2 \leq 1$) and $[Bel(B), Pl(B)] =]b_1, b_2]$ is also a left open interval (excluding $b_1 = 0$ and with $b_1 \leq b_2 \leq 1$). The credibilistic independence conditions of C-Indep Theorem can however be satisfied even[8] if $Bel(A) = 0$, or $Bel(B) = 0$, but in this case the Fagin-Halpern formulas yield 0/0 indeterminate form, which is perfectly normal.

[6]Substitute $Bel(\bar{B})$ by $1 - Pl(B)$, $Pl(\bar{B})$ by $1 - Bel(B)$ and rearrange terms.

[7]Substitute $Bel(\bar{A})$ by $1 - Pl(A)$, $Pl(\bar{A})$ by $1 - Bel(A)$ and rearrange terms.

[8]This is similar to probabilistic independence condition, where the condition $P(A \cap B) = P(A)P(B)$ is valid even if $P(A) = 0$, or $P(B) = 0$, or if both equalities hold.

B. Discussion

The propositions A and B can be credibilistically independent even if some of their lower or upper bounds equal respectively to zero or one as it will be shown in the next section. In this case, one can make a preliminary simple (pre-filtering) test to check the necessary condition that the left (lower) bound of belief interval must always be less (or equal) to right bound. For establishing such a test, it is worth noting that the following implications are true.

$$A \subseteq B \Rightarrow Bel(A) \leq Bel(B) \tag{39}$$

$$A \subseteq B \Rightarrow Pl(A) \leq Pl(B) \tag{40}$$

Proof: Indeed, if $A \subseteq B$, then $B - A$ (the complement of A in B) is also a subset of B. Since we have $B = A \cup (B - A)$ and $A \cap (B - A) = \emptyset$, from the definition of $Bel(.)$ function, one can write

$$\begin{aligned} Bel(B) &= \sum_{X \subseteq B} m(X) \\ &= \sum_{X \subseteq A \cup (B-A)} m(X) \\ &= \sum_{X \subseteq A} m(X) + \sum_{X \subseteq B-A} m(X) \end{aligned}$$

which is obviously greater (or equal) to $Bel(A) = \sum_{X \subseteq A} m(X)$. Therefore (39) is true.

Because[9] $A \subseteq B \Rightarrow \bar{B} \subseteq \bar{A}$ where $\bar{A} \triangleq \Theta - A$ and $\bar{B} \triangleq \Theta - B$ (the complements of A and of B in the FoD Θ), one always has $Bel(\bar{B}) \leq Bel(\bar{A})$. Hence, $-Bel(\bar{A}) \leq -Bel(\bar{B})$, and thus $[Pl(A) = 1 - Bel(\bar{A})] \leq [Pl(B) = 1 - Bel(\bar{B})]$. Therefore (40) is also true.

Because $A \cap B$ is always included in A and in B, one always has $Bel(A \cap B) \leq Bel(A)$ and $Bel(A \cap B) \leq Bel(B)$. For the same reason, $Pl(A \cap B) \leq Pl(A)$ and $Pl(A \cap B) \leq Pl(B)$. Therefore the following inequalities always hold

$$Bel(A \cap B) \leq \min\{Bel(A), Bel(B)\} \tag{41}$$

$$Pl(A \cap B) \leq \min\{Pl(A), Pl(B)\} \tag{42}$$

Let's examine the bounds of the belief interval for the condition C_1 given in (33), which is

$$[Bel(A \cap \bar{B}), Pl(A \cap \bar{B})] = [Pl(A)Bel(\bar{B}), Bel(A)Pl(\bar{B})]$$

