An Effective Measure of Uncertainty of Basic Belief Assignments

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***Abstract*—This paper presents a new effective measure of uncertainty (MoU) of basic belief assignments. This new continuous measure is effective in the sense that it satisfies a small number of very natural and essential desiderata. Our new simple mathematical definition of MoU captures well the interwoven link of randomness and imprecision inherent to basic belief assignments. Its numerical value is easy to calculate. This new effective MoU characterizes efficiently any source of evidence used in the belief functions framework. Because this MoU coincides with Shannon entropy for any Bayesian basic belief assignment, it can be also interpreted as an effective generalization of Shannon entropy. We also provide several examples to show how this new MoU works.**

Keywords: Measure of Uncertainty, MoU, belief functions, Shannon entropy.

I. Introduction

In the classical probabilistic framework of the theory of communication developed by Shannon in 1948 [1], [2], the measure of uncertainty (MoU), also called entropy, for characterizing a source of information (from signal transmission standpoint) is represented by Shannon entropy. This entropy measures the randomness of a probability mass function. Shannon entropy has played a very important role in the development of modern communication systems during the second half of the 20th century, and in signal and image coding, data compression, and cryptography [3] until today. Shannon theory does not concern the semantic aspects of the content of a message but only its transmission.

From 1980s and until now, many research works have been proposed to try to extend Shannon measure of uncertainty (i.e. entropy) in the belief functions framework since their introduction by Shafer in the mid of 1970s [5]. In parallel, other research works have been done on the characterization of particular aspects of the uncertainty which are related to the set consistency (or non-specificity) of basic belief assignments (BBAs). Recently Jousselme et al. [6] proposed an interesting attempt of mathematical unification of existing MoU formulations. In our recent survey paper [7], we did analyze in details 40 years of research works on MoUs. Our deep analysis of forty-eight MoUs reveals that only very few of them can be considered as effective in the mathematical sense defined in Section III. Unfortunately, these existing effective MoUs are conceptually flawed. The main contribution of this paper is to provide a clear positive answer with a new well-justified mathematical solution to the fundamental challenging question stated in the conclusion of [7]:

Is there a better conceptual effective measure of uncertainty for the basic belief assignments?

This paper is organized as follows. Section II presents the basics of belief functions. Section III presents and justifies the four essential desiderata that a MoU must satisfy in order to be effective. In the section IV we list the existing effective MoUs and we explain their conceptual flaws. Section V presents the new effective MoU for BBA (i.e. generalized Shannon entropy) with some examples in the section VI. Concluding remarks and perspectives are given in the section VII.

II. Belief functions

The belief functions (BF) were introduced by Shafer [5] for modeling epistemic uncertainty, reasoning about uncertainty and combining distinct sources of evidence. The answer of the problem under concern is assumed to belong to a known finite discrete frame of discernement (FoD) $\Theta = \{\theta_1, \ldots, \theta_N\}$ where all elements (i.e. members) of Θ are exhaustive and exclusive. The set of all subsets of Θ (including empty set $\emptyset$, and Θ) is the power-set of Θ denoted by 2^Θ. The number of elements (i.e. the cardinality) of the power-set is $2^{|\Theta|}$. A (normalized) basic belief assignment (BBA) associated with a given source of evidence is a mapping $m^\Theta(\cdot) : 2^\Theta \rightarrow [0, 1]$ such that $m^\Theta(\emptyset) = 0$ and $\sum_{X\in 2^\Theta} m^\Theta(X) = 1$. A BBA $m^\Theta(\cdot)$ characterizes a source of evidence related with a FoD Θ. For notation shorthand, we can omit the superscript Θ in $m^\Theta(\cdot)$ notation if there is no ambiguity on the FoD we work with. The quantity $m(X)$ is called the mass of belief of X. $X \in 2^\Theta$ is called a focal element (FE) of $m(\cdot)$ if $m(X) > 0$. The set of all focal elements of $m(\cdot)$ is denoted[1] by $\mathcal{F}_\Theta(m) \triangleq \{X \in 2^\Theta | m(X) > 0\}$. The belief and the plausibility of X are respectively defined for any $X \in 2^\Theta$ by [5]

$$Bel(X) = \sum_{Y\in 2^\Theta | Y\subseteq X} m(Y) \tag{1}$$

$$Pl(X) = \sum_{Y\in 2^\Theta | X\cap Y\neq\emptyset} m(Y) = 1 - \text{Bel}(\bar{X}). \tag{2}$$

where $\bar{X} \triangleq \Theta \setminus \{X\}$ is the complement of X in Θ.

[1] $\triangleq$ means *equal by definition*.

One has always $0 \leq Bel(X) \leq Pl(X) \leq 1$, see [5]. For $X = \emptyset$, $Bel(\emptyset) = Pl(\emptyset) = 0$, and for $X = \Theta$ one has $Bel(\Theta) = Pl(\Theta) = 1$. $Bel(X)$ and $Pl(X)$ are often interpreted as the lower and upper bounds of unknown probability $P(X)$ of X, that is $Bel(X) \leq P(X) \leq Pl(X)$. To quantify the uncertainty (i.e. the imprecision) of $P(X) \in [Bel(X), Pl(X)]$, we use $u(X) \in [0, 1]$ defined by

$$u(X) \triangleq Pl(X) - Bel(X) \tag{3}$$

The quantity $u(X) = 0$ if $Bel(X) = Pl(X)$ which means that $P(X)$ is known precisely, and one has $P(X) = Bel(X) = Pl(X)$. One has $u(\emptyset) = 0$ because $Bel(\emptyset) = Pl(\emptyset) = 0$, and one has $u(\Theta) = 0$ because $Bel(\Theta) = Pl(\Theta) = 1$. If all focal elements of $m(\cdot)$ are singletons of 2^Θ the BBA $m(\cdot)$ is a Bayesian BBA because $\forall X \in 2^\Theta$ one has $Bel(X) = Pl(X) = P(X)$ and $u(X) = 0$. Hence the belief and plausibility of X coincide with a probability measure $P(X)$ defined on the FoD Θ. The vacuous BBA characterizing a totally ignorant source of evidence is defined by $m_v(X) = 1$ for $X = \Theta$, and $m_v(X) = 0$ for all $X \in 2^\Theta$ different of Θ. This very particular BBA plays a major role in the establishment of a new effective measure of uncertainty for BBA.

III. Essential desiderata for a MoU

Before defining our new effective measure of uncertainty, denoted by $U(m)$, for any basic belief assignment $m(\cdot)$ related to a (non-empty) FoD Θ, we present the four essential and very natural desiderata that an effective MoU must satisfy [7].

