# Analytical Solution of the Simplest Entropiece Inversion Problem

Jean Dezert[1], Florentin Smarandache[2], and Albena Tchamova[3]

[1] The French Aerospace Lab, Palaiseau, France.
jean.dezert@onera.fr
[2] Dept. of Mathematics, Univ. of New Mexico, Gallup, NM, USA.
smarand@unm.edu
[3] IICT, Bulgarian Academy of Sciences, Sofia, Bulgaria.
tchamova@bas.bg

**Abstract.** In this paper, we present a method to solve analytically the simplest Entropiece Inversion Problem (EIP). This theoretical problem consists in finding a method to calculate a Basic Belief Assignment (BBA) from the knowledge of a given entropiece vector which quantifies effectively the measure of uncertainty of a BBA in the framework of the theory of belief functions. We give an example of the calculation of EIP solution for a simple EIP case, and we show the difficulty to establish the explicit general solution of this theoretical problem that involves transcendental Lambert's functions.

**Keywords:** Belief functions · Entropy · Measure of Uncertainty.

## 1 Introduction

In this paper, we suppose the reader to be familiar with the theory of Belief Functions (BF) introduced by Shafer in [1], and we do not present in details the basics of BF. We just recall that a frame of discernement (FoD) $\Theta = \{\theta_1, \theta_2, \dots, \theta_N\}$ is a finite exhaustive set of $N > 1$ mutually exclusive elements $\theta_i$ $(i = 1, \dots, N)$, and its power set (i.e. the set of all subsets) is denoted by $2^\Theta$. A FoD represents a set of potential solutions of a decision-making problem under consideration. A Basic Belief Assignment (BBA)[4] is a mapping $m : 2^\Theta \to [0, 1]$ with $m(\emptyset) = 0$, and $\sum_{X \in 2^\Theta} m(X) = 1$.

A new effective entropy measure $U(m)$ for any BBA $m(\cdot)$ defined on a FoD $\Theta$ has been defined as follows [2]:

$$U(m) = \sum_{X \in 2^\Theta} s(X) \tag{1}$$

where $s(X)$ is named the *entropiece* of $X$, which is defined by

$$s(X) = -m(X)(1 - u(X)) \log(m(X)) + u(X)(1 - m(X)) \tag{2}$$

[4] For notation convenience, we denote by $m$ or $m(\cdot)$ any BBA defined implicitly on the FoD $\Theta$, and we also denote it as $m^\Theta$ to explicitly refer to the FoD when necessary.

with

$$u(X) = Pl(X) - Bel(X) = \sum_{Y \in 2^{\Theta} | X \cap Y \neq \emptyset} m(Y) - \sum_{Y \in 2^{\Theta} | Y \subseteq X} m(Y). \quad (3)$$

$Pl(X)$ and $Bel(X)$ are respectively the plausibility and the belief of the element $X$ of the power set of $\Theta$, see [1] for details. $u(X)$ quantifies the imprecision of the unknown probability of $X$. The vacuous BBA characterizing the total ignorant source of evidence is denoted by $m_v$, and it is such that $m_v(\Theta) = 1$ and $m_v(X) = 0$ for any $X \subset \Theta$.

This measure of uncertainty $U(m)$ (i.e. entropy measure) is effective because it satisfies the following four essential properties [2]:

1. $U(m) = 0$ for any BBA $m(\cdot)$ focused on a singleton $X$ of $2^{\Theta}$.
2. $U(m_v^{\Theta}) < U(m_v^{\Theta'})$ if $|\Theta| < |\Theta'|$.
3. $U(m) = -\sum_{X \in \Theta} m(X) \log(m(X))$ if $m(\cdot)$ is a Bayesian[5] BBA. Hence, $U(m)$ reduces to Shannon entropy [7] in this case.
4. $U(m) < U(m_v)$ for any non-vacuous BBA $m(\cdot)$ and for the vacuous BBA $m_v(\cdot)$ defined with respect to the same FoD.

The proof of the three first properties is quite simple to make, whereas the proof of $U(m) < U(m_v)$ is much more difficult, see [2] for proofs and examples. A detailed analysis of other (non-effective) entropy measures proposed in the literature during the last four decades is done in [3].