- Lower bound of belief interval: Because $Bel(A \cap \bar{B}) \leq \min\{Bel(A), Bel(\bar{B})\}$ and $Bel(A \cap \bar{B}) = Pl(A)Bel(\bar{B})$, the following condition

  $$Pl(A)Bel(\bar{B}) \leq \min\{Bel(A), Bel(\bar{B})\}$$

  must be satisfied. In fact, because $Pl(A)Bel(\bar{B}) \leq Bel(\bar{B})$ is always true because $Pl(A) \in [0, 1]$, the following coherence condition must hold

  $$Pl(A)Bel(\bar{B}) \leq Bel(A) \tag{43}$$

  or equivalently (because $Bel(\bar{B}) = 1 - Pl(B)$)

  $$Pl(A) - Bel(A) \leq Pl(A)Pl(B) \tag{44}$$

  Note that the constraint (43) is a bit less restrictive than the inequality (35) of condition C_3. This coherence constraint says that the uncertainty on A must be less than the product of plausibilities of A and of B if one wants to have equality for the lower bound of belief interval $Bel(A \cap \bar{B}) = Pl(A)Bel(\bar{B})$ possible.

- Upper bound of belief interval: Because $Pl(A \cap \bar{B}) \leq \min\{Pl(A), Pl(\bar{B})\}$ and $Pl(A \cap \bar{B}) = Bel(A)Pl(\bar{B})$, the following condition

  $$Bel(A)Pl(\bar{B}) \leq \min\{Pl(A), Pl(\bar{B})\}$$

  must be satisfied. In fact, because $Bel(A)Pl(\bar{B}) \leq Pl(\bar{B})$ is always true because $Bel(A) \in [0, 1]$, the following coherence condition must hold

  $$Bel(A)Pl(\bar{B}) \leq Pl(A) \tag{45}$$

  Using the fact that $Pl(\bar{B}) = 1 - Bel(B)$ in (45), and rearranging terms we get

  $$Bel(A)(1 - Bel(B)) \leq Pl(A) \tag{46}$$

  $$Bel(A) - Bel(A)Bel(B)) \leq Pl(A) \tag{47}$$

  $$-Bel(A)Bel(B)) \leq Pl(A) - Bel(A) \tag{48}$$

  As we see, the inequality (48) is always satisfied because $Bel(A)$, $Bel(B)$ and $Pl(A)$ belong to $[0, 1]$ and because $Pl(A) \geq Bel(A)$, so that $-Bel(A)Bel(B) \leq 0$ whereas $Pl(A) - Bel(A) \geq 0$.
  Thus, there is in fact no need for a coherence constraint for the upper bound of belief interval to allow the equality $Pl(A \cap \bar{B}) = Bel(A)Pl(\bar{B})$ possible.

Let's examine the bounds of the belief interval for the condition C_2 given in (34), which is

$$[Bel(\bar{A} \cap B), Pl(\bar{A} \cap B)] = [Bel(\bar{A})Pl(B), Pl(\bar{A})Bel(B)]$$

- Lower bound of belief interval: Because $Bel(\bar{A} \cap B) \leq \min\{Bel(\bar{A}), Bel(B)\}$ and $Bel(\bar{A} \cap B) = Bel(\bar{A})Pl(B)$, the following condition

  $$Bel(\bar{A})Pl(B) \leq \min\{Bel(\bar{A}), Bel(B)\}$$

  must be satisfied. In fact, because $Bel(\bar{A})Pl(B) \leq Bel(\bar{A})$ is always true because $Pl(B) \in [0, 1]$, the following coherence condition must hold

  $$Bel(\bar{A})Pl(B) \leq Bel(B) \tag{49}$$

  or equivalently (because $Bel(\bar{A}) = 1 - Pl(A)$)

  $$Pl(B) - Bel(B) \leq Pl(A)Pl(B) \tag{50}$$

  Note that the constraint (49) is a bit less restrictive than the inequality (37) of condition C_4. This coherence constraint says that the uncertainty on B must be less than

[9] Letting $x \in \bar{B}$ says that x does not belong to B, but the hypothesis $A \subseteq B$ tells us that A is included in B and hence x does not belong to A as well, or in other words $x \in \bar{A}$. Therefore we have proven $\bar{B} \subseteq \bar{A}$.

the product of plausibilities of A and of B if one wants to have equality for the lower bound of belief interval $Bel(\bar{A} \cap B) = Bel(\bar{A})Pl(B)$ possible.

