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Open Questions on Neutrosophic Inference

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This paper points out important and recent results developed by F. Smarandache in his new Neutrosophic Logic (NL) and provides a geometric interpretation of NL. The last part of this paper is devoted to the neutrosophic inference problem.

Keywords: Neutrosophy; Neutrosophic probability; Data fusion; Combination rule; Bayesian fusion; Dempster-shafer theory; Body of evidence

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1. INTRODUCTION

Recently Professor Florentin Smarandache has proposed the foundations for a new logic, called from now on Neutrosophic Logic (NL) or Smarandache's Logic, to unify all existing logics (see [128–130]) for details. The main idea of NL is to characterize each logical statement not in 1D or 2D spaces like in the existing logics but rather in 3D neutrosophic space represented by the neutrosophic cube as presented in Section 6. Each dimension of the space represents respectively the truth (T), falsehood (F) and indeterminacy (I) of the statement under consideration. Moreover, each statement is allowed to be over or under true, over or under false and over or under inderterminate by

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using hyper real numbers developed in non standard analysis theory. The neutrosophical value $\mathfrak{N}_{|W}(\text{A}) = (T(A), I(A), F(A))$ in a world of discourse W of a statement A is then defined as a subset (a volume not necessary connexe; *i.e.*, a set of disjoint volumes) of the neutrosophic space. This approach allows theoretically to consider any kinds of logical statements. For example, the fuzzy set logic or the classical modal logic (which works with statements verifying $T(\text{A})$, $I(\text{A}) \equiv 0$, $F(\text{A}) = 1 - T(\text{A})$, where T is a real number belonging to $[0; 1]$) are included in NL. The neutrosophic logic can easily handle also paradoxes and anti-paradoxes. Discussion about this topic will be brought in Section 5.

Even if the global framework of this new theory is now well defined, major open questions still remain and must be solved to give to this new very promising theory its full nobility and usefulness for solving practical problems arising in Data Fusion and Multi-Expert systems. Following the track of the development of Dempster-Shafer's rule of combination of evidence, which is considered as a generalization of classical Bayesian Inference, the NL must now leave its childhood to become more mature by answering the very difficult and important questions:

- How to construct neutrosophical belief function?
- How to construct neutrosophical basic mass assignment?
- How to recover neutrosophical basic assignment from neutrosophical belief function?
- How to construct a general Neutrosophical rule of combination?

We hope that this paper will help to clarify foundations of the NL, will bring some premisses for the answers to these questions and hence will contribute to the development and the promotion of the neutrosophic logic.

2. SHORT OVERVIEW ON NON STANDARD ANALYSIS

The theory of Non Standard Analysis (NSA) has been developed by A. Robinson in sixties [111] to handle rigourously illicite equations involving infinitesimal numbers in modern algebra [115]. A. Robinson has formally extended the set of real numbers by introducing a new set of numbers, called hyper-real set (or non-standard set), which contains infinitly small positive numbers $\varepsilon > 0$ around each point of the real line. Each of these infinitly small positive numbers have been called infinitesimal. The set of infinitesimals is denoted by $\mathbb{E}$. An infinitly small number is defined as a number smaller than any positive quantity. This is an abstract mathematical entity which can't be numerically evaluated. A real number x is said *limited* if there exists a positive real (standard) number y with $|x| < y$. A real number ε is said *infinitesimal* if and only if

$$\forall n \in \mathbb{N}^*, \quad |\varepsilon| < 1/n \tag{1}$$

The set of hyper-real numbers (*i.e.*, non-standard numbers) is an extension of the real number set $\mathbb{R}$, which includes both classes of infinite numbers and infinitesimal numbers. A non-standard finite number is defined as $a+\varepsilon$ or $a-\varepsilon$, where $a \in \mathbb{R}$ and $\varepsilon \in \mathbb{E}$. The precise value of ε is not given and ε must be considered as a symbol to represent any infinitesimal number. We will write symbolically

$$a^+ \triangleq a + \varepsilon \quad \text{and} \quad {}^-a \triangleq a - \varepsilon \tag{2}$$