The entropiece $s(X)$ given by (2) corresponds to the contribution of $X$ to the whole uncertainty measure $U(m)$. The entropiece $s(X)$ involves $m(X)$ and the imprecision $u(X) = Pl(X) - Bel(X)$ about the unknown probability of $X$ in a subtle interwoven manner named *epistemic entanglement*. The cardinality of $X$ is indirectly taken into account in the derivation of $s(X)$ thanks to $u(X)$ which requires the derivation of $Pl(X)$ and $Bel(X)$ functions that depend on the cardinality of $X$. Because $u(X) \in [0, 1]$ and $m(X) \in [0, 1]$ one has $s(X) \geq 0$, and $U(m) \geq 0$. The quantity $U(m)$ is expressed in *nats* because we use the natural logarithm. $U(m)$ can be expressed in *bits* by dividing the $U(m)$ value in *nats* by $\log(2) = 0.69314718....$ This measure of uncertainty $U(m)$ is a continuous function in its basic belief mass arguments because it is a summation of continuous functions. In formula (2), we always take $m(X)\log(m(X)) = 0$ when $m(X) = 0$ because $\lim_{m(X)\to 0^+} m(X)\log(m(X)) = 0$ which can be proved using L'Hôpital rule [4]. Note that for any BBA $m$, one has always $s(\emptyset) = 0$ because $m(\emptyset) = 0$ and $u(\emptyset) = Pl(\emptyset) - Bel(\emptyset) = 0 - 0 = 0$. For the vacuous BBA, one has $s(\Theta) = 0$ because $m_v(\Theta) = 1$ and $u(\Theta) = Pl(\Theta) - Bel(\Theta) = 1 - 1 = 0$.

As proved in [2], the entropy of the vacuous BBA on the FoD $\Theta$ is equal to

$$U(m_v) = 2^{|\Theta|} - 2 \quad (4)$$

[5] $m$ is Bayesian BBA if it has only singletons as focal elements, i.e. $m(\theta_i) > 0$ for some $\theta_i \in \Theta$ and $m(X) = 0$ for all non-singletons $X$ of $2^{\Theta}$.

This maximum entropy value $2^{|\Theta|}-2$ makes perfectly sense because for the vacuous BBA there is no information at all about the conflicts between the elements of the FoD. Actually for all $X \in 2^\Theta \setminus \{\emptyset, \Theta\}$ one has $u(X)=1$ because $[Bel(X), Pl(X)] = [0,1]$, and one has $u(\emptyset)=0$ and $u(\Theta)=0$. Hence, the sum of all imprecisions of $P(X)$ for all $X \in 2^\Theta$ is exactly equal to $2^{|\Theta|}-2$ which corresponds to $U(m_v)$ as expected. Moreover, one has always $U(m_v) > \log(|\Theta|)$ which means that the vacuous BBA has always an entropy greater than the maximum of Shannon entropy $\log(|\Theta|)$ obtained with the uniform probability mass function distributed on $\Theta$.

As a dual concept of this entropy measure $U(m)$, we have defined in [8] the measure of information content of any BBA by

$$IC(m) = U(m_v) - U(m) = (2^{|\Theta|}-2) - \sum_{X \in 2^\Theta} s(X) \tag{5}$$

From the definition (5), one sees that for $m \neq m_v^\Theta$ one has $IC(m) > 0$ because $U(m) < U(m_v)$, and for $m = m_v$ one has $IC(m_v) = 0$ (i.e. the vacuous BBA carries no information), which is what we naturally expect.

Note that the information content $IC(m^\Theta)$ of a BBA depends not only of the BBA $m(\cdot)$ itself but also on the cardinality of the frame of discernment $\Theta$ because $IC(m)$ requires the knowledge of $|\Theta| = N$ to calculate the max entropy value $U(m_v) = 2^{|\Theta|}-2$ entering in (5). This remark is important to understand that even if two BBAs (defined on different FoDs) focus entirely on a same focal element, their information contents are necessarily different. This means that the information content depends on the context of the problem, i.e. the FoD. The notions of information gain and information loss between two BBAs are also mathematically defined in [8] for readers interested in this topic.