- Upper bound of belief interval: Because $Pl(\bar{A} \cap B) \leq \min\{Pl(\bar{A}), Pl(B)\}$ and $Pl(\bar{A} \cap B) = Pl(\bar{A})Bel(B)$, the following condition

$$Pl(\bar{A})Bel(B) \leq \min\{Pl(\bar{A}), Pl(B)\}$$

must be satisfied. In fact, because $Pl(\bar{A})Bel(B) \leq Pl(\bar{A})$ is always true because $Bel(B) \in [0,1]$, the following coherence condition must hold

$$Pl(\bar{A})Bel(B) \leq Pl(B) \tag{51}$$

Using the fact that $Pl(\bar{A}) = 1 - Bel(A)$ in (51), and rearranging terms we get

$$(1 - Bel(A))Bel(B) \leq Pl(B) \tag{52}$$

$$Bel(B) - Bel(A)Bel(B)) \leq Pl(B) \tag{53}$$

$$-Bel(A)Bel(B)) \leq Pl(B) - Bel(B) \tag{54}$$

As we see, the inequality (54) is always satisfied because $Bel(A)$, $Bel(B)$ and $Pl(B)$ belong to $[0,1]$ and because $Pl(B) \geq Bel(B)$, so that $-Bel(A)Bel(B)) \leq 0$ whereas $Pl(B) - Bel(B) \geq 0$.
Thus, there is in fact no need for a coherence constraint for the upper bound of belief interval to allow the equality $Pl(\bar{A} \cap B) = Pl(\bar{A})Bel(B)$ possible.

In summary, the conditions

$$Pl(A) - Bel(A) \leq Pl(A)Pl(B) \tag{55}$$

$$Pl(B) - Bel(B) \leq Pl(A)Pl(B) \tag{56}$$

are necessary for the coherence of belief interval bounds defined in the conditions C_1 and C_2. They express the fact that the width of belief interval (i.e. the uncertainty) of the proposition A and B must be less than the product of their plausibilities. The conditions (55)–(56) are very convenient to test quickly the non credibilistic independence of A and B, because if at least one condition (55), or (56) (or both) is not satisfied, then we are sure that A and B cannot be credibilistically independent. If the inequalities (55)–(56) are satisfied, we need to check if the conditions C_1, C_2, C_3 and C_4 are also satisfied to declare the credibilistic independence of A and B.

C. Special case: Bayesian belief functions

The notion of credibilistic independence defined in the previous section is a generalization of the notion of probabilistic independence. This can be justified (and verified) by examining what provides the conditions C_1, C_2, C_3 and C_4 in the limit case when the BBA $m(\cdot)$ is Bayesian. In this case, belief function $Bel(\cdot)$ and plausibility function $Pl(\cdot)$ coincide with a probability measure $P(\cdot)$, which means that the conditions C_3 and C_4 characterized by formulas (36) and (38) are always satisfied because $Pl(A) = Bel(A)$, and $Pl(B) = Bel(B)$. Moreover, the conditions C_1 and C_2 become equalities between the following degenerate intervals

$$[P(A \cap \bar{B}), P(A \cap \bar{B})] = [P(A)P(\bar{B}), P(A)P(\bar{B})]$$
$$[P(\bar{A} \cap B), P(\bar{A} \cap B)] = [P(\bar{A})P(B), P(\bar{A})P(B)]$$

or equivalently

$$P(A \cap \bar{B}) = P(A)P(\bar{B})$$
$$P(\bar{A} \cap B) = P(\bar{A})P(B)$$

These conditions are in fact equivalent to the probabilistic independence condition $P(A \cap B) = P(A)P(B)$. This can be shown from the TPT formulas $P(A \cap \bar{B}) + P(A \cap B) = P(A)$ and $P(A \cap B) + P(\bar{A} \cap B) = P(B)$ as follows.

