Erratum of paper entitled *On the Validity of Dempster's Fusion Rule and its Interpretation as a Generalization of Bayesian Fusion Rule*

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Abstract

In this erratum we correct a mathematical mistake included in the paper entitled *On the Validity of Dempster's Fusion Rule and its Interpretation as a Generalization of Bayesian Fusion Rule* published in 2014 in [1]. In taking into account this mathematical correction the Bayesian fusion rule is associative in contrary to what is claimed in the original version of our paper. The comments in our paper remain valid for pages 223 to 238. Corrections in several pages from page 239 to the end of our paper must be done as explained next in this erratum.

In [1] page 239, the general formulas[1] #(34)–#(36) are incorrect. The correct formulas are presented here.

Based on conditional statistical independence assumption $P(Z_1,Z_2|X)=P(Z_1|X)P(Z_2|X)$, we have

$$P(X|Z_1\cap Z_2)=\frac{P(Z_1\cap Z_2|X)P(X)}{P(Z_1\cap Z_2)}=\frac{P(Z_1|X)P(Z_2|X)P(X)}{P(Z_1\cap Z_2)}=\frac{\frac{P(X|Z_1)P(Z_1)}{P(X)}\frac{P(X|Z_2)P(Z_2)}{P(X)}P(X)}{\sum_{i=1}^{N}P(X=x_i,Z_1\cap Z_2)} \tag{1}$$

which can be written as

$$P(X|Z_1\cap Z_2)=\frac{1}{\sum_{i=1}^{N}\frac{P(X=x_i|Z_1)P(X=x_i|Z_2)}{P(X=x_i)}}\frac{P(X|Z_1)P(X|Z_2)}{P(X)} \tag{2}$$

The formula (2) corresponds to formula #(24) of our original paper [1]. This formula (2) can be rewritten in a symmetrical form as follows

$$P(X|Z_1\cap Z_2)=\frac{1}{K'(Z_1,Z_2)}\cdot\frac{P(X|Z_1)}{\sqrt{P(X)}}\cdot\frac{P(X|Z_2)}{\sqrt{P(X)}}=\frac{1}{K'(Z_1,Z_2)}\cdot\frac{P(X|Z_1)}{P^{\frac{1}{2}}(X)}\cdot\frac{P(X|Z_2)}{P^{\frac{1}{2}}(X)} \tag{3}$$

where the normalization constant $K'(Z_1,Z_2)$ is given by:

$$K'(Z_1,Z_2)\triangleq\sum_{i=1}^{N}\frac{P(X=x_i|Z_1)}{\sqrt{P(X=x_i)}}\cdot\frac{P(X=x_i|Z_2)}{\sqrt{P(X=x_i)}}=\sum_{i=1}^{N}\frac{P(X=x_i|Z_1)}{P^{\frac{1}{2}}(X=x_i)}\cdot\frac{P(X=x_i|Z_2)}{P^{\frac{1}{2}}(X=x_i)} \tag{4}$$

The formulas #(24)–#(33) of [1] are correct.

If we generalize the formula (1) for $s>2$ conditioning terms, we obtain the following expression

$$P(X|Z_1\cap Z_2\cap\ldots\cap Z_s)=\frac{P(Z_1\cap Z_2\cap\ldots\cap Z_s|X)P(X)}{P(Z_1\cap Z_2\cap\ldots\cap Z_s)}=\frac{P(Z_1|X)P(Z_2|X)\ldots P(Z_s|X)P(X)}{P(Z_1\cap Z_2\cap\ldots\cap Z_s)} \tag{5}$$

$$=\frac{\frac{P(X|Z_1)P(Z_1)}{P(X)}\frac{P(X|Z_2)P(Z_2)}{P(X)}\ldots\frac{P(X|Z_s)P(Z_s)}{P(X)}P(X)}{\sum_{i=1}^{N}P(X=x_i,Z_1\cap Z_2\cap\ldots\cap Z_s)} \tag{6}$$

which can be written as

$$P(X|Z_1\cap Z_2\cap\ldots\cap Z_s)=\frac{1}{\sum_{i=1}^{N}\frac{P(X=x_i|Z_1)P(X=x_i|Z_2))\ldots P(X=x_i|Z_s)}{P^{s-1}(X=x_i)}}\frac{P(X|Z_1)P(X|Z_2)\ldots P(X|Z_s)}{P^{s-1}(X)} \tag{7}$$