This paper is organized as follows. Section 2 defines the general entropiece inversion problem (EIP). Section 3 describes the simplest entropiece inversion problem (SEIP). An analytical solution of SEIP is proposed and it is applied on a simple example also in section 3. The conclusion is made in section 4.

## 2 The general entropiece inversion problem (EIP)

The set $\{s(X), X \in 2^\Theta\}$ of the entropieces values $s(X)$ given by (2) can be represented by an entropiece vector $\mathbf{s}(m) = [s(X), X \in 2^\Theta]^T$, where any order of elements $X$ of the power set $2^\Theta$ can be chosen. For simplicity, we suggest to use the classical $N$-bits representation if $|\Theta| = N$, with the increasing order (see example in section 3). The general Entropiece Inversion Problem, or EIP for short, is an interesting theoretical problem which can be easily stated as follows:

Suppose that if the entropiece vector $\mathbf{s}(m)$ known (estimated or given), is it possible to calculate a BBA $m(\cdot)$ corresponding to this entropiece vector $\mathbf{s}(m)$? and how?

Also we would like to know if the derivation of $m(\cdot)$ from $\mathbf{s}(m)$ provides a unique BBA solution, or not?

This general entropiece inversion problem is a challenging mathematical problem, and we do not know if a general analytical solution of EIP is possible, or not. We leave it as an open mathematical question for future research. However, we present in this paper the analytical solution for the simplest case where the FoD $\Theta$ has only two elements, i.e. when $|\Theta| = N = 2$. Even in this simplest case, the EIP solution is no so easy to calculate as it will be shown in the next section. This is the main contribution of this paper.

The mathematical EIP addressed in this paper is not related (for now) to any problem for the natural world and it cannot be confirmed experimentally using data from nature because the entropy concept is not directly measurable, but only computable from the estimation of probability $p(\cdot)$ or belief mass functions $m(\cdot)$. So, why do we address this entropiece inversion problem? Because in advanced information fusion systems we can imagine to have potentially access to this type of information and it makes sense to assess the underlying BBA provided by a source of evidence to eventually modify it in some fusion systems for some aims. We could also imagine to make adjustments of entropieces values to volontarly improve (or degrade) $IC(m)$, and to generate the proper modified BBA for some tasks. At this early stage of research work it is difficult to anticipate the practical interests of the calculation of solutions of the general EIP, but to present its mathematical interest for now.

## 3 The simplest entropiece inversion problem (SEIP)

### 3.1 Example

We consider a FoD $\Theta$ with only two elements, say $\Theta = \{A, B\}$, where $A$ and $B$ are mutually exclusive and exhaustive, and the following BBA

$$m(A) = 0.5, \quad m(B) = 0.3, \quad m(A \cup B) = 0.2$$

Because $[Bel(\emptyset), Pl(\emptyset)] = [0, 1]$ one has $u(\emptyset) = 0$. Because $[Bel(A), Pl(A)] = [0.5, 0.7]$, $[Bel(B), Pl(B)] = [0.3, 0.5]$, $[Bel(\Theta), Pl(\Theta)] = [1, 1]$, one has $u(A) = 0.2$, $u(B) = 0.2$, and $u(\Theta) = 0$. Applying (2), one gets $s(\emptyset) = 0$, $s(A) \approx 0.377258$, $s(B) \approx 0.428953$ and $s(\Theta) \approx 0.321887$. Using the 2-bits representation with increasing ordering[6], we encode the elements of the power set as $\emptyset = 00$, $A = 01$, $B = 10$ and $A \cup B = 11$. The entropiece vector is

$$\mathbf{s}(m^\Theta) = \begin{bmatrix} s(\emptyset) \\ s(A) \\ s(B) \\ s(A \cup B) \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0.3773 \\ 0.4290 \\ 0.3219 \end{bmatrix} \tag{6}$$

If we use the classical 2-bits (here $|\Theta| = 2$) representation with increasing ordering (as we recommand) the first component of entropiece vector $\mathbf{s}(m)$ will

[6] Once the binary values are converted into their digit value with the most significant bit on the left (i.e the least significant bit on the right).

be $s(\emptyset)$ which is always equal to zero for any BBA $m$, hence the first component of $\mathbf{s}(m)$ is always zero and it can be dropped (i.e. removed of the vector representation actually). By summing all the components of the entropiece vector $\mathbf{s}(m)$ we obtain the entropy $U(m) \approx 1.128098$ nats of the BBA $m(\cdot)$. Note that the components $s(X)$ (for $X \neq \emptyset$) of the entropieces vector $\mathbf{s}(m)$ are not independent because they are linked to each other through the calculation of $Bel(X)$ and $Pl(X)$ values entering in $u(X)$.