- If $P(A \cap \bar{B}) = P(A)P(\bar{B})$, then $P(A \cap \bar{B}) + P(A \cap B) = P(A)P(\bar{B}) + P(A \cap B) = P(A)$, and thus $P(A \cap B) = P(A)(1 - P(\bar{B})) = P(A)P(B)$.
- If $P(\bar{A} \cap B) = P(\bar{A})P(B)$, then $P(A \cap B) + P(\bar{A} \cap B) = P(A \cap B) + P(\bar{A})P(B) = P(B)$, and thus $P(A \cap B) = (1 - P(\bar{A}))P(B) = P(A)P(B)$.

Therefore, one has proved that our notion of credibilistic independence derived from FH conditioning coincides with the notion of probabilistic independence as soon as the belief function under consideration is Bayesian.

V. Illustrative examples

For convenience (and not for significance), we give some simple examples illustrating the credibilistic independence between two propositions A and B with respect to some given basic belief assignments, so that the reader will be able to check by himself how to perform the derivations.

A. Example 1 (Bayesian case)

Let consider the FoD $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6\}$ and the (uniform) Bayesian BBA defined by $m(\theta_i) = 1/6$ for $i = 1, 2, \ldots, 6$. Consider the two propositions (subsets) A and B of Θ defined as $A \triangleq \theta_1 \cup \theta_2$ and $B \triangleq \theta_2 \cup \theta_4 \cup \theta_6$. In this case, $\bar{A} = \theta_3 \cup \theta_4 \cup \theta_5 \cup \theta_6$ and $\bar{B} = \theta_1 \cup \theta_3 \cup \theta_5$. We have also $A \cap \bar{B} = \theta_1$, and $\bar{A} \cap B = \theta_4 \cup \theta_6$. Because $m(\cdot)$ is a Bayesian BBA, $Bel(X) = Pl(X) = P(X)$ for $X \in 2^\Theta$. Here one has

$$Bel(A) = Bel(\theta_1 \cup \theta_2) = m(\theta_1) + m(\theta_2) = 1/3$$
$$Pl(A) = Pl(\theta_1 \cup \theta_2) = m(\theta_1) + m(\theta_2) = 1/3$$
$$Bel(\bar{A}) = Bel(\theta_3 \cup \theta_4 \cup \theta_5 \cup \theta_6) = 1 - Pl(A) = 2/3$$
$$Pl(\bar{A}) = Pl(\theta_3 \cup \theta_4 \cup \theta_5 \cup \theta_6) = 1 - Bel(A) = 2/3$$
$$Bel(B) = m(\theta_2) + m(\theta_4) + m(\theta_6) = 1/2$$
$$Pl(B) = m(\theta_2) + m(\theta_4) + m(\theta_6) = 1/2$$
$$Bel(\bar{B}) = Bel(\theta_1 \cup \theta_3 \cup \theta_5) = 1 - Pl(B) = 1/2$$
$$Pl(\bar{B}) = Pl(\theta_1 \cup \theta_3 \cup \theta_5) = 1 - Bel(B) = 1/2$$
$$Bel(A \cap \bar{B}) = m(\theta_1) = 1/6$$
$$Pl(A \cap \bar{B}) = m(\theta_1) = 1/6$$
$$Bel(\bar{A} \cap B) = Bel(\theta_4 \cup \theta_6) = m(\theta_4) + m(\theta_6) = 1/3$$
$$Pl(\bar{A} \cap B) = Pl(\theta_4 \cup \theta_6) = m(\theta_4) + m(\theta_6) = 1/3$$

