# Computational Complexity of JPDA: Worst Case Analysis 

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#### Abstract

This technical note discusses the complexity of the JPDA (Joint Probabilistic Data Association) used for multiple target tracking. More precisely, we are interested in the calculation of the number of joint data association matrices to be generated, that are necessary for the computation of all possible event probabilities involved in a JPDA tracking filter.


## Index Terms

JPDA, Computational complexity of JPDA.

The JPDA filter has been developed in 1980's (see, e.g. [1]) together with a procedure to generate the joint association events between several targets and the available measurements. A systematic fast procedure to generate the association events has been presented in [2], with a procedure for JPDA target clustering in [3].

Consider $N_{t}$ targets under tracking, and $N_{m}$ validated measurements available at a given time $k$. We assume that there is at most one measurement per target and some measurements may correspond to FA (false alarms). We are interested in the derivation of the number of feasible joint data associations in the worst case, i.e., all measurements are assumed validated for all targets.

For instance, consider three $\left(N_{m}=3\right)$ measurements validated for two $\left(N_{t}=2\right)$ targets. The validation matrix [1] is

$$
\Omega=\left[\begin{array}{l}
111  \tag{1}\\
111 \\
111
\end{array}\right]
$$

From this validation matrix and based on the aforementioned procedure we get thirteen possible joint association events characterized by their joint association matrices listed below:

$$
\begin{array}{ll}
\Omega_{1} & =\left[\begin{array}{l}
100 \\
100 \\
100
\end{array}\right] \quad \Omega_{2}=\left[\begin{array}{l}
010 \\
100 \\
100
\end{array}\right] \quad \Omega_{3}=\left[\begin{array}{l}
010 \\
001 \\
100
\end{array}\right] \\
\Omega_{4}=\left[\begin{array}{l}
010 \\
100 \\
001
\end{array}\right] \quad \Omega_{5}=\left[\begin{array}{l}
001 \\
100 \\
100
\end{array}\right] \quad \Omega_{6}=\left[\begin{array}{l}
001 \\
010 \\
100
\end{array}\right] \\
\Omega_{7}=\left[\begin{array}{l}
001 \\
100 \\
010
\end{array}\right] \quad \Omega_{8}=\left[\begin{array}{l}
100 \\
010 \\
100
\end{array}\right] \quad \Omega_{9}=\left[\begin{array}{l}
100 \\
010 \\
001
\end{array}\right]  \tag{2}\\
\Omega_{10}=\left[\begin{array}{l}
100 \\
001 \\
100
\end{array}\right] \quad \Omega_{11}=\left[\begin{array}{l}
100 \\
001 \\
010
\end{array}\right] \quad \Omega_{12}=\left[\begin{array}{l}
100 \\
100 \\
010
\end{array}\right] \\
\Omega_{13}=\left[\begin{array}{l}
100 \\
100 \\
001
\end{array}\right] &
\end{array}
$$

The elements of these matrices are indicators, where 1 means there is an association between a measurement and a source (a target, or FA), and 0 otherwise. By convention [1], the first column of a validation matrix $\Omega$ and feasible joint association

[^0]matrice $\Omega_{i}(i=1, \ldots, 13)$ corresponds to the FA (false alarm) origin. The second column corresponds to the first target $\left(T_{1}\right)$ origin, and the third column corresponds to the second target $\left(T_{2}\right)$ origin. The first row corresponds to the first measurement $\left(\mathbf{z}_{1}\right)$, the second row to the second measurement $\left(\mathbf{z}_{2}\right)$, and the third row to the third measurement $\left(\mathbf{z}_{3}\right)$. The generation of all feasible joint association matrices can be done by the depth-first search (DFS) procedure presented in [2]. We are interested in the direct calculation of the number $N\left(N_{m}, N_{t}\right)$ of feasible joint association events in the worst case, denoted by $N^{\text {worst }}\left(N_{m}, N_{t}\right)$.

In this case, each measurement may be associated with either the clutter (i.e. a FA) or any one of the $n_{d} \leq N_{t}$ detected targets, and there are

$$
\begin{equation*}
C_{n_{d}}^{N_{t}}=\frac{N_{t}!}{\left(N_{t}-n_{d}\right)!n_{d}!} \tag{3}
\end{equation*}
$$

possible choices (combinations) of $n_{d}$ detections among $N_{t}$ targets, where $0 \leq n_{d} \leq N_{t}$. Also, there are

$$
\begin{equation*}
A_{n_{d}}^{N_{m}}=\frac{N_{m}!}{\left(N_{m}-n_{d}\right)!} \tag{4}
\end{equation*}
$$

possible permutations (arrangements) of $n_{d}$ detected targets with $N_{m}$ measurements when $N_{m} \geq n_{d}$. One cannot have $n_{d}>N_{m}$. The number of associations is limited by the smallest between the number of measurements, $N_{m}$, and the number of targets, $N_{t}$. Based on this, the number $N^{\text {worst }}\left(N_{m}, N_{t}\right)$ of joint data associations in the worst case is given by

