A Simplified Formulation of Generalized Bayes' Theorem

Jean Dezert
The French Aerospace Lab
Palaiseau, France.
jean.dezert@onera.fr

Albena Tchamova
Inst. of I&C Tech., BAS
Sofia, Bulgaria.
tchamova@bas.bg

Deqiang Han
Inst. of Integrated Automation
Xi'an Jiaotong University, China.
deqhan@gmail.com

Thanuka Wickramarathne
ECE Dept., Univ. of Massachusetts,
Lowell, MA, USA
thanuka@uml.edu

***Abstract*—In this paper we present a simple formulation of the Generalized Bayes' Theorem (GBT) which extends Bayes' theorem in the framework of belief functions. We also present the condition under which this new formulation is valid. We illustrate our theoretical results with simple examples.**

Keywords: Generalized Bayes' Theorem (GBT), Simplified GBT (SGBT), Total Belief Theorem (TBT), belief functions.

I. Introduction

Based on Dempster's works [1], [2], Shafer did introduce Belief Functions (BF) in 1976 to model the epistemic uncertainty[1] and to reason under uncertainty [3] which is referred as Dempster-Shafer Theory (DST) in the literature. Belief functions are mathematically well defined and they are very appealing from the theoretical standpoint because of their good ability to model uncertainty interpreted as imprecise probability measures in Dempster's original works.

From the end of 1970's the DST has however been cast in doubts because Dempster's rule of combination of Basic Belief Assignments (BBAs) yields counter intuitive results not only in high conflicting situations but also in low conflicting cases as well [4]–[6], and Shafer's conditioning formulas based on Dempster's rule are not consistent with conditional probability calculus [7], [8]. Discussions on the validity of DST can be found, for instance, in [4], [5], [9]–[13]. These two major concerns make DST quite risky for applications involving randomness and epistemic uncertainties and it should be replaced by better techniques to reason under uncertainty with belief functions.

In 2018 we did establish in [8], [14] two new important general results for reasoning with belief functions: the Total Belief Theorem (TBT), and the Generalized Bayes' Theorem (GBT). TBT and GBT generalize the well-known Total Probability Theorem (TPT) and Bayes' Theorem (BT) of the Probability Theory (PT). Thanks to these new theorems we have now in hands a generalized Bayesian inference mechanism for working with imprecise probability measures in the belief functions framework. Similarly to the probability theory requiring a good estimation of pdf (or pmf) involved in Bayes' formula to make a good inference, the major difficulty for applying GBT is the knowledge (or good estimation) of all[2] BBAs required in GBT. For a given size of frame of discernment, GBT requires more computations than Bayes formula (if we would prefer to work with probabilities) because we need to work with BBAs defined on the powerset of the frame of discernment.

The general formulation of GBT presented in details in [8] is not easy to apply and that is why we present in this paper a simpler and more convenient formulation of GBT providing elegant and useful mathematical expressions. The obtention of these new formulas of GBT are established from a dichotomous partitioning of the frame of discernment.

This paper is organized as follows. After a short reminder of basics of belief functions in Section II and their constructions based on Dempster's multi-valued mapping, we present briefly the Total Belief Theorem and Fagin-Halpern conditioning in Section III, and the Generalized Bayes Theorem in Section IV. In section V we establish the Simplified GBT (SGBT) drawn from GBT for working with a dichotomous partitioning of the frame of discernment. Section VI presents and discusses two examples of SGBT results. Section VII concludes this paper.

II. Basics of belief functions

A. Basic belief assignment

We consider a finite discrete frame of discernement (FoD) $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$, with $n > 1$, and where all exhaustive and exclusive elements of Θ represent the set of the potential solutions of the problem under concern. The set of all subsets of Θ (including the empty set $\emptyset$, and Θ) is the power-set of Θ denoted by 2^Θ. The number of elements (i.e. the cardinality) of 2^Θ is $2^{|\Theta|}$. A Basic Belief Assignment (BBA) associated with a given source of evidence is defined as the mapping $m(\cdot) : 2^\Theta \to [0, 1]$ satisfying the conditions $m(\emptyset) = 0$ and $\sum_{A\in 2^\Theta} m(A) = 1$. The quantity $m(A)$ is the mass of belief for subset A committed by the Source of Evidence (SoE).

B. Focal elements

A focal element X of a BBA $m(\cdot)$ is an element of 2^Θ such that $m(X) > 0$. Note that the empty set $\emptyset$ is not a focal element of a BBA because $m(\emptyset) = 0$ (closed-world assumption of Shafer's model for the FoD). The set of all focal elements of $m(\cdot)$ is denoted

$$\mathcal{F}_\Theta(m) \triangleq \{X \subseteq \Theta | m(X) > 0\} = \{X \in 2^\Theta | m(X) > 0\} \quad (1)$$

[1] Also called sometimes the cognitive uncertainty by some authors.
[2] possibly joint BBAs if we work on Cartesian product spaces [8].

The set of focal elements of $m(\cdot)$ included in $A \subseteq \Theta$ is denoted, where $\triangleq$ means *equal by definition*, by

$$\mathcal{F}_A(m) \triangleq \{X \in \mathcal{F}_\Theta(m) | X \cap A = X\} \tag{2}$$

$\mathcal{F}_\Theta(m)$ can be partitioned as $\{\mathcal{F}_A(m), \mathcal{F}_{\bar{A}}(m), \mathcal{F}_{A^*}(m)\}$ with

$$\mathcal{F}_{A^*}(m) \triangleq \mathcal{F}_\Theta(m) - \mathcal{F}_A(m) - \mathcal{F}_{\bar{A}}(m) \tag{3}$$

which represents the set of focal elements of $m(\cdot)$ which are not subsets of A, and not subsets of the complement of A in Θ which is $\bar{A} \triangleq \Theta - \{A\}$. The minus symbol in $\Theta - \{A\}$ denotes the set difference operator.

C. Belief, plausibility and uncertainty

Belief and plausibility functions are defined as[3]

$$Bel(A) \triangleq \sum_{\substack{X \in 2^\Theta \\ X \subseteq A}} m(X) = \sum_{\substack{X \in \mathcal{F}_\Theta(m) \\ X \subseteq A}} m(X) = \sum_{X \in \mathcal{F}_A(m)} m(X) \tag{4}$$

$$Pl(A) \triangleq \sum_{\substack{X \in 2^\Theta \\ X \cap A \neq \emptyset}} m(X) \tag{5}$$

$$= \sum_{\substack{X \in \mathcal{F}_\Theta(m) \\ X \cap A \neq \emptyset}} m(X) = 1 - \sum_{X \in \mathcal{F}_{\bar{A}}(m)} m(X) \tag{6}$$

$$= 1 - \mathrm{Bel}(\bar{A}). \tag{7}$$

The length of the belief interval $[Bel(A), Pl(A)]$ is usually called by abuse of terminology the *uncertainty on* A committed by the SoE. In fact it represents the imprecision on the (possibly subjective) probability of A granted by the SoE which provides the BBA $m(\cdot)$. We denote it $U(A^*)$, and it is defined as

$$U(A^*) \triangleq Pl(A) - Bel(A) = \sum_{X \in \mathcal{F}_{A^*}(m)} m(X) \tag{8}$$

