Examples where Dempster's rule is insensitive to the conflict level between the sources of evidence

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***Abstract*—In this short note, we present two classes of examples showing that Dempster's rule of combination is insensitive to the conflict level between the sources of evidence. This behavior is intuitively not satisfying because the amount of dissonance between sources should have an impact in the fusion result when the basic belief assignments (BBA) to combine are truly informative (not vacuous).**

Keywords: Dempster's rule, Information fusion, belief functions.

I. Introduction

In this short note, we discuss the behavior of Dempster's rule of combination used in Dempster-Shafer Theory (DST) [1], [2] to combine basic belief assignments provided by distinct sources of evidences. After a brief introduction of belief functions in Section II and a recall of Dempster's rule of combination in section III, we provide in section IV two classes of examples showing the counter-intuitive behavior of Dempster's rule. These new classes of examples generalize examples presented in [3]. The conclusion is made in section V.

II. Belief functions in short

Belief functions have been introduced by Shafer in [1] to model epistemic uncertainty. We assume that the answer[1] of the problem under concern belongs to a known (or given) finite discrete frame of discernement (FoD) $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$, with $n > 1$, and where all elements of Θ are exclusive[2]. The set of all subsets of Θ (including empty set $\emptyset$ and Θ) is the power-set of Θ denoted by 2^Θ. A basic belief assignment (BBA) associated with a given source of evidence is defined [1] as the mapping $m(\cdot) : 2^\Theta \to [0,1]$ satisfying $m(\emptyset) = 0$ and $\sum_{A\in 2^\Theta} m(A) = 1$. The quantity $m(A)$ is called the mass of A committed by the source of evidence. Belief and plausibility functions are respectively defined by

$$Bel(A) = \sum_{\substack{B\subseteq A\\ B\in 2^\Theta}} m(B), \quad \text{and} \quad Pl(A) = 1 - \text{Bel}(\bar{A}). \quad (1)$$

If $m(A) > 0$, A is called a focal element of $m(\cdot)$. The set of focal elements of a BBA m is denoted $\mathcal{F}(m)$. When all focal elements are singletons then $m(\cdot)$ is called a *Bayesian BBA* [1] and its corresponding $Bel(\cdot)$ function is homogeneous to a (subjective) probability measure. The vacuous BBA, or VBBA for short, representing a totally ignorant source is defined as[3] $m(\Theta) = 1$.

Shafer [1] proposed to combine $s \geq 2$ distinct sources of evidence represented by BBAs $m_1(.), \dots, m_s(.)$ over the same FoD with Dempster's rule. The justification and behavior of Dempster's rule have been disputed over the years from many counter-examples involving high or low conflicting sources (from both theoretical and practical standpoints) as reported in [4]–[7]. After a brief recall of Dempster's rule of combination in section II, we present new interesting examples showing the counter-intuitive behavior of this rule in section IV.

III. Dempster's rule of combination

Dempster's rule of combination can be seen as a normalized version of the conjunctive rule. So, let's recall at first what is the conjunctive rule (CR) of combination. Mathematically, CR of $s \geq 2$ BBAs $m_i(\cdot)$, $i = 1, \dots, s$ defined with respect to same FoD Θ is defined for any $X \in 2^\Theta$ by

$$m^{CR}_{12\dots s}(X) \triangleq \sum_{\substack{X_1,\dots,X_s\in 2^\Theta\\ X_1\cap X_2\cap\dots\cap X_s = X}} \prod_{i=1}^{s} m_i(X_i). \quad (2)$$

[1] i.e. the solution, or the decision to take.

[2] This is so-called Shafer's model of FoD [2].

[3] The complete ignorance is denoted Θ in Shafer's book [1].

The conjunction (intersection) of two (or more) sources of evidence only keeps the items of information asserted by both (all) sources. This rule has been justified by Dempster [8] in statistical terms on the basis of the independence of the sources which provide m_i with $\mathcal{F}(m_i)$, $i = 1, \ldots, s$. The set of focal elements of $m^{CR}_{12\ldots s}(\cdot)$ is given by

$$\mathcal{F}(m^{CR}_{12\ldots s}) = \{X_1 \cap \ldots \cap X_s | X_i \in \mathcal{F}(m_i), i = 1, \ldots, s\}.$$