Desideratum D1: For any non-empty frame of discernment Θ and for any BBA $m(\cdot)$ focused on a singleton X of 2^Θ one must have

$$U(m) = 0 \tag{4}$$

Justification of D1: this desideratum is natural and intuitive because any particular BBA for which a singleton X has $m(X) = 1$ characterizes its certainty, which means that there is no uncertainty about the choice of this element since it does not include other smaller element in it. So, in this case $U(m)$ must take zero value.

Desideratum D2: The measure of uncertainty of a total ignorant source of evidence must increase with the cardinality of the frame of discernment. That is

$$U(m_v^\Theta) < U(m_v^{\Theta'}), \quad \text{if } |\Theta| < |\Theta'|. \tag{5}$$

Justification of D2: this second desideratum makes perfect sense because the total ignorant source of evidence on $\Theta = \{\theta_1, \ldots, \theta_N\}$ for which $m_v^\Theta(\Theta) = 1$ knows absolutely nothing about only N elements, whereas the total ignorant source of evidence on $\Theta' = \{\theta_1, \ldots, \theta_N, \theta_{N+1}, \ldots, \theta_{N'}\}$ with $m_v^{\Theta'}(\Theta') = 1$ knows absolutely nothing about more elements because $N' > N$. This clearly indicates that $m_v^{\Theta'}$ must be in fact more ignorant than m_v^Θ.

Desideratum D3: The measure of uncertainty $U(m)$ must coincide with Shannon entropy [1]–[3] if the BBA $m(\cdot)$ is a Bayesian BBA. This desideratum is mathematically expressed for any Bayesian BBA $m(\cdot)$ defined on the FoD Θ by the condition[2]

$$U(m) = -\sum_{X \in \Theta} m(X) \log(m(X)) \tag{6}$$

Justification of D3: this third desideratum is also very natural because Shannon entropy is the most well-known and justified [9] measure used to characterize the uncertainty (the randomness, or variability) of a probability mass function. Because any Bayesian BBA induces belief and plausibility functions that coincide with a probability measure, one must have a coherence of $U(m)$ with Shannon entropy when the BBA is Bayesian.

Desideratum D4: For any non-vacuous BBA $m(\cdot)$ and for the vacuous BBA $m_v(\cdot)$ defined with respect to the same FoD one must have

$$U(m) < U(m_v) \tag{7}$$

Justification of D4: this last desideratum is also a very important one and it makes perfect sense because the total ignorant source is always characterized by the vacuous BBA $m_v(\cdot)$, and obviously no source of evidence can be more uncertain than the total ignorant source.

Effectiveness of a measure of uncertainty: A measure of uncertainty (MoU) is said effective if and only if it satisfies the four essential desiderata D1, D2, D3, and D4.

Any MoU that fails to satisfy at least one of these four desiderata is said non-effective, and in this case it cannot be considered seriously as a good measure of uncertainty for characterizing a basic belief assignment of a source of evidence. Consequently, a non-effective MoU should not be used in applications involving MoU.

As justified in [7], we voluntarily do not include the sub-additivity desideratum in the list of our desiderata for the search of an effective MoU in the belief function framework because this desideratum does not make sense when working with general (i.e. non-Bayesian) BBAs, and it is incompatible with the essential desideratum D4. We recall that the sub-additivity condition is defined by $U(m^{\Theta\times\Theta'}) \leq U(m^{\downarrow\Theta}) + U(m^{\downarrow\Theta'})$ or any joint BBA defined on the cartesian product $\Theta \times \Theta'$ of FoDs Θ and Θ', where $m^{\downarrow\Theta}$ is the marginal (i.e. projection) of $m^{\Theta\times\Theta'}(\cdot)$ on the power-set 2^Θ, and $m^{\downarrow\Theta'}$ is the marginal (i.e. projection, see [10], [11]) of $m^{\Theta\times\Theta'}(\cdot)$ on the power-set $2^{\Theta'}$. To justify our choice, just consider a simple example with $|\Theta| = 5$ and $|\Theta'| = 8$, which means that the cartesian product space $\Theta \times \Theta'$ has $|\Theta \times \Theta'| = 40$ elements. Why the MoU of the vacuous BBA $m_v^{\Theta\times\Theta'}$ related to 40 elements of $\Theta \times \Theta'$ should be less (or equal) to the sum of MoU of vacuous BBA m_v^Θ related to only 5 elements of Θ and the MoU of the vacuous BBA $m_v^{\Theta'}$ only related to the 8 elements of Θ'? We do not see any solid theoretical reason, nor intuitive reason, for justifying and requiring the

[2]Shannon entropy [1] is given here in *nats*, and we take $0 \log(0) = 0$ because $\lim_{x\to 0^+} x \log(x) = 0$ which is proved using L'Hôpital's rule [4].

subadditivity desideratum in the general framework of belief functions, and to select it as an axiom to satisfy in general as done in [12]. Unlike Vejnarova and Klir opinions [15] (p.28) (and some authors following them), we do not consider that the meaningful (or effective) measure of uncertainty of basic belief assignment must satisfy the sub-additivity desideratum in general.

IV. Existing effective MoUs

Before presenting our new effective MoU (or generalized entropy) in the next section, we must discuss a bit of the few existing effective measures of uncertainty proposed in the literature. As shown in [7], most[3] of existing MoUs are actually non-effective, and only eight MoUs can be considered as effective in the mathematical sense defined in the previous section. Most of effective MoUs share two basic principles: 1) approximate the BBA m by a probability measure (i.e. a Bayesian BBA) P_m based on some method of approximation and evaluate its Shannon entropy to estimate the randomness (or conflict) inherent to the BBA, and 2) add a term to Shannon entropy that characterizes the level of ambiguity (or non-specificity) inherent of the BBA (usually thanks to Dubois & Prade U-uncertainty [16]). For instance in [7] the BetP and DSmP transformations are used, in [17] the Cobb-Shenoy transformation [18] is used, and in [19] the authors suggest to use[4] the Bayesian BBA compatible with belief intervals drawn from $m(\cdot)$ that maximizes Shannon entropy. This general 2-steps principle is rather simple and quite intuitive but it seriously lacks of theoretical justification. We consider that such type of effective MoU construction is conceptually flawed and not very satisfactory for two main reasons:

Reason 1: these effective MoUs highly depend on the method of approximation whose choice is quite arbitrary. Worse, a method of approximation of a BBA $m(\cdot)$ to a Bayesian BBA can be totally misleading as for instance Cobb-Shenoy $PlPr_m$ transformation [18] because for this transformation the evaluation of probabilities can be inconsistent with belief interval values. More precisely, one can have $PlPr_m(\theta_i) \notin [Bel(\theta_i), Pl(\theta_i)]$ with Cobb-Shenoy method, which is obviously not reasonable, nor acceptable at all, see discussion in [7] with example. We emphasize the fact that if a method of approximation of a BBA m by a probability measure P_m is chosen, it must be at least consistent with belief interval values generated by the BBA m under concern. Clearly, we cannot recommend Cobb-Shenoy transformation for building an effective MoU based on aforementioned principles 1) and 2) as proposed recently by Jiroušek and Shenoy in [17].