### 3.2 Analytical solution of SEIP

Because we suppose $\Theta = \{A, B\}$, the expression of three last components[7] of the entropiece vector $\mathbf{s}(m)$ are given by (2), and we have

$$\begin{aligned}
s(A) &= -m(A)(1-u(A))\log(m(A)) + u(A)(1-m(A)) \\
s(B) &= -m(B)(1-u(B))\log(m(B)) + u(B)(1-m(B)) \\
s(A\cup B) &= -m(A\cup B)(1-u(A\cup B))\log(m(A\cup B)) + u(A\cup B)(1-m(A\cup B))
\end{aligned}$$

Because $u(A) = Pl(A) - Bel(A) = (m(A) + m(A\cup B)) - m(A) = m(A\cup B)$, $u(B) = Pl(B) - Bel(B) = (m(B) + m(A\cup B)) - m(B) = m(A\cup B)$ and $u(A\cup B) = Pl(A\cup B) - Bel(A\cup B) = 1 - 1 = 0$, one gets the following system of equations to solve

$$s(A) = -m(A)(1-m(A\cup B))\log(m(A)) + m(A\cup B)(1-m(A)) \quad (7)$$

$$s(B) = -m(B)(1-m(A\cup B))\log(m(B)) + m(A\cup B)(1-m(B)) \quad (8)$$

$$s(A\cup B) = -m(A\cup B)\log(m(A\cup B)) \quad (9)$$

The set of equations (7), (8) and (9) is called the EIP *transcendental equation system* for the case $|\Theta| = 2$.

The plot of function $s(A\cup B) = -m(A\cup B)\log(m(A\cup B))$ is given in Figure 1 for convenience. By derivating the function $-m(A\cup B)\log(m(A\cup B))$ we see that its maximum value is obtained for $m(A\cup B) = 1/e \approx 0.3679$ for which

$$s(A\cup B) = -\frac{1}{e}\log(1/e) = \frac{1}{e}\log(e) = \frac{1}{e}$$

Therefore, the numerical value of $s(A\cup B)$ always belongs to the interval $[0, 1/e]$.

Without loss of generality, we assume $0 < s(A\cup B) \leq 1/e$ because if $s(A\cup B) = 0$ then one deduces directly without ambiguity that either $m(A\cup B) = 1$ (which means that the BBA $m(\cdot)$ is the vacuous BBA) if $s(A) = s(B) = 1$, or $m(A\cup B) = 0$ otherwise. With the assumption $0 < s(A\cup B) \leq 1/e$, the equation (9) is of the general transcendental form

$$ye^y = a \Leftrightarrow \log(m(A\cup B))m(A\cup B) = -s(A\cup B) \quad (10)$$

[7] We always omit the 1st component $s(\emptyset)$ of entropiece vector $\mathbf{s}(m)$ which is always equal to zero and not necessary in our analysis.

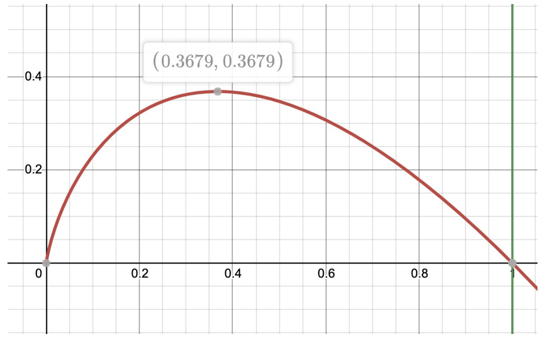


**Fig. 1.** Plot of $s(A \cup B) = -m(A \cup B) \log(m(A \cup B))$ (in red) with $X$-axis $= m(A \cup B) \in [0, 1]$, and $y$-axis $= s(A \cup B)$ in nats.

by considering the known value as $a = -s(A \cup B)$ in $[-\frac{1}{e}, 0[$, and the unknown as $y = \log(m(A \cup B))$.