The term $m^{CR}_{12\ldots s}(\emptyset)$ reflects the amount of dissonance between the sources [9] (also called the level or degree of conflict between the sources of evidence). Its management gives rise to many debates on the choice of possible rules to combine distinct and reliable sources of evidence. In DST, Shafer proposed Dempster's rule[4] in which the positive value $m^{CR}_{12\ldots s}(\emptyset)$ (if any) committed to the empty set (impossible event) is removed through a simple normalization technique. Mathematically Dempster's rule of combination of $s \geq 2$ basic belief assignments is defined by $m^{DS}_{12\ldots s}(\emptyset) = 0$, and for any $X \neq \emptyset \in 2^\Theta$

$$m^{DS}_{12\ldots s}(X) = [m_1 \oplus \ldots \oplus m_s](X) \triangleq \frac{m^{CR}_{12\ldots s}(X)}{1 - m^{CR}_{12\ldots s}(\emptyset)} \quad (3)$$

Dempster's rule is commutative and associative and preserves the neutrality of vacuous BBA in the fusion process, which makes Dempster's rule an appealing method to fuse BBAs from implementation standpoint, even if the validity of its result has been highly disputed since its first criticism made by Zadeh in [10] over last decades in case of high conflicting situations, and more recently in [4]–[7] for the case of low conflicting situations.

In the next section we present two classes of examples where Dempster's rule is insensitive to the conflict level.

IV. New classes of examples

A. First class of examples

Let's consider a finite frame of discernment Θ and two BBAs $m_1(\cdot)$ and $m_2(\cdot)$ with focal elements in 2^Θ given by

$$\mathcal{F}(m_1) = \{A_1, A_2, \ldots, A_n\}$$

$$\mathcal{F}(m_2) = \{A, B_1, \ldots, B_m\}$$

where $A_i \subseteq A$ for $1 \leq i \leq n$, and $A_i \cap B_j = \emptyset$ for $1 \leq i \leq n$ and $1 \leq j \leq m$.

[4]This rule has been introduced by Dempster in [8]. It has been denoted and popularized with the operator symbol $\oplus$ by Shafer in [1].

The mass of each focal element is denoted by its corresponding lowercase letter, that is $m_1(A_i) = a_i$ for $1 \leq i \leq n$, and $m_2(A) = a$ and $m_2(B_j) = b_j$ for $1 \leq j \leq m$. Because $m_1(\cdot)$ and $m_2(\cdot)$ are normalized BBAs, one has $\sum_{i=1}^n a_i = 1$ and $a + \sum_{j=1}^m b_j = 1$.

In applying the conjunctive rule of combination of $m_1(\cdot)$ with $m_2(\cdot)$, we get

$$m^{CR}_{12}(A_i) = a \cdot a_i \quad \text{for } 1 \leq i \leq n$$

and

$$m^{CR}_{12}(\emptyset) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j$$

Obviously $m^{CR}_{12}(\emptyset) + \sum_{i=1}^n m^{CR}_{12}(A_i) = 1$, which means that the following equality holds

$$m^{CR}_{12}(\emptyset) = 1 - \sum_{i=1}^{n} a \cdot a_i = 1 - a \sum_{i=1}^{n} a_i = 1 - a \quad (4)$$

because $\sum_{i=1}^n a_i = 1$.

To get Dempster's rule result, we need to normalize the BBA $m^{CR}_{12}(\cdot)$ by dividing the masses $m^{CR}_{12}(A_i)$ by $1 - m^{CR}_{12}(\emptyset)$, or equivalently just by dividing $m^{CR}_{12}(A_i)$ by the value a because from (4) one always has $1 - m^{CR}_{12}(\emptyset) = 1 - (1 - a) = a$.

After the normalization by division of masses $m^{CR}_{12}(A_i)$ by a, one gets as Dempster-Shafer fusion result

$$m^{DS}_{12}(A_i) = [m_1 \oplus m_2](A_i) = m_1(A_i) = a_i \quad (5)$$

Therefore, it is clear in such class of examples that the BBA $m_2(.)$ has absolutely no impact in Dempster-Shafer fusion result even if $m_2(\cdot)$ is truly informative (not vacuous) and conflicting with the BBA $m_1(\cdot)$.

The conflict level $m^{CR}_{12}(\emptyset) = \sum_{i=1}^n \sum_{j=1}^m a_i b_j$ can be as high (close to one) or as low (close to zero) as we want, Dempster's rule provides in this class of examples always the same result $m^{DS}_{12}(\cdot) = m_1(\cdot)$, which is a counter-intuitive behavior not very recommended for fusion applications.