If all the elements of $\mathcal{F}_\Theta(m)$ are singletons, $m(\cdot)$ is called a *Bayesian BBA* [3], and its corresponding $Bel(\cdot)$ and $Pl(\cdot)$ functions are homogeneous to a same (subjective) probability measure $P(\cdot)$. In this case $\mathcal{F}_{A^*}(m) = \mathcal{F}_{\bar{A}^*}(m) = \emptyset$. Shafer did prove in [3] (p.39) that $m(\cdot)$, $Bel(\cdot)$ and $Pl(\cdot)$ are one-to-one, and for any $A \subseteq \Theta$, $m(\cdot)$ is obtained from $Bel(\cdot)$ by Möbius inverse formula

$$m(A) = \sum_{B \subseteq A \subseteq \Theta} (-1)^{|A-B|} Bel(B) \tag{9}$$

D. Interpretation and construction of belief functions

In original Dempster's works [1] belief $Bel(A)$ and plausibility $Pl(A)$ are interpreted as lower and upper bounds of an unknown probability $P(A)$, and so $Bel(A) \leq P(A) \leq Pl(A)$. The construction of $m(A)$, $Bel(A)$ and $Pl(A)$ are mathematically well defined from an underlying random variable with a known probability measure and a given multi-valued mapping as follows:

- Consider a random variable x with its set of possible values in $\mathcal{X} = \{x_1, \ldots, x_m\}$ with known probabilities $p_j = P(x = x_j)$, $j = 1, \ldots, m$;
- Consider a FoD $\Theta = \{\theta_1, \ldots, \theta_n\}$ for the variable θ under concern;
- Consider/learn a multi-valued mapping $\Gamma : \mathcal{X} \mapsto 2^\Theta$ such that if $x = x_i$ then $\theta \in A$, so that $A = \Gamma(x_i) \in 2^\Theta$;
- The belief (lower proba) and plausibility (upper proba) that $\theta \in A$ are given by [1]

$$\begin{aligned} P_*(A) &= Bel(A) = Bel(\theta \in A) \\ &= P(\{x \in \mathcal{X} | \Gamma(x) \neq \emptyset, \Gamma(x) \subseteq A\}) \end{aligned} \tag{10}$$

$$\begin{aligned} P^*(A) &= Pl(A) = Pl(\theta \in A) \\ &= P(\{x \in \mathcal{X} | \Gamma(x) \cap A \neq \emptyset\}) \end{aligned} \tag{11}$$

The mass of belief that θ belongs to A is given by

$$m(A) = P(\{x \in \mathcal{X} | \Gamma(x) \neq \emptyset, \Gamma(x) = A\}) \tag{12}$$

Example for multi-valued mapping: Paul has been killed and Police asks a witness W: Who did you see killing Paul? Witness answer is Mary. To estimate the confidence of this testimony report one has to consider if this witness W is more or less precise when he is reliable, or if he is not reliable. So the state of W can belong to $\mathcal{X} = \{x_1, x_2, x_3\}$ where x_1 means W is precise, x_2 means W is approximate, and x_3 means W is not reliable. We suppose that the a priori probabilities of the state of W are $P(x_1) = 0.3$, $P(x_2) = 0.1$ and $P(x_3) = 0.6$. As FoD Θ, we consider a set of three suspects that includes the unknown killer

$$\Theta = \{\theta_1 = \text{Mary}, \theta_2 = \text{Peter}, \theta_3 = \text{John}\}$$

If we define the multivalued mapping $\Gamma(.)$ as follows

$$\begin{aligned} &\Gamma(x_1 = W \text{ is precise}) = \theta_1 \\ &\Gamma(x_2 = W \text{ is approximate}) = \{\theta_1, \theta_2\} \\ &\Gamma(x_3 = W \text{ is not reliable}) = \{\theta_1, \theta_2, \theta_3\} = \Theta \end{aligned}$$

$\Gamma(x_1 = W \text{ is precise}) = \theta_1$ means that if W is precise then Mary has killed Paul. $\Gamma(x_2 = W \text{ is approximate}) = \{\theta_1, \theta_2\}$ means that if W is less precise then Mary or Peter have killed Paul. $\Gamma(x_3 = W \text{ is not reliable}) = \Theta$ means that if W is not reliable then we have no useful information about the killer.

Applying formulas (10) and (11), one gets

$$\begin{aligned} Bel(\emptyset) &= P(\{x | \Gamma(x) \subseteq \emptyset\}) = P(\emptyset) = 0 = 1 - Pl(\Theta) \\ Bel(\theta_1) &= P(\{x | \Gamma(x) \subseteq \theta_1\}) \\ &= P(x_1) = 0.3 = 1 - Pl(\theta_2 \cup \theta_3) \\ Bel(\theta_2) &= P(\{x | \Gamma(x) \subseteq \theta_2\}) \\ &= 0 = 1 - Pl(\theta_1 \cup \theta_3) \\ Bel(\theta_3) &= P(\{x | \Gamma(x) \subseteq \theta_3\}) \\ &= 0 = 1 - Pl(\theta_1 \cup \theta_2) \\ Bel(\theta_1 \cup \theta_2) &= P(\{x | \Gamma(x) \subseteq \theta_1 \cup \theta_2\}) = P(\{x_1, x_2\}) \\ &= P(x_1) + P(x_2) = 0.4 = 1 - Pl(\theta_3) \end{aligned}$$

[3]By convention, a *sum of non existing terms* (if it occurs in formulas depending on the given BBA) is always set to zero.

$$\begin{aligned}
Bel(\theta_1 \cup \theta_3) &= P(\{x|\Gamma(x) \subseteq \theta_1 \cup \theta_3\}) \\
&= P(x_1) = 0.3 = 1 - Pl(\theta_2) \\
Bel(\theta_2 \cup \theta_3) &= P(\{x|\Gamma(x) \subseteq \theta_2 \cup \theta_3\}) \\
&= 0 = 1 - Pl(\theta_1) \\
Bel(\Theta) &= P(\{x|\Gamma(x) \subseteq \Theta\}) = P(\{x_1, x_2, x_3\}) \\
&= P(x_1) + P(x_2) + P(x_3) = 1 = 1 - Pl(\emptyset)
\end{aligned}$$