Unfortunately the solution of the transcendental equation (10) does not have an explicit expression involving simple functions. Actually, the solution of this equation is actually given by the Lambert's $W$-function which is a multivalued function (called also the omega function or product logarithm in mathematics) [6]. It can however be calculated[8] with a good precision by some numerical methods - see [5] for details. The equation $ye^y = a$ admits real solution(s) only if $a \geq -\frac{1}{e}$. For $a \geq 0$, the solution of $ye^y = a$ is $y = W_0(a)$, and for $-\frac{1}{e} \leq a < 0$ there are two possible real values of $W(a)$ - see Figure 1 of [5] which are denoted respectively $y_1 = W_0(a)$ and $y_2 = W_{-1}(a)$. The principal branch of the Lambert's function $W(x)$ satisfying $-1 \leq W(x)$ is denoted $W_0(x)$, and the branch satisfying $W(x) \leq -1$ is denoted by $W_{-1}(x)$ by Corless et al. in [5]. In our context because we have $a \in [-\frac{1}{e}, 0[$, the solutions of $ye^y = a$ are given by

$$y_1 = W_0(a) = W_0(-s(A \cup B))$$
$$y_2 = W_{-1}(a) = W_{-1}(-s(A \cup B))$$

Hence we get two possible solutions for the value of $m(A \cup B)$, which are

$$m_1(A \cup B) = e^{y_1} = e^{W_0(-s(A \cup B))} \tag{11}$$
$$m_2(A \cup B) = e^{y_2} = e^{W_{-1}(-s(A \cup B))} \tag{12}$$

Of course, at least one of these solutions is necessarily correct but we do not know which one. So, at this current stage, we must consider[9] and the two solutions $m_1(A \cup B)$ and $m_1(A \cup B)$ for $m(A \cup B)$ as acceptable, and we must

[8] Lambert's $W$-function is implemented in Matlab$^{\text{TM}}$ as *lambertw* function.
[9] If the two masses values are admissible, that is if $m_1(A \cup B) \in [0, 1]$ and if $m_2(A \cup B) \in [0, 1]$. If one of them is non-admissible it is eliminated.

continue to solve equations (7) and (8) to determine the mass values $m(A)$ and $m(B)$.

Let's now determine $m(A)$ at first by solving (7). Suppose we set the value of $m(A\cup B)$ is known and taken either as $m_1(A\cup B)$, or as $m_2(A\cup B)$, then we can rearrange the equation (7) as

$$-\frac{s(A)-m(A\cup B)}{1-m(A\cup B)} = m(A)[\log(m(A)) + \frac{m(A\cup B)}{1-m(A\cup B)}]$$

which can be rewritten as the general equation of the form

$$(y+a)e^y = b \tag{13}$$

by taking

$$y = \log(m(A)) \tag{14}$$

$$a = \frac{m(A\cup B)}{1-m(A\cup B)} \tag{15}$$

$$b = -\frac{s(A)-m(A\cup B)}{1-m(A\cup B)} \tag{16}$$

The solution of (13) are given by [5]

$$y = W(be^a) - a \tag{17}$$

Once $y$ is calculated by formula (17) and since $y = \log(m(A))$ we obtain the solution for $m(A)$ given by

$$m(A) = e^y = e^{W(be^a)-a} \tag{18}$$

Similarly, the solution for $m(B)$ will be given by

$$m(B) = e^y = e^{W(be^a)-a} \tag{19}$$

by solving the equation $(y+a)e^y = b$ with

$$y = \log(m(B)) \tag{20}$$

$$a = \frac{m(A\cup B)}{1-m(A\cup B)} \tag{21}$$

$$b = -\frac{s(B)-m(A\cup B)}{1-m(A\cup B)} \tag{22}$$

We must however check if there is one solution only $m(A) = e^{W_0(be^a)-a}$, or in fact two solutions $m_1(A) = e^{W_0(be^a)-a}$ and $m_2(A) = e^{W_{-1}(be^a)-a}$, and similarly for the solution for $m(B)$. This depends on the parameters $a$ and $b$ with respect to $[-1/e, 0[$ interval and $[0, \infty[$.