Reason 2: In fact, there is no solid reason or evidence that necessitates to approximate any (non-Bayesian) BBA by a Bayesian BBA (for using Shannon entropy measure) in the construction of MoU. Also, there is no reason why we need (or request) to make the distinction of the two aspects of uncertainty (conflict and non-specificity), and to consider them as additively separable. This is conceptually very disputable because the randomness (or conflict) and ambiguity (or non-specificity) are actually interwoven through the mass value of the focal elements of the BBA and their belief intervals.

Very recently, Zhang et al. in [22] did propose three new effective MoUs not directly based on the aforementioned 2-steps principle approach, and that is why they have attracted our attention. These MoUs are denoted by $H^1(m)$, $H^2(m)$ and $H^3(m)$ and they are respectively defined by[5]

$$H^1(m) = -\sum_{X\subseteq\Theta} m(X)\log_2(Pl(X)) + \sum_{X\subseteq\Theta} m(X) 2\log_2(|X|)$$
$$H^2(m) = -\sum_{X\subseteq\Theta} m(X)\log_2(Pl(X)) + \sum_{X\subseteq\Theta} m(X)\log_2(2^{|X|}-1)$$
$$H^3(m) = -\sum_{X\subseteq\Theta} m(X)\log_2(Pl(X)) + \sum_{\substack{X\subseteq\Theta\\|X|>1}} m(X)|X|$$

Unfortunately, Zhang et al. fail to capture well the interwoven link between conflict and non-specificity (or imprecision). Actually the authors set arbitrarily the range of their MoU as a simple parameter, either taken arbitrarily as $[0, 2\log_2(|\Theta|)]$, $[0, \log_2(2^{|\Theta|}-1)]$ or $[0, |\Theta|]$, to define their $H^1(m)$, $H^2(m)$ and $H^3(m)$ measures of uncertainty. Zhang's approach is very questionable, and actually other ranges could have been chosen instead. Moreover Zhang et al. do not identify (nor propose) the best MoU to use between $H^1(m)$, $H^2(m)$ and $H^3(m)$. The other serious problem with Zhang's approach is its lack of solid justification for using the plausibility function in the summation $-\sum_{X\subseteq\Theta} m(X)\log_2(Pl(X))$. Although effective in the mathematical sense defined in section III, Zhang's new MoUs are ill-justified and they can also be considered as conceptually flawed. That is why we present a better conceptual effective measure of uncertainty for BBA in the next section.

V. A New Effective Measure of Uncertainty

A. Mathematical definition

The new effective measure of uncertainty we propose is given by the following simple formula

$$U(m) = \sum_{X\in 2^\Theta} s(X) \quad (8)$$

with

$$s(X) \triangleq -(1-u(X))m(X)\log(m(X)) + u(X)(1-m(X)) \quad (9)$$

$s(X)$ is the uncertainty contribution of X in the MoU $U(m)$. We call $s(X)$ the *entropiece* of X. Because $u(X) \in [0,1]$ and $m(X) \in [0,1]$ one has $s(X) \geq 0$, and $U(m) \geq 0$. The

[3] Forty-eight MoUs have been analyzed in [7].

[4] found using a complicate optimization method, see [20], [21] for details.

[5] We have corrected here the definition of $H^3(m)$ which is mathematically ill-formulated in [22].

entropiece $s(X)$ takes into account the belief mass $m(X)$, and the uncertainty (or imprecision) $u(X) = Pl(X) - Bel(X)$ about the unknown probability of X in a subtle interwoven manner. The cardinality of X enters indirectly (i.e. not explicitly) in the derivations of $Bel(X)$ and $Pl(X)$, and thus in the calculation of $u(X)$ and in the entropiece $s(X)$. The quantity $-(1-u(X))\log(m(X)) = (1-u(X))\log(1/m(X))$ entering in $s(X)$ in (9) is the surprisal [8] $\log(1/m(X))$ of X discounted by the confidence $(1-u(X))$ one has about the precision of $P(X)$. The term $-m(X)(1-u(X))\log(m(X))$ is the weighted discounted surprisal of X. The second term $u(X)(1-m(X))$ corresponds to the imprecision of $P(X)$ discounted by $(1-m(X))$ because the greater $m(X)$ the less one should take into account the imprecision $u(X)$ in the MoU. As we will prove next, this new very simple MoU $U(m)$ satisfies the four essential desiderata, and thus it is effective and conceptually well justified, and it presents several advantages over existing effective MoUs given in Section VII.

Because for $X = \emptyset$, one has $m(\emptyset) = 0$ and $u(\emptyset) = 0$ the entropiece of the empty set $\emptyset$ is $s(\emptyset) = 0$. Hence the expression of $U(m)$ can be written equivalently as

$$U(m) = s(\emptyset) + \sum_{X\in 2^\Theta | X\neq\emptyset} s(X) = \sum_{X\in 2^\Theta | X\neq\emptyset} s(X) \quad (10)$$

It is worth noting that for any BBA focused on $X \neq \emptyset$ with $m(X) = 1$, we have $m(X) = Bel(X) = Pl(X) = 1$, and thus $u(X) = 0$. In this case, the entropiece of X is[6]

$$\begin{aligned} s(X) &= -(1-u(X))m(X)\log(m(X)) + u(X)(1-m(X)) \\ &= -(1-0)1\log(1) + 0(1-1) = 0 \end{aligned}$$

In particular, if $m(\Theta) = 1$ (which corresponds to the vacuous BBA) we have the entropiece $s(\Theta) = 0$.

$U(m)$ is expressed in *nats* because we use the natural logarithm which makes derivations simpler, specially for making some proofs in the sequel. $U(m)$ can be expressed in *bits* by dividing the $U(m)$ value in *nats* by $\log(2) = 0.69314718...$. This measure of uncertainty $U(m)$ is a continuous function in its basic belief mass arguments because it is a summation of continuous functions.