Conditions C_1 and C_2 are satisfied because

$$C_1\colon \begin{cases} [Bel(A\cap\bar{B}), Pl(A\cap\bar{B})] = [\frac{1}{6}, \frac{1}{6}] \\ [Pl(A)Bel(\bar{B}), Bel(A)Pl(\bar{B})] = [\frac{1}{3}\cdot\frac{1}{2}, \frac{1}{3}\cdot\frac{1}{2}] = [\frac{1}{6}, \frac{1}{6}] \end{cases}$$

$$C_2\colon \begin{cases} [Bel(\bar{A}\cap B), Pl(\bar{A}\cap B)] = [\frac{1}{3}, \frac{1}{3}] \\ [Bel(\bar{A})Pl(B), Pl(\bar{A})Bel(B)] = [\frac{2}{3}\cdot\frac{1}{2}, \frac{2}{3}\cdot\frac{1}{2}] = [\frac{1}{3}, \frac{1}{3}] \end{cases}$$

The condition $C_3 : Pl(A)Bel(\bar{B}) \leq Bel(A)Pl(\bar{B})$ is satisfied because $Pl(A)Bel(\bar{B}) = \frac{1}{3}\cdot\frac{1}{2} = Bel(A)Pl(\bar{B})$. The condition $C_4 : Bel(\bar{A})Pl(B) \leq Pl(\bar{A})Bel(B)$ is also satisfied because $Bel(\bar{A})Pl(B) = \frac{2}{3}\cdot\frac{1}{2} = Pl(\bar{A})Bel(B)$.

Because the conditions C_1, C_2, C_3 and C_4 are satisfied, the propositions A and B are credibilistically independent. In fact, in this Bayesian case, A and B are also probabilistically independent because $P(A\cap B) = P(\theta_2) = P(A)P(B)$. Note that the coherence conditions (55) and (56) are of course satisfied because

$$[Pl(A) - Bel(A) = 0] \leq [Pl(A)Pl(B) = (1/3)\cdot(1/2)]$$
$$[Pl(B) - Bel(B) = 0] \leq [Pl(A)Pl(B) = (1/3)\cdot(1/2)]$$

B. Example 2 (Non Bayesian case)

Let consider the FoD $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5\}$ and the two propositions (subsets) A and B of Θ defined as $A \triangleq \theta_1\cup\theta_2\cup\theta_3$ and $B \triangleq \theta_3\cup\theta_4$. In this case, $\bar{A} = \theta_4\cup\theta_5$ and $\bar{B} = \theta_1\cup\theta_2\cup\theta_5$. We have also $A\cap\bar{B} = (\theta_1\cup\theta_2\cup\theta_3)\cap(\theta_1\cup\theta_2\cup\theta_5) = \theta_1\cup\theta_2$ and $\bar{A}\cap B = (\theta_4\cup\theta_5)\cap(\theta_3\cup\theta_4) = \theta_4$. Suppose that the BBA $m(\cdot)$ is simply defined as[10]

$$m(\theta_1) = 0.5 \qquad m(\theta_3) = 0.1 \qquad m(\theta_1\cup\theta_3) = 0.4$$

Based on the BBA $m(\cdot)$, the belief and plausibilities of propositions involved in the derivations are

$$[Bel(A), Pl(A)] = [1, 1], \quad [Bel(\bar{A}), Pl(\bar{A})] = [0, 0],$$
$$[Bel(B), Pl(B)] = [0.1, 0.5], \quad [Bel(\bar{B}), Pl(\bar{B})] = [0.5, 0.9],$$
$$[Bel(A\cap B), Pl(A\cap B)] = [0.1, 0.5],$$
$$[Bel(A\cap\bar{B}), Pl(A\cap\bar{B})] = [0.5, 0.9],$$
$$[Bel(\bar{A}\cap B), Pl(\bar{A}\cap B)] = [0, 0].$$

The condition C_1 is satisfied because

$$C_1\colon \begin{cases} [Bel(A\cap\bar{B}), Pl(A\cap\bar{B})] = [0.5, 0.9] \\ [Pl(A)Bel(\bar{B}), Bel(A)Pl(\bar{B})] = [1\cdot 0.5, 1\cdot 0.9] \end{cases}$$