We illustrate in the next subsection how to calculate the SEIP solution from these analytical formulas for the previous exemple.

### 3.3 SEIP solution of the previous example

We recall that we have for this example $s(\emptyset) = 0$, $s(A) \approx 0.3773$, $s(B) \approx 0.4290$ and $s(\Theta) \approx 0.3219$. If we apply formulas (11)-(12) for this example, we have $a = -s(A \cup B) = -0.3219$ and therefore

$$\begin{aligned} y_1 &= W_0(-0.3219) = -0.5681 \\ y_2 &= W_{-1}(-0.3219) = -1.6094 \end{aligned}$$

Hence the two potential solutions for the mass $m(A \cup B)$ are

$$\begin{aligned} m_1(A \cup B) &= e^{y_1} \approx 0.5666 \\ m_2(A \cup B) &= e^{y_2} = 0.2000 \end{aligned}$$

It can be easily verified that

$$\begin{aligned} -m_1(A \cup B)\log(m_1(A \cup B)) &= 0.3219 = s(A \cup B) \\ -m_2(A \cup B)\log(m_2(A \cup B)) &= 0.3219 = s(A \cup B) \end{aligned}$$

We see that the second potential solution $m_2(A \cup B) = 0.2000$ is the solution that corresponds to the original mass of $A \cup B$ of the BBA $m(A \cup B)$ of our example.

Now, we examine what would be the values of $m(A)$ and $m(B)$ given respectively by (18) and (19) by taking either $m(A \cup B) = m_1(A \cup B) = 0.5666$ or $m(A \cup B) = m_2(A \cup B) = 0.20$.

- Let's examine the 1st possibility with the potential solution

$$m(A \cup B) = m_1(A \cup B) = 0.5666$$

For determining $m(A)$, we have to solve $(y + a)e^y = b$ with the unknown $y = \log(m(A))$ and with

$$\begin{aligned} a &= \frac{m(A \cup B)}{1 - m(A \cup B)} \approx \frac{0.5666}{1 - 0.5666} = 1.3073 \\ b &= -\frac{s(A) - m(A \cup B)}{1 - m(A \cup B)} \approx -\frac{0.3773 - 0.5666}{1 - 0.5666} = 0.4369 \end{aligned}$$

Hence, $be^a = 0.4368 \cdot e^{1.3073} \approx 1.6148$.
Applying formula (18), one gets[10]

$$\begin{aligned} m_1(A) &= e^{W_0(be^a)-a} = 0.5769 \\ m_2(A) &= e^{W_{-1}(be^a)-a} = -0.0216 + 0.0924i \end{aligned}$$

[10] Using *lambertw* Matlab function.

For determing $m(B)$ we have to solve $(y+a)e^y = b$ with the unknown $y = \log(m(B))$ and with

$$a = \frac{m(A \cup B)}{1 - m(A \cup B)} \approx \frac{0.5666}{1 - 0.5666} = 1.3073$$
$$b = -\frac{s(B) - m(A \cup B)}{1 - m(A \cup B)} \approx -\frac{0.4290 - 0.5666}{1 - 0.5666} = 0.3176$$

Hence, $be^a = 0.3176 \cdot e^{1.3073} \approx 1.1739$.
Applying formula (19), one gets

$$m_1(B) = e^{W_0(be^a)-a} = 0.5065$$
$$m_2(B) = e^{W_{-1}(be^a)-a} = -0.0204 + 0.0657i$$

One sees that there is no effective choice for the values of $m(A)$ and $m(B)$ if we suppose $m(A \cup B) = m_1(A \cup B) = 0.5666$ because if one takes as real values solutions $m(A) = m_1(A) = 0.5769$ and $m(B) = m_1(B) = 0.5065$ one would get

$$m(A) + m(B) + m(A \cup B) = 0.5769 + 0.5065 + 0.5666 = 1.65$$

which is obviously greater than one. This generates an improper BBA.