B. Entropy of the vacuous BBA

Consider the FoD Θ of cardinality $|\Theta| = N$ greater than zero, and the vacuous BBA m_v defined on this FoD for which $m_v(\Theta) = 1$ and $m_v(X) = 0$ for any $X \neq \Theta$ in 2^Θ. For this vacuous BBA one always has $Bel(\Theta) = Pl(\Theta) = 1$ and thus $u(\Theta) = Pl(\Theta) - Bel(\Theta) = 0$, and one has also $u(\emptyset) = 0$. For all elements $X \neq \Theta$ with $X \in 2^\Theta \setminus \{\emptyset\}$ one has also necessarily $Bel(X) = 0$, $Pl(X) = 1$ and thus $u(X) = Pl(X) - Bel(X) = 1$. Consequently, the expression (10) with the BBA m_v becomes[7]

$$\begin{aligned} U(m_v) = &- \sum_{X\in 2^\Theta | X\neq\emptyset} (1-u(X))m_v(X)\log(m_v(X)) \\ &+ \sum_{X\in 2^\Theta | X\neq\emptyset} u(X)(1-m_v(X)) \\ =& -(1-u(\Theta))m_v(\Theta)\log(m_v(\Theta)) \\ &- \sum_{X\in 2^\Theta | (X\neq\emptyset)\wedge(X\neq\Theta)} (1-u(X))m_v(X)\log(m_v(X)) \\ &+ u(\Theta)(1-m_v(\Theta)) \\ &+ [\sum_{X\in 2^\Theta | (X\neq\emptyset)\wedge(X\neq\Theta)} u(X)(1-m_v(X))] \end{aligned}$$

In this expression of $U(m_v)$ we have[8]

$$\begin{cases} -(1-u(\Theta))m_v(\Theta)\log(m_v(\Theta)) = -(1-0)1\log(1) = 0 \\ -\sum_{X\in 2^\Theta | (X\neq\emptyset)\wedge(X\neq\Theta)} (1-u(X))m_v(X)\log(m_v(X)) = 0 \\ u(\Theta)(1-m_v(\Theta)) = 0(1-1) = 0 \\ \sum_{X\in 2^\Theta | (X\neq\emptyset)\wedge(X\neq\Theta)} u(X)(1-m_v(X)) = 2^N - 2 \end{cases}$$

Therefore, it comes finally for the vacuous BBA m_v defined on a FoD of size $N > 0$ the following MoU value

$$U(m_v) = 2^N - 2 \quad (11)$$

The entropy $U(m)$ makes perfect sense because for the vacuous BBA $m_v(\cdot)$ there is no information about the conflicts between the elements of the FoD. One has $u(\emptyset) = 0$ because $[Bel(\emptyset), Pl(\emptyset)] = [0,0]$, $u(\Theta) = 0$ because $[Bel(\Theta), Pl(\Theta)] = [1,1]$, and for all $X \in 2^\Theta \setminus \{\emptyset, \Theta\}$ one has $u(X) = 1$ because $[Bel(X), Pl(X)] = [0,1]$. Hence, the sum of all imprecisions of $P(X)$ for all $X \in 2^\Theta$ is exactly equal to $2^N - 2$ when $|\Theta| = N$. In the degenerate case where $|\Theta| = N = 1$, one has $U(m_v) = 2^1 - 2 = 0$ which indicates that there is absolutely no uncertainty in this very particular case. This result makes perfect sense also. For non-degenerate FoD (i.e. when $|\Theta| > 1$) one has always $U(m_v) > \log(N)$ which means that the vacuous BBA representing the totally ignorant source of evidence has an entropy greater than the maximum of Shannon entropy $\log(N)$ obtained with the uniform probability mass function distributed on Θ. This is an expected result because no BBA can represent the total ignorance, but the vacuous BBA.

C. Effectiveness of $U(m)$

In this subsection we establish the effectiveness of our new generalized entropy $U(m)$ defined in (8). For this, we prove the following four lemmas.

Lemma 1: $U(m)$ satisfies the desideratum D1.

Proof: Consider at first the very special case where Θ includes only one element θ, that is $\Theta = \{\theta\}$ and $|\Theta| = 1$. In this case there exists only one possible

[6]because $\log(1) = 0$.

[7]The notation $a \wedge b$ means that the conditions a and b are both satisfied.

[8]For $X \neq \Theta$, $m_v(X) = 0$ and $m_v(X)\log(m_v(X)) = 0\log(0) = 0$.

BBA over $2^\Theta = \{\emptyset, \theta\}$ defined by $m(\emptyset) = 0$ and $m(\theta) = 1$. Hence $Bel(\theta) = Pl(\theta) = 1$, $u(\theta) = 0$, and $s(\theta) = (1 - u(\theta))m(\theta)\log(m(\theta)) + u(\theta)(1 - m(\theta)) = 0$. Therefore $U(m) = s(\emptyset) + s(\theta) = 0$. In more general (i.e. when $|\Theta| > 1$) if X is a singleton of 2^Θ (i.e. $|X| = 1$) and if $m(X) = 1$ then $Bel(X) = Pl(X) = 1$ and $u(X) = 0$. For the elements Y of $2^\Theta \setminus \{\emptyset\}$ containing X one has also $Bel(Y) = Pl(Y) = 1$ and therefore $u(Y) = 0$. For all elements Y of $2^\Theta \setminus \{\emptyset\}$ not containing X one has always $Bel(Y) = Pl(Y) = 0$ and therefore $u(Y) = 0$. In summary, one has: 1) $m(X) = 1$, $u(X) = 0$, $s(X) = 0$, 2) $m(Y) = 0$, $u(Y) = 0$, $s(Y) = 0$ for all $Y \neq X$, $Y \in 2^\Theta \setminus \{\emptyset\}$, and 3) $s(\emptyset) = 0$. Applying formula (8) (or (10)) we obtain $U(m) = 0$, which completes the proof of lemma 1.

Lemma 2: $U(m)$ satisfies the desideratum D2.

Proof: Consider two FoD Θ et Θ' with $|\Theta| = N$ and $|\Theta'| = N'$ greater than zero, and suppose $N < N'$. For the vacuous BBA m_v^Θ defined on the FoD Θ, one has $U(m_v^\Theta) = 2^N - 2$. Similarly, for the vacuous BBA $m_v^{\Theta'}$ defined on the FoD Θ', one has $U(m_v^{\Theta'}) = 2^{N'} - 2$. Because the exponential function is an increasing function, one has always $2^N < 2^{N'}$, and also $2^N - 2 < 2^{N'} - 2$. Therefore $U(m_v^\Theta) < U(m_v^{\Theta'})$ when $|\Theta| < |\Theta'|$, which completes the proof of lemma 2.

Lemma 3: $U(m)$ satisfies the desideratum D3.

Proof: When the BBA m is Bayesian, its focal elements are only singletons of 2^Θ and $Bel(X) = Pl(X)$ for all $X \in 2^\Theta$. Hence $u(X) = 0$ for all $X \in 2^\Theta$. Thus, in the expression (9) of $s(X)$ one has always $-(1 - u(X))m(X)\log(m(X)) = -m(X)\log(m(X))$ and $u(X)(1 - m(X)) = 0(1 - m(X)) = 0$, so that $s(X) = -m(X)\log(m(X))$. Therefore $U(m) = \sum_{X \in 2^\Theta} s(X) = -\sum_{X \in 2^\Theta} m(X)\log(m(X))$. Because the masses of all non-singleton elements of 2^Θ are zero, we finally obtain $U(m) = -\sum_{X \in 2^\Theta ||X|=1} m(X)\log(m(X)) = -\sum_{X \in \Theta} m(X)\log(m(X))$, and this is Shannon entropy. This completes the proof of lemma 3.