$$\begin{aligned}
Pl(\emptyset) &= P(\{x|\Gamma(x) \cap \emptyset \neq \emptyset\}) = P(\emptyset) = 0 = 1 - Bel(\Theta) \\
Pl(\theta_1) &= P(x|\Gamma(x) \cap \theta_1 \neq \emptyset\}) = P(\{x_1, x_2, x_3\}) \\
&= P(x_1) + P(x_2) + P(x_3) = 1 = 1 - Bel(\theta_2 \cup \theta_3) \\
Pl(\theta_2) &= P(\{x|\Gamma(x) \cap \theta_2 \neq \emptyset\}) = P(\{x_2, x_3\} \\
&= P(x_2) + P(x_3) = 0.7 = 1 - Bel(\theta_1 \cup \theta_3) \\
Pl(\theta_3) &= P(\{x|\Gamma(x) \cap \theta_3 \neq \emptyset\}) \\
&= P(x_3) = 0.6 = 1 - Bel(\theta_1 \cup \theta_2) \\
Pl(\theta_1 \cup \theta_2) &= P(\{x|\Gamma(x) \cap (\theta_1 \cup \theta_2) \neq \emptyset\}) = P(\{x_1, x_2, x_3\}) \\
&= P(x_1) + P(x_2) + P(x_3) = 1 = 1 - Bel(\theta_3) \\
Pl(\theta_1 \cup \theta_3) &= P(\{x|\Gamma(x) \cap (\theta_1 \cup \theta_3) \neq \emptyset\}) = P(\{x_1, x_2, x_3\}) \\
&= P(x_1) + P(x_2) + P(x_3) = 1 = 1 - Bel(\theta_2) \\
Pl(\theta_2 \cup \theta_3) &= P(\{x|\Gamma(x) \cap (\theta_2 \cup \theta_3) \neq \emptyset\}) = P(\{x_2, x_3\}) \\
&= P(x_2) + P(x_3) = 0.7 = 1 - Bel(\theta_1) \\
Pl(\Theta) &= P(\{x|\Gamma(x) \cap (\theta_1 \cup \theta_2 \cup \theta_3) \neq \emptyset\}) = P(\{x_1, x_2, x_3\}) \\
&= P(x_1) + P(x_2) + P(x_3) = 1 = 1 - Bel(\emptyset)
\end{aligned}$$

In applying formula (12), one gets finally the BBA

$$\begin{aligned}
m(\emptyset) &= P(\{x|\Gamma(x) = \emptyset\}) = P(\emptyset) = 0 \\
m(\theta_1) &= P(\{x|\Gamma(x) = \theta_1\}) = P(x_1) = 0.3 \\
m(\theta_2) &= P(\{x|\Gamma(x) = \theta_2\}) = 0 \\
m(\theta_3) &= P(\{x|\Gamma(x) = \theta_3\}) = 0 \\
m(\theta_1 \cup \theta_2) &= P(\{x|\Gamma(x) = \theta_1 \cup \theta_2\}) = P(x_2) = 0.1 \\
m(\theta_1 \cup \theta_3) &= P(\{x|\Gamma(x) = \theta_1 \cup \theta_3\}) = 0 \\
m(\theta_2 \cup \theta_3) &= P(\{x|\Gamma(x) = \theta_2 \cup \theta_3\}) = 0 \\
m(\Theta) &= P(\{x|\Gamma(x) = \Theta\}) = P(x_3) = 0.6
\end{aligned}$$

Some authors have proposed different interpretations of belief functions to escape the probabilistic framework introduced by Dempster to save DST of its inherent contradiction mainly due to the choice of Dempster's rule of combination and Shafer's conditioning approach based on Dempster's rule. The most important attempt has been done in 1990's by Smets in [15] with his axiomatic Transferable Belief Model (TBM). It however remains disputable because of the ambiguous (or inconsistent/double) interpretation of the empty set.

In this paper we adopt the original Dempster's interpretation and construction of belief functions because it is mathematically well defined, clear and consistent.

III. TBT and Fagin-Halpern Conditioning

A. Total Belief Theorem

In [8], we have generalized the Total Probability Theorem (TPT) [16] for working with belief functions and we proved the following simple and important theorem.

Total Belief Theorem (TBT): Let's consider a FoD Θ with $|\Theta| \geq 2$ elements and a BBA $m(\cdot)$ defined on 2^Θ with the set of focal elements $\mathcal{F}_\Theta(m)$. For any chosen partition $\{A_1, \ldots, A_k\}$ of Θ and for any $B \subseteq \Theta$, one has

$$Bel(B) = \sum_{i=1,\ldots,k} Bel(A_i \cap B) + U(A^* \cap B) \tag{13}$$

where

$$U(A^* \cap B) \triangleq \sum_{X \in \mathcal{F}_{A^*}(m) | X \in \mathcal{F}_B(m)} m(X) \tag{14}$$

and $\mathcal{F}_{A^*}(m) \triangleq \mathcal{F}_\Theta(m) - \mathcal{F}_{A_1}(m) - \ldots - \mathcal{F}_{A_k}(m)$.

Proof of TBT: see [8], with example.

From (14), one sees that $U(A^* \cap B) \in [0, 1]$. If one applies TBT with $B = \Theta$, we get $\sum_{i=1,\ldots,k} Bel(A_i) + U(A^*) = 1$ where $U(A^*) \triangleq \sum_{X \in \mathcal{F}_{A^*}(m)} m(X)$. This equality corresponds to TPT if $U(A^*) = 0$ (i.e. there is no imprecision on the value of probabilities of A_i, $i = 1, \ldots, k$).

In spite of its apparent simplicity the TBT is very important because it provides a strong theoretical justification of Fagin-Halpern (FH) belief and plausibility conditioning formulas [7], [17] proposed in 1990's as a very serious alternative to Shafer's conditioning formulas. Indeed, it can be easily proved with a simple counter-example (e.g. Ellsberg's urn example - see [8]) that conditioning formulas established by Shafer from Dempster's rule of combination are not consistent with bounds of the conditional probabilities. The main advantage of FH conditioning formulas is that they provide exact bounds of imprecise conditional probability and they coincide exactly with the conditional probability when the belief functions involved in FH formulas are Bayesian.

B. Fagin-Halpern belief conditioning formulas

In [8] we have proved that the TBT justifies the following FH conditioning formulas (assuming $Bel(B) > 0$)

$$Bel(A|B) = \frac{Bel(A \cap B)}{Bel(A \cap B) + Pl(\bar{A} \cap B)} \tag{15}$$

$$Pl(A|B) = \frac{Pl(A \cap B)}{Pl(A \cap B) + Bel(\bar{A} \cap B)} \tag{16}$$

Fagin and Halpern in [7] proved that $Bel(\cdot|B)$ is a true belief function and so FH belief conditioning is an appealing solution for belief and plausibility conditioning. A proof that FH formulas are belief functions has been also given by Sundberg and Wagner in [18]. Hence TBT provides a complete justification of FH formulas which offers a full compatibility with the conditional probability calculus [18], [19].

Similarly, by interchanging notations A and B and assuming $Bel(A) > 0$, the previous FH formulas can be expressed as

$$Bel(B|A) = \frac{Bel(A \cap B)}{Bel(A \cap B) + Pl(\bar{B} \cap A)} \tag{17}$$

$$Pl(B|A) = \frac{Pl(A \cap B)}{Pl(A \cap B) + Bel(\bar{B} \cap A)} \tag{18}$$

When $m(\cdot)$ is Bayesian $Bel(\cdot) = Pl(\cdot) = P(\cdot)$, and so $Pl(A\cap B) = Bel(A\cap B) = P(A\cap B)$, $Pl(\bar{A}\cap B) = Bel(\bar{A}\cap B) = P(\bar{A}\cap B)$ and $Pl(\bar{B}\cap A) = Bel(\bar{B}\cap A) = P(\bar{B}\cap A)$. FH formulas above reduce to

$$Bel(A|B) = Pl(A|B) = \frac{P(A\cap B)}{P(A\cap B) + P(\bar{A}\cap B)}$$

From TPT [16]) $P(A\cap B) + P(\bar{A}\cap B) = P(B)$, thus

$$Bel(A|B) = Pl(A|B) = P(A\cap B)/P(B) = P(A|B) \quad (19)$$

Similarly, one can also easily verify that

$$Bel(B|A) = Pl(B|A) = P(A\cap B)/P(A) = P(B|A) \quad (20)$$

Hence from (19) and (20) one obtains the well-known equality

$$P(A\cap B) = P(A|B)P(B) = P(B|A)P(A) \quad (21)$$

IV. Generalized Bayes' Theorem

In [8] we did also establish from TBT the following Generalized Bayes' Theorem (GBT) and lemma.