– Let's consider the 2nd possibility with the potential solution

$$m(A \cup B) = m_2(A \cup B) = 0.20$$

For determinating $m(A)$, we have to solve $(y+a)e^y = b$ with the unknown $y = \log(m(A))$ and with

$$a = \frac{m(A \cup B)}{1 - m(A \cup B)} = \frac{0.20}{1 - 0.20} = 0.25$$
$$b = -\frac{s(A) - m(A \cup B)}{1 - m(A \cup B)} \approx -\frac{0.3773 - 0.20}{1 - 0.20} = -0.2216$$

Hence, $be^a = -0.2216 \cdot e^{0.25} \approx -0.2845$.

$$m_1(A) = e^{W_0(be^a)-a} = 0.5000$$
$$m_2(A) = e^{W_{-1}(be^a)-a} = 0.1168$$

For determinating $m(B)$ we have to solve $(y+a)e^y = b$ with the unknown $y = \log(m(B))$ and with

$$a = \frac{m(A \cup B)}{1 - m(A \cup B)} \approx \frac{0.20}{1 - 0.20} = 0.25$$
$$b = -\frac{s(B) - m(A \cup B)}{1 - m(A \cup B)} \approx -\frac{0.4290 - 0.20}{1 - 0.20} = -0.2862$$

Hence, $be^a = -0.2862 \cdot e^{0.25} \approx -0.3675$.
Applying formula (19), one gets

$$\begin{aligned}
m_1(B) &= e^{W_0(be^a)-a} = 0.3000\\
m_2(B) &= e^{W_{-1}(be^a)-a} = 0.2732
\end{aligned}$$

Based on this 2nd possibility for potential solution $m(A \cup B) = 0.20$, one sees that the only possible effective choice of mass values $m(A)$ and $m(B)$ is to take $m(A) = m_1(A) = 0.50$ and $m(B) = m_1(B) = 0.30$ which gives the proper sought BBA such that $m(A) + m(B) + m(A \cup B) = 1$ which exactly corresponds to the orignal BBA that has been used to generate the entropiece vector $\mathbf{s}(m)$ for this example.

In summary, for the case $|\Theta| = 2$ it is always possible to calculate the BBA $m(\cdot)$ from the knowledge of the entropiece vector, and the solution of SEIP is obtained by analytical formulas.

### 3.4 Remark

In the very particular case where $s(A \cup B) = 0$ the equation (9) reduces to

$$-m(A \cup B)\log(m(A \cup B)) = 0 \tag{23}$$

which has two possible solutions $m(A \cup B) = m_1(A \cup B) = 1$, and $m(A \cup B) = m_2(A \cup B) = 0$.

If $m(A \cup B) = 1$, then it means that necessarily the BBA is the vacuous BBA, and so $m(A) = m(B) = 0$, $u(A) = Pl(A) - Bel(A) = 1$, $u(B) = Pl(B) - Bel(B) = 1$. Therefore[11]

$$\begin{aligned}
s(A) &= -m(A)(1-u(A))\log(m(A)) + u(A)(1-m(A))\\
&= -m(A)(1-m(A\cup B))\log(m(A)) + m(A\cup B)(1-m(A))\\
&= 0(1-1)\log(0) + 1(1-0) = 1\\
s(B) &= -m(B)(1-u(B))\log(m(B)) + u(B)(1-m(B))\\
&= -m(B)(1-m(A\cup B))\log(m(B)) + m(A\cup B)(1-m(B))\\
&= 0(1-1)\log(0) + 1(1-0) = 1
\end{aligned}$$

So the choice of $m(A \cup B) = m_1(A \cup B) = 1$ is the only possible if the entropiece vector is $\mathbf{s}(m) = [110]^T$.

[11] We use the formal notation $\log(0)$ even if $\log(0)$ is $-\infty$ because in our derivations we have always a $0\log(0)$ product which is equal to zero due to L'Hôpital's rule [4].