[1] For avoiding confusion with formula number in this erratum, we denote the formula number appearing in the original paper [1] by #(xx), where xx the number under concern.

or equivalently as

$$P(X|Z_1 \cap \ldots \cap Z_s) = \frac{1}{K(X, Z_1, \ldots, Z_s)} \cdot \prod_{k=1}^{s} P(X|Z_k) \tag{8}$$

where the coefficient $K(X, Z_1, \ldots, Z_s)$ is defined by

$$K(X, Z_1, \ldots, Z_s) \triangleq P^{s-1}(X) \sum_{i=1}^{N} \frac{(\prod_{k=1}^{s} P(X = x_i|Z_k))}{P^{s-1}(X = x_i)} \tag{9}$$

The formula #(34) of [1] must be replaced by the formula (9) above.

The symmetrized form of Eq. (7) is:

$$P(X|Z_1 \cap \ldots \cap Z_s) = \frac{1}{K'(Z_1, \ldots, Z_s)} \cdot \prod_{k=1}^{s} \frac{P(X|Z_k)}{\sqrt[s]{P^{s-1}(X)}} = \frac{1}{K'(Z_1, \ldots, Z_s)} \cdot \prod_{k=1}^{s} \frac{P(X|Z_k)}{P^{\frac{s-1}{s}}(X)} \tag{10}$$

with the normalization constant $K'(Z_1, \ldots, Z_s)$ given by:

$$K'(Z_1, \ldots, Z_s) \triangleq \sum_{i=1}^{N} \prod_{k=1}^{s} \frac{P(X = x_i|Z_k)}{\sqrt[s]{P^{s-1}(X = x_i)}} = \sum_{i=1}^{N} \prod_{k=1}^{s} \frac{P(X = x_i|Z_k)}{P^{\frac{s-1}{s}}(X = x_i)} \tag{11}$$

Hence the incorrect expression #(35) of $P(X|Z_1 \cap \ldots \cap Z_s)$ in [1] must be replaced by the formula (10) above, and the incorrect expression #(36) of $K'(Z_1, \ldots, Z_s)$ must be replaced by the formula (11).

The agreement $A_s(X)$ of order s, the global agreement GA_s, and the global conflict GC_s for s sources must be also corrected as follows:

$$A_s(X = x_i) \triangleq \prod_{k=1}^{s} \frac{P(X = x_i|Z_k)}{\sqrt[s]{P^{s-1}(X = x_i)}}$$

$$GA_s \triangleq \sum_{i_1,\ldots,i_s=1|i_1=\ldots=i_s}^{N} \frac{P(X = x_{i_1}|Z_1)}{\sqrt[s]{P^{s-1}(X = x_{i_1})}} \cdots \frac{P(X = x_{i_s}|Z_s)}{\sqrt[s]{P^{s-1}(X = x_{i_s})}}$$

$$GC_s \triangleq \sum_{i_1,\ldots,i_s=1}^{N} \frac{P(X = x_{i_1}|Z_1)}{\sqrt[s]{P^{s-1}(X = x_{i_1})}} \cdots \frac{P(X = x_{i_s}|Z_s)}{\sqrt[s]{P^{s-1}(X = x_{i_s})}} - GA_s$$

The first consequence of this correction is that the property P1 stated in [1] page 242 must be corrected as *(P1): The PMF P(X) is a neutral element of Bayes fusion rule*. Remark 2 and formula #(45) on page 242 must be removed.