a is the standard part of non-standard finite number a^+ or ${}^-a$ and ε corresponds to the non-standard part of the number. One has ${}^-a < a$ and $a^+ > a$. Originally, the construction of hyper-real numbers proposed by Robinson was based on first order logic and on the acceptation of Zermerlo's Choice Axiom (ZCA) [38]. This ZCA tell us that if we consider a family of non empty sets $\{A_1, \ldots, A_n\}$, there exists a function f which picks up exactly one element $a_i \in A_i$ in each set A_i, $i = 1, \ldots, n$. Even if one admits the ZCA, the construction of such (choice) function f is not determinable. This axiom is very important in the axiomatic of the foundations of the Zemerlo-Fraenkel (ZFT) set theory. We recall that all basic entities in ZFT are sets and the following correspondances hold $0 = \emptyset$, $1 = \{\emptyset\}$, $2 = \{\emptyset, \{\emptyset\}\} = \{0, 1\}, \ldots, n+1 = \{0, \ldots, n\}$ *etc.* In seventies, E. Nelson has developed in his milestone paper [99], a new set theory, called Internal Set Theory (IST) which extends the ZFT by introducing a new predicat denoted by $st(\cdot)$ (st stands for standard part of $(\cdot)$). st predicat indicates that a number x is standard if $x = st(x)$. The significance of st predicat which is governed by three axioms (Idealization, Standardization and Transfer) in IST, allows to distinguish infinitesimal from standard numbers. We will say that "x is non standard" (*i.e.*, charmed) whenever one has $x \neq st(x)$. By introducing such new predicat, Nelson puts in a prominent new element in traditional set of integer numbers $\mathbb{N}$. In the IST, the infinitesimals are only real numbers which do not satisfy *st* predicat. The development of IST yields a better understanding of infinite quantities considered as whole entities (rather than infinite succession of elements). Three important consequences follow from ZF/IST [116, 110].

- any infinite set includes a non standard component (from idealization axiom)
- if all elements of a set A are standard, then A is a finite set (corollary)
- there exists a finite set which include all standard entities of mathematical universe

According to E. Nelson, there is infinitesimal numbers because among the infinity of real numbers there are numbers which cannot be mathematically formally expressed (in numerical sense). All these non standard numbers cannot be assigned precisely. From this point of view assignable entities are closely related with finitude whereas unassignable entities are related with infinitude.

2.1. Operations with Non Standard Numbers

We remind now some useful definitions introduced in NSA and basic algebraic operations with non-standard numbers. More details can be found in [110].

- Monads: Monads are set of hyper-real numbers. In the following we will consider

$$\mu(^-a) \triangleq \{a-\varepsilon|\varepsilon\in\mathbb{E}\} \tag{3}$$

$$\mu(b^+) \triangleq \{b+\varepsilon|\varepsilon\in\mathbb{E}\} \tag{4}$$

- Binad: A binad $\mu(^-c^+)$ is defined as a collection of opened punctured neighbourhood of c [130]

$$\mu(^-c^+) \triangleq \{c-\varepsilon|\varepsilon\in\mathbb{E}\}\cup\{c+\varepsilon|\varepsilon\in\mathbb{E}\} \tag{5}$$

  $c\notin\mu(^-c^+)$. There is no order between c and $^-c^+$.

- Non-standard interval: A non standard open interval is an interval of the form $]^{-}a; b^{+}[$. The left and right border of a non-standard interval $]^{-}a; b^{+}[$ are vague because they correspond to (sub)sets $\mu(^{-}a)$ and $\mu(b^{+})$. Inferior and superior bounds of $]^{-}a; b^{+}[$ are respectively $-a$ and b^{+}. In other words,

$$^{-}a = \inf(]^{-}a; b^{+}[) \quad \text{and} \quad b^{+} = \sup(]^{-}a; b^{+}[)$$

The non-standard *unit* interval is $]0^{-};1^{+}[$. 0 and 1 belong to $]0^{-};1^{+}[$ and also all non-standard numbers infinitely small but less than 0 or greater than 1.