Generalized Bayes' Theorem (GBT): For any partition $\{A_1, \ldots, A_k\}$ of a FoD Θ, any belief function $Bel(\cdot) : 2^\Theta \mapsto [0, 1]$, and any subset B of Θ with $Bel(B) > 0$, one has for $i \in \{1, \ldots, k\}$

$$Bel(A_i|B) = \frac{Bel(B|A_i)q(A_i, B)}{\sum_{i=1}^{k} Bel(B|A_i)q(A_i, B) + U((\bar{A}_i \cap B)^*)} \quad (22)$$

where

$$q(A_i, B) \triangleq Bel(A_i) + U((\bar{B}\cap A_i)^*) - U(B^*\cap A_i) \quad (23)$$

with

$$U((\bar{B}\cap A_i)^*) \triangleq Pl(\bar{B}\cap A_i) - Bel(\bar{B}\cap A_i) \quad (24)$$

$$U(B^*\cap A_i) \triangleq \sum_{X\in\mathcal{F}_{B^*}(m)|X\in\mathcal{F}_{A_i}(m)} m(X) \quad (25)$$

and where

$$U((\bar{A}_i\cap B)^*) \triangleq Pl(\bar{A}_i\cap B) - Bel(\bar{A}_i\cap B) \quad (26)$$

Lemma 1: GBT degenerates to Bayes' theorem formula if $Bel(\cdot)$ is a Bayesian BF, that is

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{i=1}^{k} P(B|A_i)P(A_i)} \quad (27)$$

V. Simplified formulation of GBT

In this section we establish a simplified formulation of GBT which will be denoted SGBT for short in the sequel. Because the GBT formula (22) is not very easy to use and quite difficult to compute in applications, we propose a more useful simplified formulation of GBT which is drawn from (22) when considering only a simple dichotomous partitioning of the frame of discernment Θ. More precisely we consider a partition $\{A, \bar{A}\}$ of Θ with $A \subseteq \Theta$ and $\bar{A}$ is the complement of A in Θ, that is $\bar{A} = \Theta - \{A\}$. We establish the following theorem which is the main contribution of this paper.

Simplified Generalized Bayes' Theorem (SGBT): For any partition $\{A, \bar{A}\}$ of a FoD Θ, any belief function $Bel(\cdot) : 2^\Theta \mapsto [0, 1]$, and any subset B of Θ, one has

- If $Pl(A\cap\bar{B}) > 0$ (Condition C_1)

$$Bel(A|B) = \frac{Bel(B|A)Pl(A\cap\bar{B})}{Bel(B|A)Pl(A\cap\bar{B}) + Pl(\bar{B}|A)Pl(\bar{A}\cap B)} \quad (28)$$

- If $Bel(A\cap\bar{B}) > 0$ (Condition C_2)

$$Pl(A|B) = \frac{Pl(B|A)Bel(A\cap\bar{B})}{Pl(B|A)Bel(A\cap\bar{B}) + Bel(\bar{B}|A)Bel(\bar{A}\cap B)} \quad (29)$$

and if the denominators involved in formulas (28) and (29) are strictly positive.

Note that if condition C_2 is satisfied then the condition C_1 is also satisfied, but not necessarily the converse.

Proof of SGBT: From GBT formula (22), we replace the terms by their expressions to obtain SGBT formulas (28)–(29). For notation convenience, we denote $A_1 \triangleq A$ and $A_2 \triangleq \bar{A}$. Hence the GBT formula reduces to

$$\begin{aligned} Bel(A|B) &= \frac{Num}{Den} \\ &= \frac{Bel(B|A_1)q(A_1, B)}{\sum_{i=1}^{2} Bel(B|A_i)q(A_i, B) + U((\bar{A}_1\cap B)^*)} \end{aligned}$$

where

$$\begin{aligned} Num &\triangleq Bel(B|A_1)q(A_1, B) \\ Den &\triangleq Bel(B|A_1)q(A_1, B) \\ &\quad + Bel(B|A_2)q(A_2, B) + U((\bar{A}_1\cap B)^*) \end{aligned}$$

and

$$\begin{aligned} q(A_1, B) &= Bel(A_1) + U((\bar{B}\cap A_1)^*) - U(B^*\cap A_1) \\ &= Bel(A_1) + \underbrace{Pl(\bar{B}\cap A_1) - Bel(\bar{B}\cap A_1)}_{U((\bar{B}\cap A_1)^*)} - U(B^*\cap A_1) \end{aligned}$$

Because $\mathcal{F}_{B^*}(m) = \mathcal{F}_\Theta(m) - \mathcal{F}_B(m) - \mathcal{F}_{\bar{B}}(m)$ one has

$$\begin{aligned} U(B^*\cap A_1) &= \sum_{X\in\mathcal{F}_{B^*}(m)|X\in\mathcal{F}_{A_1}(m)} m(X) \\ &= \sum_{X\in\mathcal{F}_\Theta(m)-\mathcal{F}_B(m)-\mathcal{F}_{\bar{B}}(m)|X\in\mathcal{F}_{A_1}(m)} m(X) \\ &= \sum_{X\in\mathcal{F}_\Theta(m)|X\in\mathcal{F}_{A_1}(m)} m(X) \\ &\quad - \sum_{X\in\mathcal{F}_B(m)|X\in\mathcal{F}_{A_1}(m)} m(X) \\ &\quad - \sum_{X\in\mathcal{F}_{\bar{B}}(m)|X\in\mathcal{F}_{A_1}(m)} m(X) \\ &= Bel(A_1) - Bel(A_1\cap B) - Bel(A_1\cap\bar{B}) \end{aligned}$$

Therefore

$$\begin{aligned} q(A_1, B) &= Bel(A_1) + \underbrace{Pl(\bar{B}\cap A_1) - Bel(\bar{B}\cap A_1)}_{U((\bar{B}\cap A_1)^*)} \\ &\quad - \underbrace{[Bel(A_1) - Bel(A_1\cap B) - Bel(A_1\cap\bar{B})]}_{U(B^*\cap A_1)} \\ &= Pl(A_1\cap\bar{B}) + Bel(A_1\cap B) \end{aligned}$$

Similarly, one has

$$\begin{aligned} q(A_2,B) &= Bel(A_2) + U((\bar{B}\cap A_2)^*) - U(B^*\cap A_2) \\ &= Bel(A_2) + \underbrace{Pl(\bar{B}\cap A_2) - Bel(\bar{B}\cap A_2)}_{U((\bar{B}\cap A_2)^*)} \\ &\quad - \underbrace{[Bel(A_2) - Bel(A_2\cap B) - Bel(A_2\cap\bar{B})]}_{U(B^*\cap A_2)} \\ &= Pl(A_2\cap\bar{B}) + Bel(A_2\cap B) \end{aligned}$$