The remark 3 on page 242 of [1] is incorrect. Indeed, if we take $P(X|Z_k) = P(X)$ for $k = 1, \ldots, s$ and based on the correct formula (10), we get actually

$$Bayes(P(X), P(X), ..., P(X); P(X)) = P(X)$$

and for any type of pmf $P(X)$ (i.e. uniform, and non-uniform pmf).

The property *(P3) : The Bayes fusion rule is in general not associative* stated in [1] on page 242 is incorrect and it must be corrected as *(P3) : The Bayes fusion rule is associative.*

Proof of the property P3 (Associativity of Bayes rule): The expression of $P(X|Z_1 \cap \ldots \cap Z_{s-1})$ is given by formula (10) when using $s-1$ conditioning terms. Hence we have

$$P(X|Z_1 \cap \ldots \cap Z_{s-1}) = \frac{1}{K'(Z_1, \ldots, Z_{s-1})} \cdot \prod_{k=1}^{s-1} \frac{P(X|Z_k)}{P^{\frac{s-2}{s-1}}(X)} \tag{12}$$

with the normalization constant $K'(Z_1, \ldots, Z_{s-1})$ given by

$$K'(Z_1, \ldots, Z_{s-1}) = \sum_{i=1}^{N} \prod_{k=1}^{s-1} \frac{P(X = x_i|Z_k)}{P^{\frac{s-2}{s-1}}(X = x_i)} \tag{13}$$

To calculate $P(X|(Z_1 \cap \ldots \cap Z_{s-1}) \cap Z_s)$ from $P(X|Z_1 \cap \ldots \cap Z_{s-1})$ and $P(X|Z_s)$, we use Bayes formula with the conditional statistical independence assumption, and we get

$$P(X|(Z_1 \cap \ldots \cap Z_{s-1}) \cap Z_s) = \frac{P(Z_1 \cap \ldots \cap Z_{s-1}|X)P(Z_s|X)P(X)}{\sum_{i=1}^{N} P(Z_1 \cap \ldots \cap Z_{s-1}|X = x_i)P(Z_s|X = x_i)P(X = x_i)} \tag{14}$$

Because

$$P(Z_1 \cap \ldots \cap Z_{s-1}|X) = \frac{P(X|Z_1 \cap \ldots \cap Z_{s-1})P(Z_1 \cap \ldots \cap Z_{s-1})}{P(X)}$$

and

$$P(Z_s|X) = \frac{P(X|Z_s)P(Z_s)}{P(X)}$$

The expression of $P(X|(Z_1 \cap \ldots \cap Z_{s-1}) \cap Z_s)$ given by (14) can be rewritten as

$$P(X|(Z_1 \cap \ldots \cap Z_{s-1}) \cap Z_s) = \frac{\frac{P(X|Z_1 \cap \ldots \cap Z_{s-1})P(Z_1 \cap \ldots \cap Z_{s-1})}{P(X)} \frac{P(X|Z_s)P(Z_s)}{P(X)} P(X)}{\sum_{i=1}^{N} \frac{P(X=x_i|Z_1 \cap \ldots \cap Z_{s-1})P(Z_1 \cap \ldots \cap Z_{s-1})}{P(X=x_i)} \frac{P(X=x_i|Z_s)P(Z_s)}{P(X=x_i)} P(X = x_i)} \tag{15}$$

After simplification by $P(Z_1 \cap \ldots \cap Z_{s-1})P(Z_s)$ it comes

$$P(X|(Z_1 \cap \ldots \cap Z_{s-1}) \cap Z_s) = \frac{\frac{P(X|Z_1 \cap \ldots \cap Z_{s-1})}{P(X)} \frac{P(X|Z_s)}{P(X)} P(X)}{\sum_{i=1}^{N} \frac{P(X=x_i|Z_1 \cap \ldots \cap Z_{s-1})}{P(X=x_i)} \frac{P(X=x_i|Z_s)}{P(X=x_i)} P(X = x_i)} \tag{16}$$