Lemma 4: $U(m)$ satisfies the desideratum D4.

Proof: see the appendix.

Theorem: $U(m)$ is an effective measure of uncertainty of a basic belief assignment.

Proof: Because $U(m)$ satisfies all desiderata D1, D2, D3, and D4 as proved in lemmas 1–4, the measure of uncertainty $U(m)$ defined in (8) is effective.

D. Remarks about $U(m)$

Remark 1: It is worth noting that we do not have specified a priori what should be the range of an effective MoU in contrary to some axiomatic attempts made by different authors as reported, for instance, in [12]. We consider that the choice of the range must not be chosen a priori. The maximum range must result of the effective MoU mathematical definition. We only request the satisfaction of the desideratum D4, which is much more general, natural and essential.

Remark 2: The choice of the desideratum D3 (compatibility with Shannon probabilistic entropy) could be disputed because other entropy definitions and generalizations exist in the probabilistic framework (as those defined by Rényi [13], Tsallis [14], etc). We think however that Shannon entropy is still the most used and preferred one for engineers working in information fusion. The measure of uncertainty $U(m)$ presented in this paper could be (hopefully) generalized by replacing the desideratum D3 by another one using another choice of generalized entropy definition, which would obviously necessitate a modification of the definition of $U(m)$. This theoretical question has not yet been explored, and is left for future research.

Remark 3: It can be proved[9] that $U(m)$ verifies the monotonicity property. More precisely, if m_Y and m_Z are two distinct BBAs defined on the same FoD Θ and respectively focused on Y and on Z in 2^Θ, then one has always $U(m_Y) < U(m_Z)$ if $|Y| < |Z|$. As a special case, one has $U(m_Y) < U(m_Z)$ if $Y \subset Z$.

Remark 4: Consider a BBA m^Θ defined on a FoD Θ. Its *zero-extension* $m^{\Theta'}$ on a FoD Θ' including Θ (i.e. $\Theta \subseteq \Theta'$) is defined by $m^{\Theta'}(X) = 0$ for all $X \in 2^{\Theta'}$ not included in 2^Θ, and $m^{\Theta'}(X) = m^\Theta(X)$ for all $X \in 2^\Theta$. It means that $[Bel(\theta_i), Pl(\theta_i)] = [0, 0]$ for all $\theta_i \in \Theta' \setminus \Theta$. Under this condition, one has always $U(m^\Theta) \leq U(m^{\Theta'})$ because $u^{\Theta'}(X) \geq 0$ if $X \cap Y \neq \emptyset$ for some $Y \in 2^\Theta$. Hence there exists at least an extra term $s^{\Theta'}(X) > 0$ entering in $U(m^{\Theta'})$ calculation (w.r.t. $U(m^\Theta)$) if $m^\Theta \neq m_v^{\Theta'}$. Therefore, the extendability property of Shannon entropy for probability measures must be extended as $U(m^\Theta) \leq U(m^{\Theta'})$ for (non-Bayesian) basic belief assignments. The equality $U(m^\Theta) = U(m^{\Theta'})$ holds if m^Θ is a Bayesian BBA because $U(m^\Theta)$ coincides with Shannon entropy in this case.

VI. Examples

In this section we give several simple numerical examples of the value of the measure of uncertainty $U(m)$ expressed in nats. The examples are given in Table I and they correspond to different BBAs m_i ($i = 1, 2, \ldots, 6$), and to the vacuous BBA m_v defined on a FoD Θ. For $|\Theta| = 2$, we have only one possible union/disjunction $\theta_1 \cup \theta_2$ in 2^Θ which makes the examples too simple and not very interesting. Because for $|\Theta| \geq 4$ we have $2^4 = 16$ elements of 2^Θ to list, and due to paper length restriction we just give here some examples for $|\Theta| = 3$ with $\Theta = \{\theta_1, \theta_2, \theta_3\}$.

The numerical values of $U(m)$ have been truncated to their third decimal. m_1 and m_2 are Bayesian BBAs, and m_2 is the uniform Bayesian BBA. Hence we have $U(m_2) = \log(|\Theta|) = \log(3) \approx 1.098$ which is the maximum of Shannon entropy for this FoD. The BBAs m_3, …, m_6 and m_v are non-Bayesian

[9] Sketch of proof: prove that $U(m_Y) = 2^{|\Theta|} - 1 - |\{X \in 2^\Theta | Y \subseteq X\}| - |\{X \in 2^\Theta | X \cap Y = \emptyset\}|$ and $U(m_Z) = 2^{|\Theta|} - 1 - |\{X \in 2^\Theta | Z \subseteq X\}| - |\{X \in 2^\Theta | X \cap Z = \emptyset\}|$, and compare $U(m_Y)$ and $U(m_Z)$ when $|Y| < |Z|$ to complete the proof.

BBAs, and $U(m_v) = 2^3 - 2 = 6$ is the maximum value of the new proposed generalized entropy.

$X \in 2^\Theta$	m_1	m_2	m_3	m_4	m_5	m_6	m_v
$\emptyset$	0	0	0	0	0	0	0
θ_1	0.2	1/3	0.1	0.1	1/7	0	0
θ_2	0.3	1/3	0.2	0.2	1/7	0	0
$\theta_1 \cup \theta_2$	0	0	0.7	0.05	1/7	1	0
θ_3	0.5	1/3	0	0.3	1/7	0	0
$\theta_1 \cup \theta_3$	0	0	0	0.03	1/7	0	0
$\theta_2 \cup \theta_3$	0	0	0	0.02	1/7	0	0
Θ	0	0	0	0.3	1/7	0	1
$U(m_i)$	1.029	1.098	3.005	3.100	3.435	4	6

Table I
EXAMPLES FOR $U(m_i)$, $i = 1, 2 \ldots, 6$ AND $U(m_v)$.

It is worth noting that a non-Bayesian BBA m can have an entropy value $U(m)$ smaller than the maximum of Shannon entropy, which is normal and not surprising. For instance, if we consider $\Theta = \{\theta_1, \theta_2, \theta_3\}$ and the BBA $m(\theta_1) = 0.1$, $m(\theta_2) = 0.8$ and $m(\theta_1 \cup \theta_2) = 0.1$, we get $U(m) \approx 0.909$ which is smaller than $\log(|\Theta|) = \log(3) \approx 1.098$. Therefore, the condition $U(m) < \log(|\Theta|)$ does not imply that the BBA m is necessarily a Bayesian BBA, but if $U(m) > \log(|\Theta|)$ we are sure that m is a non-Bayesian BBA. We recall also that any BBA focused on a singleton has always zero uncertainty because lemma 1 holds.