The value $U((\bar{A}_1\cap B)^*)$ is given by

$$U((\bar{A}_1\cap B)^*) = Pl(\bar{A}_1\cap B) - Bel(\bar{A}_1\cap B)$$

Therefore the numerator and denominator of $Bel(A|B)$ are

$$\begin{aligned} Num &\triangleq Bel(B|A_1)q(A_1,B) \\ &= Bel(B|A_1)[Pl(A_1\cap\bar{B}) + Bel(A_1\cap B)] \\ &= Bel(B|A)[Pl(A\cap\bar{B}) + Bel(A\cap B)] \end{aligned}$$

$$\begin{aligned} Den &\triangleq Bel(B|A_1)q(A_1,B) + Bel(B|A_2)q(A_2,B) \\ &\quad + U((\bar{A}_1\cap B)^*) \\ &= Bel(B|A_1)[Pl(A_1\cap\bar{B}) + Bel(A_1\cap B)] \\ &\quad + Bel(B|A_2)[Pl(A_2\cap\bar{B}) + Bel(A_2\cap B)] \\ &\quad + [Pl(\bar{A}_1\cap B) - Bel(\bar{A}_1\cap B)] \\ &= Bel(B|A)[Pl(A\cap\bar{B}) + Bel(A\cap B)] \\ &\quad + Bel(B|\bar{A})[Pl(\bar{A}\cap\bar{B}) + Bel(\bar{A}\cap B)] \\ &\quad + [Pl(\bar{A}\cap B) - Bel(\bar{A}\cap B)] \end{aligned}$$

Because $Bel(B|A) = Bel(A\cap B)/[Bel(A\cap B)+Pl(A\cap\bar{B})]$ and $Bel(B|\bar{A}) = Bel(\bar{A}\cap B)/[Bel(\bar{A}\cap B)+Pl(\bar{A}\cap\bar{B})]$ based on FH formulas, after basic algebra one can verify that $Num = Bel(A\cap B)$ and $Den = Bel(A\cap B) + Pl(\bar{A}\cap B)$.

Because $Bel(B|\bar{A}) = Bel(\bar{A}\cap B)/[Bel(\bar{A}\cap B)+Pl(\bar{A}+\bar{B})]$, the term $Bel(B|\bar{A})[Pl(\bar{A}\cap\bar{B})+Bel(\bar{A}\cap B)]$ involved in Den equals $Bel(\bar{A}\cap B)$. Hence the expression of Den reduces to

$$\begin{aligned} Den &= Bel(B|A)[Pl(A\cap\bar{B}) + Bel(A\cap B)] \\ &\quad + \underbrace{Bel(B|\bar{A})[Pl(\bar{A}\cap\bar{B}) + Bel(\bar{A}\cap B)]}_{Bel(\bar{A}\cap B)} \\ &\quad + [Pl(\bar{A}\cap B) - Bel(\bar{A}\cap B)] \\ &= Bel(B|A)[Pl(A\cap\bar{B}) + Bel(A\cap B)] + Pl(\bar{A}\cap B) \end{aligned}$$

If $Pl(\bar{B}|A) = Pl(A\cap\bar{B})/[Pl(A\cap\bar{B})+Bel(A\cap B)] > 0$ and if we multiply the expressions of Num and Den by $Pl(\bar{B}|A)$ one gets

$$\begin{aligned} Bel(A|B) &= \frac{Num}{Den} = \frac{Num\cdot Pl(\bar{B}|A)}{Den\cdot Pl(\bar{B}|A)} \\ &= \frac{Num\cdot\frac{Pl(A\cap\bar{B})}{Pl(A\cap\bar{B})+Bel(A\cap B)}}{Den\cdot\frac{Pl(A\cap\bar{B})}{Pl(A\cap\bar{B})+Bel(A\cap B)}} \\ &= \frac{Bel(B|A)Pl(A\cap\bar{B})}{Bel(B|A)Pl(A\cap\bar{B}) + Pl(\bar{B}|A)Pl(\bar{A}\cap B)} \end{aligned}$$

which corresponds exactly to the SGBT formula (28).

The SGBT formula (29) can also be obtained similarly from GBT by expressing at first $Bel(\bar{A}|B) = Bel(A_2|B)$ as

$$\begin{aligned} Bel(\bar{A}|B) &= \frac{Num'}{Den'} = \frac{Bel(B|A_2)q(A_2,B)}{\sum_{i=1}^{2} Bel(B|A_i)q(A_i,B) + U((\bar{A}_2\cap B)^*)} \\ &= \frac{Bel(B|A_2)q(A_2,B)}{Bel(B|A_1)q(A_1,B) + Bel(B|A_2)q(A_2,B) + U((\bar{A}_2\cap B)^*)} \end{aligned}$$

where[4]

$$\begin{aligned} Num' &\triangleq Bel(B|A_2)q(A_2,B) \\ &= Bel(B|\bar{A})[Pl(\bar{A}\cap\bar{B}) + Bel(\bar{A}\cap B)] \\ &= Bel(\bar{A}\cap B) \\ Den' &\triangleq Bel(B|A_1)q(A_1,B) + Bel(B|A_2)q(A_2,B) \\ &\quad + U((\bar{A}_2\cap B)^*) \\ &= Bel(B|A)[Pl(A\cap\bar{B}) + Bel(A\cap B)] \\ &\quad + Bel(B|\bar{A})[Pl(\bar{A}\cap\bar{B}) + Bel(\bar{A}\cap B)] \\ &\quad + [Pl(A\cap B) - Bel(A\cap B)] \\ &= Bel(\bar{A}\cap B) + Pl(A\cap B) \end{aligned}$$

If $Bel(\bar{B}|A) = Bel(A\cap\bar{B})/[Bel(A\cap\bar{B})+Pl(A\cap B)] > 0$ and if we multiply Num' and Den' by $Bel(\bar{B}|A)$ one gets[5]

$$\begin{aligned} Bel(\bar{A}|B) &= \frac{Num'}{Den'} = \frac{Num'\cdot Bel(\bar{B}|A)}{Den'\cdot Bel(\bar{B}|A)} \\ &= \frac{Bel(\bar{A}\cap B)Bel(\bar{B}|A)}{Bel(\bar{A}\cap B)Bel(\bar{B}|A) + Pl(A\cap B)Bel(\bar{B}|A)} \\ &= \frac{Bel(\bar{A}\cap B)Bel(\bar{B}|A)}{Bel(\bar{A}\cap B)Bel(\bar{B}|A) + Pl(B|A)Bel(A\cap\bar{B})} \end{aligned}$$

Hence

$$\begin{aligned} Pl(A|B) &= 1 - Bel(\bar{A}|B) \\ &= \frac{Pl(B|A)Bel(A\cap\bar{B})}{Pl(B|A)Bel(A\cap\bar{B}) + Bel(\bar{B}|A)Bel(\bar{A}\cap B)} \end{aligned}$$

which corresponds to SGBT formula (29).

Therefore, one has proved that expression (28) can be obtained from GBT if $Pl(A\cap\bar{B}) > 0$, and expression (29) can be obtained from GBT if $Bel(A\cap\bar{B}) > 0$. This completes the proof of SGBT.