The condition C_2 is also satisfied because

$$C_2\colon \begin{cases} [Bel(\bar{A}\cap B), Pl(\bar{A}\cap B)] = [0, 0] \\ [Bel(\bar{A})Pl(B), Pl(\bar{A})Bel(B)] = [0\cdot 0.5, 0\cdot 0.1] \end{cases}$$

The condition C_3 given by $Pl(A)Bel(\bar{B}) \leq Bel(A)Pl(\bar{B})$ is satisfied because $Pl(A)Bel(\bar{B}) = 1\cdot 0.5 = 0.5$ and $Bel(A)Pl(\bar{B}) = 1\cdot 0.9 = 0.9$.

[10]All other elements of 2^Θ which are not focal elements of the BBA $m(\cdot)$ receive a zero value.

The condition C_4 given by $Bel(\bar{A})Pl(B) \leq Pl(\bar{A})Bel(B)$ is satisfied because $Bel(\bar{A})Pl(B) = 0\cdot 0.5 = 0$ and $Pl(\bar{A})Bel(B) = 0\cdot 0.1 = 0$.
Therefore the propositions A and B are credibilistically independent. One can easily verify using Fagin-Halpern formulas that $[Bel(A|B), Pl(A|B)] = [Bel(A), Pl(A)]$ and $[Bel(B|A), Pl(B|A)] = [Bel(B), Pl(B)]$. Indeed, in applying (12) and (13) one gets

$$\begin{aligned} Bel(A|B) &= \frac{Bel(A\cap B)}{Bel(A\cap B) + Pl(\bar{A}\cap B)} = \frac{0.1}{0.1+0} \\ &= 1 = Bel(A) \\ Pl(A|B) &= \frac{Pl(A\cap B)}{Pl(A\cap B) + Bel(\bar{A}\cap B)} = \frac{0.5}{0.5+0} \\ &= 1 = Pl(A) \end{aligned}$$

and in applying (14) and (15), one gets

$$\begin{aligned} Bel(B|A) &= \frac{Bel(A\cap B)}{Bel(A\cap B) + Pl(A\cap\bar{B})} = \frac{0.1}{0.1+0.9} \\ &= 0.1 = Bel(B) \\ Pl(B|A) &= \frac{Pl(A\cap B)}{Pl(A\cap B) + Bel(A\cap\bar{B})} = \frac{0.5}{0.5+0.5} \\ &= 0.5 = Pl(B) \end{aligned}$$

Note that the coherence conditions (55) and (56) are of course satisfied because

$$[Pl(A) - Bel(A) = 0] \leq [Pl(A)Pl(B) = 1\cdot 0.5 = 0.5]$$
$$[Pl(B) - Bel(B) = 0.4] \leq [Pl(A)Pl(B) = 1\cdot 0.5 = 0.5]$$

C. Example 3 (Non Bayesian case)

Here we consider a more interesting example where the widths of belief intervals are not all restricted to zero. Consider the frame of discernment $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$ and the very simple BBA $m(\cdot)$ defined by $m(\theta_3) = 0.7$ and $m(\theta_1\cup\theta_3) = 0.3$. We consider the propositions $A \triangleq \theta_1\cup\theta_3$ and $B \triangleq \theta_2\cup\theta_3$. In this case, we have $\bar{A} = \theta_2\cup\theta_4$, $\bar{B} = \theta_1\cup\theta_4$, $A\cap B = \theta_3$, $A\cap\bar{B} = \theta_1$ and $\bar{A}\cap B = \theta_2$. Based on the BBA $m(\cdot)$, the belief and plausibilities of propositions involved in the derivations are

$$[Bel(A), Pl(A)] = [1, 1], \quad [Bel(\bar{A}), Pl(\bar{A})] = [0, 0],$$
$$[Bel(B), Pl(B)] = [0.7, 1], \quad [Bel(\bar{B}), Pl(\bar{B})] = [0, 0.3],$$
$$[Bel(A\cap B), Pl(A\cap B)] = [0.7, 1],$$
$$[Bel(A\cap\bar{B}), Pl(A\cap\bar{B})] = [0, 0.3],$$
$$[Bel(\bar{A}\cap B), Pl(\bar{A}\cap B)] = [0, 0].$$