After the simplification by $P(X)$ in the numerator of (16) and the simplification by $P(X = x_i)$ in the denominator of (16) it comes

$$P(X|(Z_1 \cap \ldots \cap Z_{s-1}) \cap Z_s) = \frac{P(X|Z_1 \cap \ldots \cap Z_{s-1}) \frac{P(X|Z_s)}{P(X)}}{\sum_{i=1}^{N} P(X = x_i|Z_1 \cap \ldots \cap Z_{s-1}) \frac{P(X=x_i|Z_s)}{P(X=x_i)}} \tag{17}$$

Replacing $P(X|Z_1 \cap \ldots \cap Z_{s-1})$ by its expression given in (12), we have

$$P(X|(Z_1 \cap \ldots \cap Z_{s-1}) \cap Z_s) = \frac{\left[\frac{1}{K'(Z_1,\ldots,Z_{s-1})} \cdot \prod_{k=1}^{s-1} \frac{P(X|Z_k)}{P^{\frac{s-2}{s-1}}(X)}\right] \frac{P(X|Z_s)}{P(X)}}{\sum_{i=1}^{N} \left[\frac{1}{K'(Z_1,\ldots,Z_{s-1})} \cdot \prod_{k=1}^{s-1} \frac{P(X=x_i|Z_k)}{P^{\frac{s-2}{s-1}}(X=x_i)}\right] \frac{P(X=x_i|Z_s)}{P(X=x_i)}} \tag{18}$$

After simplification by the constant $K'(Z_1, \ldots, Z_{s-1})$ one gets

$$P(X|(Z_1 \cap \ldots \cap Z_{s-1}) \cap Z_s) = \frac{\left[\prod_{k=1}^{s-1} \frac{P(X|Z_k)}{P^{\frac{s-2}{s-1}}(X)}\right] \frac{P(X|Z_s)}{P(X)}}{\sum_{i=1}^{N} \left[\prod_{k=1}^{s-1} \frac{P(X=x_i|Z_k)}{P^{\frac{s-2}{s-1}}(X=x_i)}\right] \frac{P(X=x_i|Z_s)}{P(X=x_i)}} = \frac{\frac{1}{P^{s-1}(X)} \prod_{k=1}^{s} P(X|Z_k)}{\sum_{i=1}^{N} \frac{1}{P^{s-1}(X=x_i)} \prod_{k=1}^{s} P(X = x_i|Z_k)} \tag{19}$$

The formula (19) can be rewritten with an equivalent symmetrical form as

$$P(X|(Z_1 \cap \ldots \cap Z_{s-1}) \cap Z_s) = \frac{\prod_{k=1}^{s} \frac{P(X|Z_k)}{P^{\frac{s-1}{s}}(X)}}{\sum_{i=1}^{N} \prod_{k=1}^{s} \frac{P(X=x_i|Z_k)}{P^{\frac{s-1}{s}}(X=x_i)}} = \frac{1}{K'(Z_1, \ldots, Z_s)} \cdot \prod_{k=1}^{s} \frac{P(X|Z_k)}{P^{\frac{s-1}{s}}(X)} \tag{20}$$

where $K'(Z_1, \ldots, Z_s) = \sum_{i=1}^{N} \prod_{k=1}^{s} \frac{P(X=x_i|Z_k)}{P^{\frac{s-1}{s}}(X=x_i)}$.

Therefore, we have proved that expression of $P(X|(Z_1 \cap \ldots \cap Z_{s-1}) \cap Z_s)$ given by (20) is equal to the expression of $P(X|Z_1 \cap \ldots \cap Z_{s-1} \cap Z_s)$ given by (10). This proves the associativity of Bayes fusion rule, i.e. the validity of the property P3. Note that the equality $P(X|(Z_1 \cap \ldots \cap Z_{s-1}) \cap Z_s) = P(X|Z_1 \cap \ldots \cap Z_{s-1} \cap Z_s)$ does not depend on a particular choice of the intersection of $s - 1$ subsets involved in the conditioning because the intersection operator is associative. Hence the conditioning terms $(Z_1 \cap \ldots \cap Z_{s-1}) \cap Z_s$ and $Z_1 \cap \ldots \cap Z_{s-1} \cap Z_s$ are equal. This implies that the two conditional probabilities must be necessary equal, which is proved by our previous derivations.