Lemma 2: SGBT formulas (28) and (29) coincide with conditional probability formula $P(A|B) = P(B|A)P(A)/P(B) = P(A\cap B)/P(B)$ if the belief function is Bayesian.

Proof: Replacing $Bel(\cdot)$ and $Pl(\cdot)$ by $P(\cdot)$ in (28) and (29) we get $P(A|B) = \frac{P(B|A)P(A\cap\bar{B})}{P(B|A)P(A\cap\bar{B})+P(\bar{B}|A)P(\bar{A}\cap B)} = \frac{P(B|A)P(A\cap\bar{B})}{\frac{P(A\cap B)}{P(A)}P(A\cap\bar{B})+\frac{P(A\cap\bar{B})}{P(A)}P(\bar{A}\cap B)} = \frac{P(B|A)P(A)}{P(A\cap B)+P(\bar{A}\cap B)} = \frac{P(B|A)P(A)}{P(B)}$ because $P(A\cap B) + P(\bar{A}\cap B) = P(B)$. This completes the proof of lemma 2.

In appendix we also prove that $Bel(A|B) \leq Pl(A|B)$ when using SGBT formulas (28) and (29).

[4]Here $U((\bar{A}_2\cap B)^*) = Pl(\bar{A}_2\cap B) - Bel(\bar{A}_2\cap B) = Pl(A\cap B) - Bel(A\cap B)$ because $\bar{A}_2 = \bar{\bar{A}} = A$.

[5]From FH formulas $Pl(A\cap B)Bel(\bar{B}|A) = Pl(B|A)Bel(A\cap\bar{B})$.

VI. EXAMPLES

In this section we give two simple interesting examples of application of SGBT. Example 1 shows that GBT and SGBT works fine because conditions C_1 and C_2 are satisfied, whereas the example 2 shows that GBT works fine but SGBT doesn't work because of violation of condition C_1.

A. Example 1

We consider $\Theta = \{x_1, x_2, x_3, x_4\}$ and the BBA chosen as follows $m(x_1) = 0.05$, $m(x_2) = 0.03$, $m(x_1 \cup x_2) = 0.02$, $m(x_3) = 0.04$, $m(x_4) = 0.06$, $m(x_3 \cup x_4) = 0.10$, $m(x_2 \cup x_3) = 0.30$ and $m(x_1 \cup x_2 \cup x_3 \cup x_4) = m(\Theta) = 0.40$. We also consider the partition $\Theta = \{A = \{x_1, x_2\}, \bar{A} = \{x_3, x_4\}\}$ and the subset $B = \{x_2, x_3\}$. Hence one has

$$\Theta = \{\underbrace{x_1, \overbrace{x_2}^{}}_{A}\overbrace{,}^{B}\underbrace{\overbrace{x_3}^{}, x_4}_{\bar{A}}\}$$

with $A = \{x_1, x_2\} = x_1 \cup x_2$, $\bar{A} = \{x_3, x_4\} = x_3 \cup x_4$, $B = \{x_2, x_3\} = x_2 \cup x_3$, and $\bar{B} = \{x_1, x_4\} = x_1 \cup x_4$.

The set of focal elements in this example is

$$\mathcal{F}_\Theta(m) = \{x_1, x_2, x_1 \cup x_2, x_3, x_4, x_3 \cup x_4, \\ x_2 \cup x_3, x_1 \cup x_2 \cup x_3 \cup x_4\}$$

The sets of focal elements included in A and in $\bar{A}$ are $\mathcal{F}_A(m) = \{x_1, x_2, x_1 \cup x_2\}$, and $\mathcal{F}_{\bar{A}}(m) = \{x_3, x_4, x_3 \cup x_4\}$, and one has $\mathcal{F}_{A*}(m) = \mathcal{F}_\Theta(m) - \mathcal{F}_A(m) - \mathcal{F}_{\bar{A}}(m) = \{x_2 \cup x_3, x_1 \cup x_2 \cup x_3 \cup x_4\}$. The sets of focal elements included in B and in $\bar{B}$ are $\mathcal{F}_B(m) = \{x_2, x_3, x_2 \cup x_3\}$, $\mathcal{F}_{\bar{B}}(m) = \{x_1, x_4\}$, and one has $\mathcal{F}_{B*}(m) = \mathcal{F}_\Theta(m) - \mathcal{F}_B(m) - \mathcal{F}_{\bar{B}}(m) = \{x_1 \cup x_2, x_3 \cup x_4, x_1 \cup x_2 \cup x_3 \cup x_4\}$. From the BBA $m(\cdot)$ we get the following belief and plausibility values listed in Table I which are useful for making derivations of FH, GBT and SGBT formulas.

Subsets of Θ	$Bel(\cdot)$	$Pl(\cdot)$
$A = x_1 \cup x_2$	$Bel(A) = 0.10$	$Pl(A) = 0.80$
$\bar{A} = x_3 \cup x_4$	$Bel(\bar{A}) = 0.20$	$Pl(\bar{A}) = 0.90$
$B = x_2 \cup x_3$	$Bel(A) = 0.37$	$Pl(B) = 0.89$
$\bar{B} = x_1 \cup x_4$	$Bel(\bar{B}) = 0.11$	$Pl(\bar{B}) = 0.63$
$A \cap B = x_2$	$Bel(A \cap B) = 0.03$	$Pl(A \cap B) = 0.75$
$A \cap \bar{B} = x_1$	$Bel(A \cap \bar{B}) = 0.05$	$Pl(A \cap \bar{B}) = 0.47$
$\bar{A} \cap B = x_3$	$Bel(\bar{A} \cap B) = 0.04$	$Pl(\bar{A} \cap B) = 0.84$
$\bar{A} \cap \bar{B} = x_4$	$Bel(\bar{A} \cap \bar{B}) = 0.06$	$Pl(\bar{A} \cap \bar{B}) = 0.56$

Table I
BELIEF AND PLAUSIBILITY VALUES USED FOR THE DERIVATIONS.

- Application of FH formulas: with (15)-(16) one gets

$$Bel(A|B) = \frac{Bel(A \cap B)}{Bel(A \cap B) + Pl(\bar{A} \cap B)} = \frac{0.03}{0.03 + 0.84} \\ \approx 0.03448275$$

$$Pl(A|B) = \frac{Pl(A \cap B)}{Pl(A \cap B) + Bel(\bar{A} \cap B)} = \frac{0.75}{0.75 + 0.04} \\ \approx 0.94936708$$

With FH formulas (17)-(18), one gets

$$Bel(B|A) = \frac{Bel(A \cap B)}{Bel(A \cap B) + Pl(\bar{B} \cap A)} = \frac{0.03}{0.03 + 0.47} \\ = 0.06$$

$$Pl(B|A) = \frac{Pl(A \cap B)}{Pl(A \cap B) + Bel(\bar{B} \cap A)} = \frac{0.75}{0.75 + 0.05} \\ = 0.9375$$