Abellán and Moral's example revisited

We revisit Abellán and Moral's example [23] with the FoD $\Theta = \{\theta_1, \theta_2, \theta_3\}$ and the BBAs $m(\cdot)$ and $m'(\cdot)$ defined by

$$\begin{cases} m(\theta_1) = m(\theta_2) = m(\theta_3) = 0.2 \\ m(\Theta) = 0.4 \end{cases}$$

$$\begin{cases} m'(\theta_1) = m'(\theta_2) = m'(\theta_3) = 0.161 \\ m'(\theta_2 \cup \theta_3) = 0.317 \\ m'(\Theta) = 0.2 \end{cases}$$

Abellán and Moral's intuitively think it is reasonable that m should represent more uncertainty than m' as m is completely symmetrical and m' points to $\theta_2 \cup \theta_3$. We disagree with this intuition because the authors did not take into account the changes of masses values between m and m', nor the imprecisions of all unknown probabilities $P(X)$ generated by m, and the imprecisions of $P'(X)$ generated by m'.

If we analyze more carefully these two basic belief assignments we get the belief intervals $[Bel(X), Pl(X)]$ based on m, and the belief intervals $[Bel'(X), Pl'(X)]$ based on m' listed in Table II. Based on the belief interval values listed in Table II, it is clear that m' generates in fact globally more uncertainty (imprecisions on probabilities of elements of the power set of Θ) than m if we compare $u(X)$ and $u'(X)$ values. If we apply our new effective MoU definition, we obtain $U(m) = 3.1059$ nats, and $U(m') = 3.3384$ nats. One sees that $U(m) < U(m')$, which well reflects that m' is actually a bit more uncertain than m, contrary to what one would expect based on an incorrect intuition. This simple example is very interesting because it shows clearly how a simplistic intuition can easily fail.

$X \in 2^\Theta$	$[Bel(X), Pl(X)]$	$u(X)$	$[Bel'(X), Pl'(X)]$	$u'(X)$
$\emptyset$	[0,0]	0	[0,0]	0
θ_1	[0.2,0.6]	0.4	[0.161,0.361]	0.200
θ_2	[0.2,0.6]	0.4	[0.161,0.678]	0.517
$\theta_1 \cup \theta_2$	[0.4,0.8]	0.4	[0.322,0.839]	0.517
θ_3	[0.2,0.6]	0.4	[0.161,0.678]	0.517
$\theta_1 \cup \theta_3$	[0.4,0.8]	0.4	[0.322,0.839]	0.517
$\theta_2 \cup \theta_3$	[0.4,0.8]	0.4	[0.639,0.839]	0.200
Θ	[1,1]	0	[1,1]	0

Table II
BELIEF INTERVALS DRAWN FROM m AND m'.

Entropic surface for all BBAs $m(.)$ defined on $\Theta = \{\theta_1, \theta_2\}$

The figure 1 shows the entropic surface corresponding to $U(m)$ when $m(\theta_1) \in [0,1]$, $m(\theta_2) \in [0,1]$ such that $m(\theta_1) + m(\theta_2) \leq 1$, and with $m(\theta_1 \cup \theta_2) = 1 - m(\theta_1) - m(\theta_2)$.

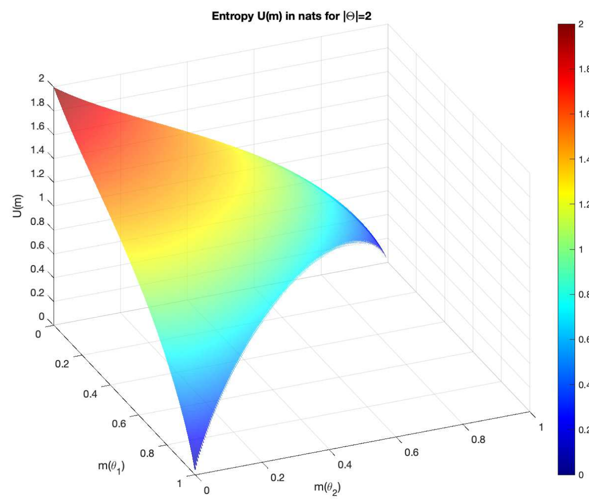

Figure 1. Entropy value $U(m)$ for all $m(.)$ defined on $\Theta = \{\theta_1, \theta_2\}$.

One verifies visually that $U(m)$ surface is smooth. Its border in the vertical plane passing through the points $(m(\theta_1) = 1, m(\theta_2) = 0)$ and $(m(\theta_1) = 0, m(\theta_2) = 1)$ corresponds to Shannon entropy curve whose maximum value is $\log(2) \approx 0.6931$, which is what we naturally expect. The unique maximum value of $U(m)$ is for the vacuous BBA m_v, and it is $U(m_v) = 2^{|\Theta|} - 2 = 2$.

VII. CONCLUSION

In this paper we have presented a new effective measure of uncertainty for basic belief assignments which is conceptually better justified than the few existing effective measures defined so far. This new generalized entropy measure verifies all the four very natural and essential desiderata, and presents the main advantages of simplicity, continuity, monotonicity and it also responds to the change of dimension of the frame of discernment. It is based on the interwoven link between the randomness and the imprecision of unknown probabilities of all elements of the power set of the frame of discernment which is inherent to any basic belief assignment.

This new entropy measure makes a clear distinction between the maximum uncertainty of the vacuous BBA, and the uncertainties related to all non-vacuous BBAs, in particular with respect to Bayesian BBAs. Hence, we have answered positively to the challenging question about the existence of a better conceptual effective measure of uncertainty for BBAs. We hope that this new effective entropy measure will arouse the interest of users of belief functions who need an effective entropy measure in their own applications. It is worth mentioning that a dual of this new measure of entropy can be defined to characterize the information content of any BBA, as well as the notion of information gain and information loss between two BBAs. This will be reported in a future publication.

As a first perspective of this theoretical work, this new entropy measure could be useful to develop advanced methods for performance evaluation of information fusion techniques, and for reasoning under uncertainty using the belief functions. As a second perspective, this new entropy could also serve to measure the uncertainty of quantitative possibility measures in the possibility theory because any quantitative possibility measure is a special case of a plausibility function which is one-to-one with a consonant belief mass function (i.e. a BBA having nested focal elements).