B. Second class of examples

This second class of example is a bit more general than the previous one. We consider a finite frame of discernment Θ and two BBAs $m_1(\cdot)$ and $m_2(\cdot)$ with focal elements in 2^Θ given by

$$\mathcal{F}(m_1) = \{A_1, A_2, \ldots, A_n, B\}$$

$$\mathcal{F}(m_2) = \{B, C_1, \ldots, C_m\}$$

such that

- $A_i \subseteq B$ for $1 \leq i \leq n$,
- $B \cap C_j = \emptyset$ for $1 \leq j \leq m$, and
- $A_i \cap C_j = \emptyset$ for $1 \leq i \leq n$ and $1 \leq j \leq m$.

The mass of each focal element is denoted by its corresponding lowercase letter for all elements A_i, that is $m_1(A_i) = a_i$ for $1 \leq i \leq n$ and by $m_1(B) = b_1$. Similarly $m_2(C_j) = c_j$ for $1 \leq j \leq m$ and $m_2(B) = b_2$. Because $m_1(\cdot)$ and $m_2(\cdot)$ are normalized BBAs, one has $b_1 + \sum_{i=1}^{n} a_i = 1$ and $b_2 + \sum_{j=1}^{m} c_j = 1$.

In applying the conjunctive rule of combination of $m_1(\cdot)$ with $m_2(\cdot)$, we get

$$\begin{aligned} m_{12}^{CR}(A_i) &= b_2 a_i \quad \text{for } 1 \leq i \leq n \\ m_{12}^{CR}(B) &= b_1 b_2 \\ m_{12}^{CR}(\emptyset) &= \sum_{i=1}^{n}\sum_{j=1}^{m} a_i c_j + \sum_{j=1}^{m} b_1 c_j \end{aligned}$$

Obviously $m_{12}^{CR}(\emptyset) + \sum_{i=1}^{n} m_{12}^{CR}(A_i) + m_{12}^{CR}(B) = 1$, which means that the following equality holds

$$\begin{aligned} m_{12}^{CR}(\emptyset) &= 1 - \sum_{i=1}^{n} m_{12}^{CR}(A_i) - m_{12}^{CR}(B) \\ &= 1 - \sum_{i=1}^{n} b_2 \cdot a_i - b_1 b_2 \\ &= 1 - b_2(b_1 + \sum_{i=1}^{n} a_i) \\ &= 1 - b_2 \end{aligned}$$

because $b_1 + \sum_{i=1}^{n} a_i = 1$.

To get Dempster's rule result, we need to normalize the BBA $m_{12}^{CR}(\cdot)$ by dividing the masses $m_{12}^{CR}(A_i)$ and $m_{12}^{CR}(B)$ by $1 - m_{12}^{CR}(\emptyset) = 1 - (1 - b_2) = b_2$.

After the normalization by division of masses $m_{12}^{CR}(A_i)$ and $m_{12}^{CR}(B)$ by $b_2 \neq 0$, one gets the Dempster-Shafer fusion result for $i = 1, \ldots, n$

$$m_{12}^{DS}(A_i) = [m_1 \oplus m_2](A_i) = m_1(A_i) = a_i \qquad (6)$$

and

$$m_{12}^{DS}(B) = [m_1 \oplus m_2](B) = m_1(B) = b_1 \qquad (7)$$

Therefore, it is clear in such second class of examples that the BBA $m_2(.)$ has also absolutely no impact in Dempster-Shafer fusion result even if $m_2(\cdot)$ is truly informative (not vacuous) and conflicting with the BBA $m_1(\cdot)$.

The conflict level $m_{12}^{CR}(\emptyset) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i c_j + \sum_{j=1}^{m} b_1 c_j = 1 - b_2$ can be as high (close to one) or as low (close to zero) as we want. Dempster's rule provides in this second class of examples always the same result $m_{12}^{DS}(\cdot) = m_1(\cdot)$, which is a counter-intuitive behavior not recommended for fusion applications.

V. Conclusions

We have given two classes of counter-examples to Dempster's Rule, where this rule is insensitive to the fusion, in the sense that combining two different conflicting sources of information characterized by the basic belief assignments $m_1(\cdot)$ and $m_2(\cdot)$, the fusion result is equal to $m_1(\cdot)$. Therefore $m_2(\cdot)$ has no impact in the fusion, although $m_2(\cdot)$ is different from the uninformative source characterized by the vacuous basic belief assignment $m(\Theta) = 1$.

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