- Application of SGBT formulas: with (28) and (29) one gets[6]

$$\begin{aligned} Bel(A|B) &= \frac{Bel(B|A)Pl(A \cap \bar{B})}{Bel(B|A)Pl(A \cap \bar{B}) + Pl(\bar{B}|A)Pl(\bar{A} \cap B)} \\ &= \frac{Bel(B|A)Pl(A \cap \bar{B})}{Bel(B|A)Pl(A \cap \bar{B}) + [1 - Bel(B|A)]Pl(\bar{A} \cap B)} \\ &= \frac{0.06 \cdot 0.47}{0.06 \cdot 0.47 + [1 - 0.06]0.84} = \frac{0.0282}{0.0282 + 0.7896} \\ &\approx 0.03448275 \end{aligned}$$

$$\begin{aligned} Pl(A|B) &= \frac{Pl(B|A)Bel(A \cap \bar{B})}{Bel(\bar{B}|A)Bel(\bar{A} \cap B) + Pl(B|A)Bel(A \cap \bar{B})} \\ &= \frac{Pl(B|A)Bel(A \cap \bar{B})}{[1 - Pl(B|A)]Bel(\bar{A} \cap B) + Pl(B|A)Bel(A \cap \bar{B})} \\ &= \frac{0.9375 \cdot 0.05}{[1 - 0.9375]0.04 + 0.9375 \cdot 0.05} \\ &= \frac{0.046875}{0.0025 + 0.046875} \approx 0.94936708 \end{aligned}$$

- Application of GBT formula (22): we denote $A_1 = A = x_1 \cup x_2$ and $A_2 = \bar{A} = x_3 \cup x_4$. Here GBT formula (22) becomes

$$Bel(A|B) = \frac{Bel(B|A_1)q(A_1, B)}{\sum_{i=1}^{2} Bel(B|A_i)q(A_i, B) + U((\bar{A}_1 \cap B)^*)}$$

where $Bel(B|A_1)$ and $Bel(B|A_2)$ terms are given by

$$\begin{aligned} Bel(B|A_1) \equiv Bel(B|A) &= \frac{Bel(A \cap B)}{Bel(A \cap B) + Pl(\bar{B} \cap A)} \\ &= \frac{0.03}{0.03 + 0.47} = 0.06 \\ Bel(B|A_2) \equiv Bel(B|\bar{A}) &= \frac{Bel(\bar{A} \cap B)}{Bel(\bar{A} \cap B) + Pl(\bar{B} \cap \bar{A})} \\ &= \frac{0.04}{0.04 + 0.56} \approx 0.06666667 \end{aligned}$$

The terms $q(A_1, B)$ and $q(A_2, B)$ are given by

$$\begin{aligned} q(A_1, B) &= Bel(A_1) + U((\bar{B} \cap A_1)^*) - U(B^* \cap A_1) \\ &= 0.10 + 0.42 - 0.02 = 0.50 \\ q(A_2, B) &= Bel(A_2) + U((\bar{B} \cap A_2)^*) - U(B^* \cap A_2) \\ &= 0.20 + 0.50 - 0.10 = 0.60 \end{aligned}$$

[6]It is worth noting that conditions C_1 and C_2 are satisfied in this example because $Pl(A \cap \bar{B}) = 0.47$ and $Bel(A \cap \bar{B}) = 0.05$.

because

$$\begin{aligned}
U((\bar{B}\cap A_1)^*) &= Pl(\bar{B}\cap A_1) - Bel(\bar{B}\cap A_1)\\
&= Pl(\bar{B}\cap A) - Bel(\bar{B}\cap A)\\
&= 0.47 - 0.05 = 0.42\\
U(B^*\cap A_1) &= \sum_{X\in\mathcal{F}_{B^*}(m)|X\in\mathcal{F}_{A_1}(m)} m(X)\\
&= \sum_{X\in\mathcal{F}_{B^*}(m)|X\in\mathcal{F}_{A}(m)} m(X)\\
&= m(x_1\cup x_2) = 0.02
\end{aligned}$$

and

$$\begin{aligned}
U((\bar{B}\cap A_2)^*) &= Pl(\bar{B}\cap A_2) - Bel(\bar{B}\cap A_2)\\
&= Pl(\bar{B}\cap \bar{A}) - Bel(\bar{B}\cap \bar{A})\\
&= 0.56 - 0.06 = 0.50\\
U(B^*\cap A_2) &= \sum_{X\in\mathcal{F}_{B^*}(m)|X\in\mathcal{F}_{A_2}(m)} m(X)\\
&= \sum_{X\in\mathcal{F}_{B^*}(m)|X\in\mathcal{F}_{\bar{A}}(m)} m(X)\\
&= m(x_3\cup x_4) = 0.10
\end{aligned}$$

The value $U((\bar{A}_1\cap B)^*)$ involved in the denominator of $Bel(A|B)$ expression is given by

$$\begin{aligned}
U((\bar{A}_1\cap B)^*) &= Pl(\bar{A}\cap B) - Bel(\bar{A}\cap B)\\
&= 0.84 - 0.04 = 0.80
\end{aligned}$$

Replacing all these values in GBT formula of $Bel(A|B)$ we get

$$\begin{aligned}
Bel(A|B) &\equiv Bel(A_1|B)\\
&= \frac{Bel(B|A_1)q(A_1,B)}{\sum_{i=1}^{2} Bel(B|A_i)q(A_i,B) + U((\bar{A}_1\cap B)^*)}\\
&\approx \frac{0.06\cdot 0.50}{0.06\cdot 0.50 + 0.06666667\cdot 0.60 + 0.80}\\
&\approx \frac{0.03}{0.870000002} \approx 0.03448275
\end{aligned}$$

As shown, $Bel(A|B)$ calculated by GBT and by SGBT formulas are consistent with the value calculated directly from FH formulas. For calculating $Pl(A|B)$, we calculate at first $Bel(\bar{A}|B) = Bel(A_2|B)$ and then $Pl(A|B) = 1 - Bel(\bar{A}|B)$. Applying GBT formula for calculating $Bel(A_2|B)$, one has

$$\begin{aligned}
Bel(\bar{A}|B) &= Bel(A_2|B)\\
&= \frac{Bel(B|A_2)q(A_2,B)}{\sum_{i=1}^{2} Bel(B|A_i)q(A_i,B) + U((\bar{A}_2\cap B)^*)}
\end{aligned}$$

The values of $Bel(B|A_i)$, $q(A_i,B)$ for $i = 1,2$ have been calculated previously and $U((\bar{A}_2\cap B)^*)$ is given by

$$\begin{aligned}
U((\bar{A}_2\cap B)^*) &= Pl(\bar{A}_2\cap B) - Bel(\bar{A}_2\cap B)\\
&= Pl(A\cap B) - Bel(A\cap B)\\
&= 0.75 - 0.03 = 0.72
\end{aligned}$$

Therefore,

$$\begin{aligned}
Bel(\bar{A}|B) &\equiv Bel(A_2|B)\\
&= \frac{Bel(B|A_2)q(A_2,B)}{\sum_{i=1}^{2} Bel(B|A_i)q(A_i,B) + U((\bar{A}_2\cap B)^*)}\\
&\approx \frac{0.06666667\cdot 0.60}{0.06\cdot 0.50 + 0.06666667\cdot 0.60 + 0.72}\\
&\approx \frac{0.040000002}{0.790000002} \approx 0.05063292
\end{aligned}$$

and finally we get

$$Pl(A|B) = 1 - Bel(\bar{A}|B) \approx 0.94936708$$

From this very simple example we have verified that FH formulas, GBT formula and simplified GBT formula are all consistent because the conditions C_1 and C_2 are satisfied.