The condition C_1 is satisfied because

$$C_1\colon \begin{cases} [Bel(A\cap\bar{B}), Pl(A\cap\bar{B})] = [0, 0.3] \\ [Pl(A)Bel(\bar{B}), Bel(A)Pl(\bar{B})] = [1\cdot 0, 1\cdot 0.3] \end{cases}$$

The condition C_2 is satisfied because

$$C_2\colon \begin{cases} [Bel(\bar{A}\cap B), Pl(\bar{A}\cap B)] = [0, 0] \\ [Bel(\bar{A})Pl(B), Pl(\bar{A})Bel(B)] = [0\cdot 1, 0\cdot 0.7] \end{cases}$$

The condition C_3 given by $Pl(A)Bel(\bar{B}) \leq Bel(A)Pl(\bar{B})$ is satisfied because $Pl(A)Bel(\bar{B}) = 1 \cdot 0 = 0$ and $Bel(A)Pl(\bar{B}) = 1 \cdot 0.3 = 0.3$.
The condition C_4 given by $Bel(\bar{A})Pl(B) \leq Pl(\bar{A})Bel(B)$ is satisfied because $Bel(\bar{A})Pl(B) = 0 \cdot 1 = 0$ and $Pl(\bar{A})Bel(B) = 0 \cdot 0.7 = 0$.
Therefore the propositions A and B are credibilistically independent. One can easily verify using Fagin-Halpern formulas that $[Bel(A|B), Pl(A|B)] = [Bel(A), Pl(A)]$ and $[Bel(B|A), Pl(B|A)] = [Bel(B), Pl(B)]$. Indeed, in applying (12) and (13) one gets

$$\begin{aligned} Bel(A|B) &= \frac{Bel(A \cap B)}{Bel(A \cap B) + Pl(\bar{A} \cap B)} = \frac{0.7}{0.7 + 0} \\ &= 1 = Bel(A) \\ Pl(A|B) &= \frac{Pl(A \cap B)}{Pl(A \cap B) + Bel(\bar{A} \cap B)} = \frac{1}{1 + 0} \\ &= 1 = Pl(A) \end{aligned}$$

and in applying (14) and (15), one gets

$$\begin{aligned} Bel(B|A) &= \frac{Bel(A \cap B)}{Bel(A \cap B) + Pl(A \cap \bar{B})} = \frac{0.7}{0.7 + 0.3} \\ &= 0.7 = Bel(B) \\ Pl(B|A) &= \frac{Pl(A \cap B)}{Pl(A \cap B) + Bel(A \cap \bar{B})} = \frac{1}{1 + 0} \\ &= 1 = Pl(B) \end{aligned}$$

Note that the coherence conditions (55) and (56) are of course satisfied because

$$[Pl(A) - Bel(A) = 0] \leq [Pl(A)Pl(B) = 1 \cdot 1 = 1]$$
$$[Pl(B) - Bel(B) = 0.3] \leq [Pl(A)Pl(B) = 1 \cdot 1 = 1]$$

VI. Conclusions

In this paper the notion of credibilistic independence of two propositions has been proposed in the framework of belief functions. It is a generalization of the notion of (probabilistic) independence of two events defined classically in the theory of probability. Our definition is totally consistent with the probabilistic independence when the basic belief assignment is Bayesian because it is based on Fagin-Halpern belief conditioning formulas (derived from Total Belief Theorem) which are consistent with imprecise conditional probability calculus. Simple examples of the notion of credibilistic independence have also been given to illustrate how to test easily the credibilistic independence of two propositions in practice from a given basic belief assignment.

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