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APPENDIX

Proof of Lemma 4

We first note from the expression (9) of $s(X)$ that we always have $s(X) = u(X)$ for $X \in 2^\Theta$ and $X \neq \emptyset$ if $m(X) = 0$. We have also $s(X) = 0$ for $X \in 2^\Theta$ and $X \neq \emptyset$ if $m(X) = 1$. For $X \in 2^\Theta$ and $X \neq \emptyset$, if $0 < m(X) < 1$ one has

$$\begin{aligned} s(X) &= -(1-u(X))m(X)\log(m(X)) + u(X)(1-m(X)) \\ &= (1-u(X))m(X)\log(\frac{1}{m(X)}) + u(X)(1-m(X)) \\ &< (1-u(X))m(X)(\frac{1}{m(X)} - 1) + u(X)(1-m(X)) \end{aligned}$$

This strict inequality comes from the fact that for any real number $x > 0$ with $x \neq 1$, the strict inequality $\log(x) < x-1$ holds[10] (see [24], p. 68). Because $(1-u(X))m(X)(\frac{1}{m(X)} - 1) + u(X)(1-m(X)) = 1 - m(X)$, one has finally the following inequality

$$s(X) < 1 - m(X) \tag{12}$$

[10]because the derivative $f'(x)$ of $f(x) = x - 1 - \log(x)$ is always positive for $x > 0$ except for $x = 1$ where $f'(1) = 0$.

To prove that $U(m) < U(m_v)$, we consider all the cases for the distribution of the belief masses in the BBA $m \neq m_v$ as follows:

Case 1: $0 < m(X) < 1$ for all $X \neq \emptyset$ of 2^Θ.
In this (most general) case we have

$$U(m) = \sum_{X \in 2^\Theta | X \neq \emptyset} s(X) < \sum_{X \in 2^\Theta | X \neq \emptyset} (1 - m(X))$$

The majorant $\sum_{X \in 2^\Theta | X \neq \emptyset}(1 - m(X))$ can be written as

$$\sum_{X \in 2^\Theta | X \neq \emptyset} 1 - m(X) = \sum_{X \in 2^\Theta | X \neq \emptyset} 1 - \sum_{X \in 2^\Theta | X \neq \emptyset} m(X)$$

Because one has $\sum_{X \in 2^\Theta | X \neq \emptyset} 1 = 2^N - 1$, and $\sum_{X \in 2^\Theta | X \neq \emptyset} m(X) = 1$, the majorant is given by

$$\sum_{X \in 2^\Theta | X \neq \emptyset} 1 - m(X) = 2^N - 1 - 1 = 2^N - 2$$

This majorant corresponds exactly to $U(m_v)$, therefore we have proved that

$$U(m) < U(m_v) \tag{13}$$

when $0 < m(X) < 1$ for all $X \neq \emptyset$ of 2^Θ.

Case 2: Consider the particular BBA for which $m(X) = 1$ for some $X \neq \emptyset$ and $X \neq \Theta$ in 2^Θ.

- If X is a singleton of 2^Θ then $Bel(X) = Pl(X) = 1$ and $u(X) = 0$. For the elements Y of 2^Θ including X one has $Bel(Y) = Pl(Y) = 1$ and thus $u(Y) = 0$. for the elements Y of 2^Θ not including X one always has $Bel(Y) = Pl(Y) = 0$ and thus $u(Y) = 0$. Hence, $m(X) = 1$, $u(X) = 0$, $s(X) = 0$, and also $m(Y) = 0$, $u(Y) = 0$, $s(Y) = 0$ for all $Y \neq X$. Therefore we get $U(m) = 0$ which is smaller than $U(m_v) = 2^N - 2$, i.e. $U(m) < U(m_v)$ in this case.
- If X is not a singleton of 2^Θ and if $m(X) = 1$ then $Bel(X) = Pl(X) = 1$, $u(X) = 0$ and $s(X) = 0$. We have also $s(\Theta) = 0$ because $m(\Theta) = 0$, and we have $u(\Theta) = 0$ because $Bel(\Theta) = Pl(\Theta) = 1$. For all $Y \neq \emptyset$, $Y \neq X$ and $Y \neq \Theta$ such that $X \cap Y = \emptyset$, we always have $u(Y) = 0$ because $Bel(Y) = 0$ and $Pl(Y) = 0$. For all $Y \neq \emptyset$, $Y \neq X$ and $Y \neq \Theta$ such that $X \cap Y \neq \emptyset$, we always have $u(Y) = 1$ because $Bel(Y) = 0$ and $Pl(Y) = m(X) = 1$ because X has a non-empty intersection with Y. Consequently, the expression of $U(m)$ can be reformulated as

$$\begin{aligned} U(m) = s(\emptyset) &+ s(X) + s(\Theta) \\ &+ \sum_{Y \in 2^\Theta \setminus \{\emptyset, X, \Theta\} | Y \cap X = \emptyset} s(Y) \\ &+ \sum_{Y \in 2^\Theta \setminus \{\emptyset, X, \Theta\} | Y \cap X \neq \emptyset} s(Y) \end{aligned} \tag{14}$$

We have $s(\emptyset) + s(X) + s(\Theta) = 0$ because $s(\emptyset) = 0$, $s(\Theta) = 0$ and $s(X) = 0$ when $m(X) = 1$. For $Y \in 2^\Theta \setminus \{\emptyset, X, \Theta\}$ such that $Y \cap X = \emptyset$, we have

$u(Y) = 0$ and $m(Y) = 0$, hence $s(Y) = -(1 - u(Y))m(Y)\log(m(Y)) + u(Y)(1 - m(Y)) = (1 - 0)0\log(0) + 0(1 - 0) = 0$. Consequently

$$\sum_{Y \in 2^\Theta \setminus \{\emptyset, X, \Theta\} | Y \cap X = \emptyset} s(Y) = 0$$

For $Y \in 2^\Theta \setminus \{\emptyset, X, \Theta\}$ such that $Y \cap X \neq \emptyset$, we have $u(Y) = 1$ and $m(Y) = 0$, hence $s(Y) = -(1 - u(Y))m(Y)\log(m(Y)) + u(Y)(1 - m(Y)) = (1 - 1)0\log(0) + 1(1 - 0) = 1$. Consequently,

$$\sum_{Y \in 2^\Theta \setminus \{\emptyset, X, \Theta\} | Y \cap X \neq \emptyset} 1 < 2^N - 2$$

Therefore, if a BBA is focused on any element $X \neq \Theta$ (singleton, or not), that is if $m(X) = 1$, we have proved that the strict inequality $U(m) < U(m_v)$ always holds.

Case 3: Some elements of the BBA have at least a zero mass value, and others have some strictly positive mass values strictly smaller than 1.