- Addition rules of non-standard finite numbers

$$\begin{array}{rl}
a +^{-} b = {}^{-}(a+b) & {}^{-}a + b = {}^{-}(a+b) \\
a^{+} + b = (a+b)^{+} & {}^{-}a^{+} + b = {}^{-}(a+b)^{+} \\
a + b^{+} = (a+b)^{+} & {}^{-}a +^{-} b = {}^{-}(a+b) \\
a^{+} +^{-} b = {}^{-}(a+b)^{+} & {}^{-}a^{+} +^{-} b = {}^{-}(a+b)^{+} \\
a +^{-} b^{+} = {}^{-}(a+b)^{+} & {}^{-}a + b^{+} = {}^{-}(a+b)^{+} \\
a^{+} + b^{+} = (a+b)^{+} & {}^{-}a^{+} + b^{+} = {}^{-}(a+b)^{+} \\
 & {}^{-}a +^{-} b^{+} = {}^{-}(a+b)^{+} \\
a^{+} +^{-} b^{+} = {}^{-}(a+b)^{+} & {}^{-}a^{+} +^{-} b^{+} = {}^{-}(a+b)^{+}
\end{array}$$

- Substraction rules of non-standard finite numbers

$$\begin{array}{rl}
a -^{-} b = {}^{-}(a-b) & {}^{-}a - b = {}^{-}(a-b) \\
a^{+} - b = (a-b)^{+} & {}^{-}a^{+} - b = {}^{-}(a-b)^{+} \\
a - b^{+} = (a-b)^{+} & {}^{-}a -^{-} b = {}^{-}(a-b) \\
a^{+} -^{-} b = {}^{-}(a-b)^{+} & {}^{-}a^{+} -^{-} b = {}^{-}(a-b)^{+} \\
a -^{-} b^{+} = {}^{-}(a-b)^{+} & {}^{-}a - b^{+} = {}^{-}(a-b)^{+} \\
a^{+} - b^{+} = (a-b)^{+} & {}^{-}a^{+} - b^{+} = {}^{-}(a-b)^{+} \\
 & {}^{-}a -^{-} b^{+} = {}^{-}(a-b)^{+} \\
a^{+} -^{-} b^{+} = {}^{-}(a-b)^{+} & {}^{-}a^{+} -^{-} b^{+} = {}^{-}(a-b)^{+}
\end{array}$$

- Multiplication rules of non-standard finite numbers

$$\begin{array}{rl}
a \cdot^{-} b = {}^{-}(a \cdot b) & {}^{-}a \cdot b = {}^{-}(a \cdot b) \\
a^{+} \cdot b = (a \cdot b)^{+} & {}^{-}a^{+} \cdot b = {}^{-}(a \cdot b)^{+}
\end{array}$$

$$a \cdot b^+ = (a \cdot b)^+ \quad {}^-a \cdot {}^- b = {}^-(a \cdot b)$$
$$a^+ \cdot {}^- b = {}^-(a \cdot b)^+ \quad {}^-a^+ \cdot {}^- b = {}^-(a \cdot b)^+$$
$$a \cdot {}^- b^+ = {}^-(a \cdot b)^+ \quad {}^-a \cdot b^+ = {}^-(a \cdot b)^+$$
$$a^+ \cdot b^+ = (a \cdot b)^+ \quad {}^-a^+ \cdot b^+ = {}^-(a \cdot b)^+$$
$${}^-a \cdot {}^- b^+ = {}^-(a \cdot b)^+$$
$$a^+ \cdot {}^- b^+ = {}^-(a \cdot b)^+ \quad {}^-a^+ \cdot {}^- b^+ = {}^-(a \cdot b)^+$$

- Division rules of a non standard finite number by a non null number

$$a/{}^-b = {}^-(a/b) \quad {}^-a/b = {}^-(a/b)$$
$$a^+/b = (a/b)^+ \quad {}^-a^+/b = {}^-(a/b)^+$$
$$a/b^+ = (a/b)^+ \quad {}^-a/{}^-b = {}^-(a/b)$$
$$a^+/{}^-b = {}^-(a/b)^+ \quad {}^-a^+/{}^-b = {}^-(a/b)^+$$
$$a/{}^-b^+ = {}^-(a/b)^+ \quad {}^-a/b^+ = {}^-(a/b)^+$$
$$a^+/b^+ = (a/b)^+ \quad {}^-a^+/b^+ = {}^-(a/b)^+$$
$${}^-a/{}^-b^+ = {}^-(a/b)^+$$
$$a^+/{}^-b^+ = {}^-(a/b)^+ \quad {}^-a^+/{}^-b^+ = {}^-(a/b)^+$$