With the correct formulas (10)-(11), the numerical application for example 1 on page 243 of [1] gives

$$\begin{cases} P(X = x_1|Z_1 \cap Z_2 \cap Z_3) = \frac{1}{K_{123}} \frac{0.1}{\sqrt[3]{0.2^2}} \frac{0.5}{\sqrt[3]{0.2^2}} \frac{0.6}{\sqrt[3]{0.2^2}} = 0.7273 \\ P(X = x_2|Z_1 \cap Z_2 \cap Z_3) = \frac{1}{K_{123}} \frac{0.9}{\sqrt[3]{0.8^2}} \frac{0.5}{\sqrt[3]{0.8^2}} \frac{0.4^2}{\sqrt[3]{0.8^2}} = 0.2727 \end{cases}$$

where the normalization constant $K_{123} = K'(Z_1, Z_2, Z_3)$ is given by (11) for $s = 3$, i.e.

$$K_{123} = \frac{0.1}{\sqrt[3]{0.2^2}}\frac{0.5}{\sqrt[3]{0.2^2}}\frac{0.6}{\sqrt[3]{0.2^2}} + \frac{0.9}{\sqrt[3]{0.8^2}}\frac{0.5}{\sqrt[3]{0.8^2}}\frac{0.4}{\sqrt[3]{0.8^2}} = 1.0312$$

This corrected result shows that Bayes fusion rule is actually associative because one has

$$\begin{cases} P(X|(Z_1 \cap Z_2) \cap Z_3) = P(X|Z_1 \cap Z_2 \cap Z_3) \\ P(X|Z_1 \cap (Z_2 \cap Z_3)) = P(X|Z_1 \cap Z_2 \cap Z_3) \\ P(X|Z_2 \cap (Z_1 \cap Z_3)) = P(X|Z_1 \cap Z_2 \cap Z_3) \end{cases}$$

As consequence, the property *(P4)* on page 245 of [1], although being correct, is not necessary.

On page 250 of [1], the sentence:
Indeed, in Bayes rule one divides each posterior source $m_i(x_j)$ *by* $\sqrt[s]{m_0(x_j)}$, $i = 1, 2, \dots s$, *whereas the prior source* $m_0(.)$ *is combined in a pure conjunctive manner by DS rule with the bba's* $m_i(.)$, $i = 1, 2, \dots s$, *as if* $m_0(.)$ *was a simple additional source.*
must be corrected as:
Indeed, in Bayes rule one divides each posterior source $m_i(x_j)$ *by* $\sqrt[s]{m_0^{s-1}(x_j)}$, $i = 1, 2, \dots s$, *whereas the prior source* $m_0(.)$ *is combined in a pure conjunctive manner by DS rule with the bba's* $m_i(.)$, $i = 1, 2, \dots s$, *as if* $m_0(.)$ *was a simple additional source.*

This erratum concerns also some incorrect formulas appearing in a preliminary version of [1] presented in 2013, see [2].

REFERENCES

[1] J. Dezert, A. Tchamova, *On the validity of Dempster's fusion rule and its interpretation as a generalization of Bayesian fusion rule*, International Journal of Intelligent Systems, Special Issue: Advances in Intelligent Systems, Vol. 29, Issue 3, pp. 223–252, March 2014.

[2] J. Dezert, A. Tchamova, D. Han, J.-M. Tacnet, *Why Dempster's fusion rule is not a generalization of Bayes fusion rule*, Proc. of Fusion 2013 Int. Conference on Information Fusion, Istanbul, Turkey, July 9-12, 2013.