If $s(A) < 1$, or if $s(B) < 1$ (or both) then we must choose $m(A \cup B) = m_2(A \cup B) = 0$, and in this case we have to solve the equations

$$\begin{aligned}
s(A) &= -m(A)(1-u(A))\log(m(A)) + u(A)(1-m(A)) \\
&= -m(A)(1-m(A\cup B))\log(m(A)) + m(A\cup B)(1-m(A)) \\
&= -m(A)\log(m(A)) \\
s(B) &= -m(B)(1-u(B))\log(m(B)) + u(B)(1-m(B)) \\
&= -m(B)(1-m(A\cup B))\log(m(B)) + m(A\cup B)(1-m(B)) \\
&= -m(B)\log(m(B))
\end{aligned}$$

The possible solutions of equation $s(A) = -m(A)\log(m(A))$ are given by

$$m_1(A) = e^{W_0(-s(A))} \tag{24}$$

$$m_2(A) = e^{W_{-1}(-s(A))} \tag{25}$$

and the possible solutions of equation $s(B) = -m(B)\log(m(B))$ are given by

$$m_1(B) = e^{W_0(-s(B))} \tag{26}$$

$$m_2(B) = e^{W_{-1}(-s(B))} \tag{27}$$

In this particular case where $s(A \cup B) = 0$, and $s(A) < 1$ or $s(B) < 1$, we have to select the pair of possible solutions among the four possible choices

$$\begin{aligned}
(m(A), m(B)) &= (m_1(A), m_1(B)), \\
(m(A), m(B)) &= (m_1(A), m_2(B)), \\
(m(A), m(B)) &= (m_2(A), m_1(B)), \\
(m(A), m(B)) &= (m_2(A), m_2(B)).
\end{aligned}$$

The judicious choice of pair $(m(A), m(B))$ must satisfy the proper BBA constraint $m(A)+m(B)+m(A\cup B) = 1$, where $m(A\cup B) = 0$ because $s(A\cup B) = 0$ in this particular case.

For instance, if we consider $\Theta = \{A, B\}$ and the following (bayesian) BBA

$$m(A) = 0.6, m(B) = 0.4, m(A \cup B) = 0$$

The entropiece vector $\mathbf{s}(m)$ is

$$\mathbf{s}(m) = \begin{bmatrix} s(A) \\ s(B) \\ s(A\cup B) \end{bmatrix} \approx \begin{bmatrix} 0.3065 \\ 0.3665 \\ 0 \end{bmatrix} \tag{28}$$

Hence from $\mathbf{s}(m)$ we can deduce $m(A \cup B) = 0$ because we cannot consider $m(A \cup B) = 1$ as a valid solution because $s(A) < 1$ and $s(B) < 1$. The possible solutions of equation $s(A) = -m(A)\log(m(A))$ are

$$\begin{aligned}
m_1(A) &= e^{W_0(-s(A))} = e^{W_0(-0.3065)} = 0.6000 \\
m_2(A) &= e^{W_{-1}(-s(A))} = e^{W_{-1}(-0.3065)} = 0.1770
\end{aligned}$$

and the possible solutions of equation $s(B) = -m(B)\log(m(B))$ are

$$m_1(B) = e^{W_0(-s(B))} = e^{W_0(-0.3665)} = 0.4000$$
$$m_2(B) = e^{W_{-1}(-s(B))} = e^{W_{-1}(-0.3665)} = 0.3367$$

One sees that the only effective (or judicious) choice for $m(A)$ and $m(B)$ is to take $m(A) = m_1(A) = 0.60$ and $m(B) = m_1(B) = 0.40$, which coincides with the original bayesian BBA that has been used to generate the entropiece vector $\mathbf{s}(m) = [0.3065, 0.3665, 0]^T$.

## 4 Conclusion

In this paper we have introduced for the first time the entropiece inversion problem (EIP) which consists in calculating a basic belief assignment from the knowledge of a given entropiece vector which quantifies effectively the measure of uncertainty of a BBA in the framework of the theory of belief functions. The general analytical solution of this mathematical problem is a very challenging open problem because it involves transcendental equations. We have shown however how it is possible to obtain an analytical solution for the simplest EIP involving only two elements in the frame of discernment. Even in this simplest case the analytical solution of EIP is not easy to obtain because it requires a calculation of values of the transcendental Lambert's functions. Even if no general analytical formulas are found for the solution of general EIP, it would be interesting to develop numerical methods to approximate the general EIP solution, and to exploit it in future advanced information fusion systems.

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