$$
\begin{equation*}
N^{\mathrm{worst}}\left(N_{m}, N_{t}\right)=\sum_{n_{d}=0}^{\min \left(N_{m}, N_{t}\right)} N_{d}\left(n_{d}, N_{t}\right) N_{a}\left(n_{d}, N_{m}\right) \tag{5}
\end{equation*}
$$

where $N_{d}\left(n_{d}, N_{t}\right)$ is the number of combinations of $n_{d}$ detected targets out of $N_{t}$ given by

$$
\begin{equation*}
N_{d}\left(n_{d}, N_{t}\right)=C_{n_{d}}^{N_{t}} \tag{6}
\end{equation*}
$$

and where $N_{a}\left(n_{d}, N_{m}\right)$ is the number of arrangements (permutations) of $n_{d}$ detected targets with $N_{m}$ measurements given by

$$
\begin{equation*}
N_{a}\left(n_{d}, N_{m}\right)=A_{n_{d}}^{N_{m}} \tag{7}
\end{equation*}
$$

We give in Table I the values of $N^{\text {worst }}\left(N_{m}, N_{t}\right)$ for $N_{m}=1,2, \ldots, 20$ and $N_{t}=2,3, \ldots, 7$.

| $N^{\text {worst }}\left(N_{m}, N_{t}\right)$ | 2 | 3 | 4 | 5 | 6 | 7 |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $N_{m}=1$ | 3 | 4 | 5 | 6 | 7 | 8 |
| $N_{m}=2$ | 7 | 13 | 21 | 31 | 43 | 57 |
| $N_{m}=3$ | 13 | 34 | 73 | 136 | 229 | 358 |
| $N_{m}=4$ | 21 | 73 | 209 | 501 | 1,045 | 1,961 |
| $N_{m}=5$ | 31 | 136 | 501 | 1,546 | 4,051 | 9,276 |
| $N_{m}=6$ | 43 | 229 | 1,045 | 4,051 | 13,327 | 37,633 |
| $N_{m}=7$ | 57 | 358 | 1,961 | 9,276 | 37,633 | 130,922 |
| $N_{m}=8$ | 73 | 529 | 3,393 | 19,081 | 93,289 | 394,353 |
| $N_{m}=9$ | 91 | 748 | 5,509 | 36,046 | 207,775 | $1,047,376$ |
| $N_{m}=10$ | 111 | 1,021 | 8,501 | 63,591 | 424,051 | $2,501,801$ |
| $N_{m}=11$ | 133 | 1,354 | 12,585 | 106,096 | 805,597 | $5,470,158$ |
| $N_{m}=12$ | 157 | 1,753 | 18,001 | 169,021 | $1,442,173$ | $11,109,337$ |
| $N_{m}=13$ | 183 | 2,224 | 25,013 | 259,026 | $2,456,299$ | $21,204,548$ |
| $N_{m}=14$ | 211 | 2,773 | 33,909 | 384,091 | $4,010,455$ | $38,398,641$ |
| $N_{m}=15$ | 241 | 3,406 | 45,001 | 553,636 | $6,315,001$ | $66,471,826$ |
| $N_{m}=16$ | 273 | 4,129 | 58,625 | 778,641 | $9,636,817$ | $110,676,833$ |
| $N_{m}=17$ | 307 | 4,948 | 75,141 | $1,071,766$ | $14,308,663$ | $178,134,552$ |
| $N_{m}=18$ | 343 | 5,869 | 94,933 | $1,447,471$ | $20,739,259$ | $278,295,193$ |
| $N_{m}=19$ | 381 | 6,898 | 118,409 | $1,922,136$ | $29,424,085$ | $423,470,006$ |
| $N_{m}=20$ | 421 | 8,041 | 146,001 | $2,514,181$ | $40,956,901$ | $629,438,601$ |

Table I
Number $N^{\text {worst }}\left(N_{m}, N_{t}\right)$ OF Joint data associations.

The derivation of $N^{\text {worst }}\left(N_{m}, N_{t}\right)$ done for the worst case above corresponds to the maximum number of feasible joint data associations assuming each measurement is validated for each target. When some measurements are not validated by all targets the number $N\left(N_{m}, N_{t}\right)$ of joint data associations will always be smaller than $N^{\text {worst }}\left(N_{m}, N_{t}\right)$.

The exact explicit formula of $N\left(N_{m}, N_{t}\right)$ seems very complicated to establish for the cases including some zeros in the validation matrix because it must take into account the structure of this matrix corresponding to the validations. The direct calculation of $N\left(N_{m}, N_{t}\right)$ remains a challenging open problem.

## REFERENCES

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