The measure of uncertainty $U(m)$ defined in (10) requires $2^N - 1$ terms $s(X)$ to calculate in general (i.e. when all $X \in 2^\Theta \setminus \{\emptyset\}$ are focal elements of m). If some elements X have zero mass value, this measure $U(m)$ can always be decomposed as

$$U(m) = \sum_{X \in 2^\Theta | (X \neq \emptyset) \wedge (m(X) = 0)} s(X) + \sum_{X \in 2^\Theta | (X \neq \emptyset) \wedge (0 < m(X) < 1)} s(X) \quad (15)$$

Because one has $s(X) = u(X)$ when $m(X) = 0$, the first summation of (15) is equal to $\sum_{X \in 2^\Theta | (X \neq \emptyset) \wedge (m(X) = 0)} u(X)$. Because $u(X) \leq 1$, and $s(X) < 1 - m(X)$ when $m(X) < 1$, one has the following strict inequality that holds

$$U(m) < \sum_{X \in 2^\Theta | (X \neq \emptyset) \wedge (m(X) = 0)} 1 + \sum_{X \in 2^\Theta | (X \neq \emptyset) \wedge (0 < m(X) < 1)} (1 - m(X))$$

We can have at most $2^N - 3$ elements of $2^\Theta \setminus \{\emptyset\}$ having a mass equal to zero because we must have at least $(2^N - 1) - (2^N - 3) = 2$ elements X_1 and X_2 of 2^Θ for which $0 < m(X_1) < 1$, $0 < m(X_2) < 1$ with $m(X_1) + m(X_2) = 1$. If we assume that there are $1 < M \leq 2^N - 3$ elements of $2^\Theta \setminus \{\emptyset\}$ that have zero mass value, then there exist $K = 2^N - 1 - M$ elements $X_1, X_2, \ldots, X_K$ of $2^\Theta \setminus \{\emptyset\}$ for which $0 < m(X_k) < 1$, $k = 1, \ldots, K$ and with $\sum_{k=1}^{(2^N-1)-M} m(X_k) = 1$. Hence,

$$U(m) < M + \sum_{k=1}^{(2^N-1)-M} (1 - m(X_k))$$

or equivalently,

$$U(m) < \underbrace{M + (2^N - 1) - M}_{2^N - 1} - \underbrace{\sum_{k=1}^{(2^N-1)-M} m(X_k)}_{1}$$

Hence, $U(m) < 2^N - 2$, and consequently we have $U(m) < U(m_v)$ because $U(m_v) = 2^N - 2$.

In summary, we have examined all possible cases for the distribution of the belief masses, and we have proved that we always have the strict inequality $U(m) < U(m_v)$ satisfied. This completes the proof of the Lemma 4.

References

[1] C.E. Shannon, *A mathematical theory of communication*, The Bell System Technical Journal, Vol. 27, pp. 379-423 & 623-656, July & October 1948 (reprinted in [3]).

[2] C.E. Shannon, W. Weaver, *The Mathematical Theory of Communication*, Urbana, IL: The University of Illinois Press, 1–117, 1949.

[3] N.J.A. Sloane, A.D. Wyner (Editors), *Claude Elwood Shannon - Collected Papers*, IEEE Press, 924 pages, 1993.

[4] R.E. Bradley, S.J. Petrilli, C.E. Sandifer, *L'Hôpital's analyse des infiniments petits (An annoted translation with source material by Johann Bernoulli)*, Birkhäuser, 311 pages, 2015.

[5] G. Shafer, *A mathematical theory of evidence*, Princeton University Press, 1976.

[6] A.-L. Jousselme, F. Pichon, N. Ben Abdallah, *A Note About Entropy and Inconsistency in Evidence Theory*, in Proc. of Belief 2021 Int. Conf, Shanghai, China, Oct. 15–17, 2021.

[7] J. Dezert, A. Tchamova, *On Effectiveness of Measures of Uncertainty of Basic Belief Assignments*, Information & Security Journal: An International Journal (ISIJ), Vol. 52, Feb. 2022.

[8] M. Tribus, *Rational Descriptions, Decisions and Designs, Pergamon Press*, New York, pp. 26–28, 1969.

[9] J. Aczél, Z. Daróczy, *On Measures of Information and Their Characterizations*, Academic Press, 1975.

[10] D. Dubois, H. Prade, *Additivity and monotonicity of measures of information defined in the setting of Shafer's evidence theory*, BUSEFAL 24, pp. 64–76, 1985.

[11] M.T. Lamata, S. Moral, *Measures of entropy in the theory of evidence*, Int. J. Gen. Systems, Vol. 14(4), pp. 297–305, 1988.

[12] S. Moral-García, J. Abellán, *Required mathematical properties and behaviors of uncertainty measures on belief intervals*, Int. J. Intell. Syst., pp. 1–24, 2021.

[13] A. Rényi, *On measures of entropy and information*, in Proc. of the 4th Berkeley Symposium on Mathematics, Statistics and Probability, Vol. 1, pp. 547–561, 1961.

[14] C. Tsallis, *Nonadditive entropy: The concept and its use*, The European Physical Journal A, 40(3), 2009.

[15] J. Vejnarova, G.J. Klir, *Measure of strife in Dempster-Shafer theory*, Int.. J. of General Systems, Vol. 22(1), pp. 25-42, 1993.

[16] D. Dubois, H. Prade, *A note on measures of specificity for fuzzy sets*, Int. J. Gen. Syst., Vol. 10, no. 4, pp. 279–283, 1985.

[17] R. Jiroušek, P.P. Shenoy, *A new definition of entropy of belief functions in the Dempster-Shafer theory*, Int. J. Approx. Reasoning, Vol. 92(1), pp. 49–65, Jan. 2018.

[18] B. Cobb, P.P. Shenoy, *On the plausibility transformation method for translating belief function models to probability models*, IJAR, Vol. 41, 2006.

[19] Y. Maeda, H. Ichihashi, *An uncertainty measure with monotonicity under the random set inclusion*, Int. J. of General Systems, Vol. 21(4), pp. 379–392, 1993.

[20] D. Harmanec, G.J. Klir, *Measuring total uncertainty in Dempster-Shafer theory: a novel approach*, Int J General Syst., Vol. 22(4), pp. 405–419, 1994.

[21] D. Harmanec, G. Resconi, G.J. Klir, Y. Pan, *On the computation of the uncertainty measure for the Dempster-Shafer theory*, Int. J. of General Systems, Vol. 25(2), pp. 153–163, 1996.

[22] Y. Zhang, F. Huang, X. Deng, W. Jiang, *A New Total Uncertainty Measure from A Perspective of Maximum Entropy Requirement*, Entropy 2021, 23, 1061, 2021.

[23] J. Abellán, S. Moral, *Completing a total uncertainty measure in Dempster-Shafer theory*, Int. J. of General Systems, Vol. 28(4–5), pp. 299–314, 1999.

[24] M. Abramowitz, I. Stegun, *Handbook of mathematical functions*, (9th Edition), Dover, 1970.