B. Example 2

We consider the example of [8] (Section VIII). We verify that SGBT formula (28) works because $Bel(B|A_1) = 0.0889$, $Pl(A_1\cap\bar{B}) = 0.41$, $Pl(\bar{B}|A_1) = 1 - Bel(B|A_1) = 1 - 0.0889 = 0.9111$ and $Pl(\bar{A}_1\cap B) = 0.54$ so that

$$\begin{aligned}
Bel(A_1|B) &= \frac{Bel(B|A_1)Pl(A_1\cap\bar{B})}{Bel(B|A_1)Pl(A_1\cap\bar{B}) + Pl(\bar{B}|A_1)Pl(\bar{A}_1\cap B)}\\
&= \frac{0.0889\cdot 0.41}{0.0889\cdot 0.41 + 0.9111\cdot 0.54} = 0.0690
\end{aligned}$$

which is the same value of what we get by applying directly FH formula, or GBT formula (22). The SGBT formula (28) works because the condition C_1 (i.e. $Pl(A_1\cap\bar{B}) = 0.41 > 0$) is satisfied. Similarly, using (28), one has for $Bel(A_2|B)$

$$\begin{aligned}
Bel(A_2|B) &= \frac{Bel(B|A_2)Pl(A_2\cap\bar{B})}{Bel(B|A_2)Pl(A_2\cap\bar{B}) + Pl(\bar{B}|A_2)Pl(\bar{A}_2\cap B)}\\
&= \frac{0\cdot 0.43}{0\cdot 0.43 + 1\cdot 0.80} = 0
\end{aligned}$$

which is the same value of what we get by applying directly FH formula, or GBT formula (22). Here SGBT formula (28) works because the condition C_1 (i.e. $Pl(A_2\cap\bar{B}) = 0.43 > 0$) is satisfied.

For the value $Bel(A_3|B) = 0.0625$ computed by FH conditioning formula, or by GBT formula (22) things are different because when applying SGBT formula (28) we get $0/0$ indetermination. Indeed,

$$\begin{aligned}
Bel(A_3|B) &= \frac{Bel(B|A_3)Pl(A_3\cap\bar{B})}{Bel(B|A_3)Pl(A_3\cap\bar{B}) + Pl(\bar{B}|A_3)Pl(\bar{A}_3\cap B)}\\
&= \frac{1\cdot 0}{1\cdot 0 + 0\cdot 0.75} = \frac{0}{0}
\end{aligned}$$

So one sees that SGBT formula (28) does not work for computing $Bel(A_3|B)$ in this case because the condition C_1 (i.e. $Pl(A_3\cap\bar{B}) > 0$) is not satisfied which is normal. In this case the correct value $Bel(A_3|B) = 0.0625$ must be calculated by GBT or FH formulas.

Therefore in practice a special attention must always be paid to conditions C_1 and C_2 before applying SGBT formulas, and in case of violation of one of these conditions, one needs to work back directly with FH or GBT formulas.

VII. Conclusion

The main contribution of this paper is the derivation of a simplified formulation of Generalized Bayes' Theorem, called SGBT, which extends Bayesian Theorem in the frame of belief functions. The simplification is imposed from the fact that the general formulation of GBT is not easy to apply in real world applications. It is drawn from GBT for working with a dichotomous partitioning of the frame of discernment. The conditions under which this new formulation is valid are presented. The theoretical results obtained are illustrated with simple theoretical examples. The challenging question of application of GBT and SGBT to solve real-world problems is under investigation.

Appendix

A. Proof that $Bel(A|B) \leq Pl(A|B)$ from SGBT formula

To prove that $Bel(A|B) \leq Pl(A|B)$ from SGBT formulas (28)-(29) one needs to prove the following inequality

$$\frac{Bel(B|A)Pl(A\cap\bar{B})}{Bel(B|A)Pl(A\cap\bar{B})+Pl(\bar{B}|A)Pl(\bar{A}\cap B)} \leq \frac{Pl(B|A)Bel(A\cap\bar{B})}{Bel(\bar{B}|A)Bel(\bar{A}\cap B)+Pl(B|A)Bel(A\cap\bar{B})}$$

After basic algebraic manipulations on the previous inequality, one has to prove if $R_1 \leq R_2 \cdot R_3 \cdot R_4$. where, for the notation convenience, $R_1 = Bel(B|A)/Pl(B|A)$, $R_2 = Bel(A\cap\bar{B})/Pl(A\cap\bar{B})$, $R_3 = Pl(\bar{B}|A)/Bel(\bar{B}|A)$ and $R_4 = Pl(\bar{A}\cap B)/Bel(\bar{A}\cap B)$. Our proof is done by contradiction as follows.

Let us assume that $R_2 \cdot R_3 \cdot R_4 < R_1$ is valid, that is

$$\underbrace{\frac{Bel(A\cap\bar{B})}{Pl(A\cap\bar{B})}}_{R_2}\cdot\underbrace{\frac{Pl(\bar{B}|A)}{Bel(\bar{B}|A)}}_{R_3}\cdot\underbrace{\frac{Pl(\bar{A}\cap B)}{Bel(\bar{A}\cap B)}}_{R_4} < \underbrace{\frac{Bel(B|A)}{Pl(B|A)}}_{R_1} \quad (30)$$

Because $R_2 \leq 1$, one has necessarily $R_2 \cdot R_3 \cdot R_4 \leq R_3 \cdot R_4$, so we must have (if our assumption is valid) $R_3 \cdot R_4 < R_1$, that is

$$\underbrace{\frac{1-Bel(B|A)}{1-Pl(B|A)}}_{R_3}\cdot\underbrace{\frac{Pl(\bar{A}\cap B)}{Bel(\bar{A}\cap B)}}_{R_4} < \underbrace{\frac{Bel(B|A)}{Pl(B|A)}}_{R_1} \quad (31)$$

or equivalently

$$[1-Bel(B|A)]Pl(B|A) < Bel(B|A)[1-Pl(B|A)]\underbrace{\frac{Bel(\bar{A}\cap B)}{Pl(\bar{A}\cap B)}}_{1/R_4}$$

Because $Bel(\bar{A}\cap B)/Pl(\bar{A}\cap B) \leq 1$ then

$$Bel(B|A)[1-Pl(B|A)]\underbrace{\frac{Bel(\bar{A}\cap B)}{Pl(\bar{A}\cap B)}}_{1/R_4} \leq Bel(B|A)[1-Pl(B|A)]$$

So we must have (if our assumption is valid)

$$[1-Bel(B|A)]Pl(B|A) < Bel(B|A)[1-Pl(B|A)] \quad (32)$$

which is (after rearranging terms) equivalent to have the inequality $Pl(B|A) < Bel(B|A)$ satisfied. However, from Fagin-Halpern definitions of conditional belief function and properties of belief functions the previous inequality $Pl(B|A) < Bel(B|A)$ is never satisfied. Therefore our assumption $R_2 \cdot R_3 \cdot R_4 < R_1$ is not valid and one has necessarily $R_1 \leq R_2 \cdot R_3 \cdot R_4$, which completes the proof that $Bel(A|B) \leq Pl(A|B)$ when $Bel(A|B)$ and $Pl(A|B)$ are calculated by the SGBT formulas (28) and (29).

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