- Power rules of a non standard finite number

$$a^{{}^-b} = {}^-(a^b) \quad ({}^-a)^b = {}^-(a^b)$$
$$(a^+)^b = (a^b)^+ \quad ({}^-a^+)^b = {}^-(a^b)^+$$
$$a^{b^+} = (a^b)^+ \quad ({}^-a)^{{}^-b} = {}^-(a^b)$$
$$(a^+)^{{}^-b} = {}^-(a^b)^+ \quad ({}^-a^+)^{{}^-b} = {}^-(a^b)^+$$
$$a^{{}^-b^+} = {}^-(a^b)^+ \quad ({}^-a)^{b^+} = {}^-(a^b)^+$$
$$(a^+)^{b^+} = (a^b)^+ \quad ({}^-a^+)^{b^+} = {}^-(a^b)^+$$
$$({}^-a)^{{}^-b^+} = {}^-(a^b)^+$$
$$(a^+)^{{}^-b^+} = {}^-(a^b)^+ \quad ({}^-a^+)^{{}^-b^+} = {}^-(a^b)^+$$

- Roots of non standard finite number

If $\sqrt[n]{a}$ exists (and is standard) then one defines the roots of ${}^-a$, a^+ and ${}^-a^+$ as follows

$$\sqrt[n]{{}^-a} = {}^-\sqrt[n]{a} \quad \sqrt[n]{a^+} = \sqrt[n]{a}^+ \quad \sqrt[n]{{}^-a^+} = {}^-\sqrt[n]{a}^+$$

3. SMARANDACHE'S SET OPERATIONS

Here are the four basic Smarandache's set operations defined in [128, 130] involved in the manipulation of neutrosophic events. So, lets consider S_1 and S_2 be two (unidimensional) standard or non-standard real subsets. The addition, substraction, multiplication and division (by a non null finite number) of these sets are defined as follows:

- Addition

$$S_1 \oplus S_2 = S_2 \oplus S_1 \triangleq \{x | x = s_1 + s_2, \forall s_1 \in S_1, \forall s_2 \in S_2\} \tag{6}$$

The inferior (Inf) and superior (Sup) values of $S_1 \oplus S_2$ are given by

$$\text{Inf}\,[S_1 \oplus S_2] = \text{Inf}[S_1] + \text{Inf}[S_2] \quad \text{and} \quad \text{Sup}[S_1 \oplus S_2] = \text{Sup}[S_1] + \text{Sup}[S_2]$$

Example 1 (S_1 and S_2 real standard):

$$\begin{cases} S_1 = [-1; 1] \cup \{-2, 2, 3\} \\ S_2 = [-2; 0] \cup \{-3, 1\} \end{cases} \Rightarrow S_1 \oplus S_2 = [-4; 3] \cup \{-5, 4\}$$

Example 2 (S_1 and S_2 non standard):

$$\begin{cases} S_1 =]^-(-1); 1^+[\cup \{-2, {}^-2^+, 3\} \\ S_2 =]^-(-2); {}^-0[\cup \{-3, 1^+\} \end{cases} \Rightarrow S_1 \oplus S_2 =]^-(-4); {}^-3^+[\cup \{-5, 4^+\}$$

Note that in this case, $3 \in S_1$ but surprisingly $3 \notin S_1 \oplus S_2$

- Substraction

$$S_1 \ominus S_2 = -(S_2 \ominus S_1) \triangleq \{x | x = s_1 - s_2, \forall s_1 \in S_1, \forall s_2 \in S_2\} \tag{7}$$

For *real positive* subsets, the Inf and Sup values of $S_1 \ominus S_2$ are given by

$$\text{Inf}[S_1 \ominus S_2] = \text{Inf}[S_1] - \text{Sup}[S_2] \quad \text{and} \quad \text{Sup}[S_1 \ominus S_2] = \text{Sup}[S_1] - \text{Inf}[S_2]$$

Example 1 (S_1 and S_2 real standard):

$$\begin{cases} S_1 = [-1; 1] \cup \{-2, 2, 3\} \\ S_2 = [-2; 0] \cup \{-3, 1\} \end{cases} \Rightarrow S_1 \ominus S_2 = [-2; 5] \cup \{-3, 6\}$$

Example 2 (S_1 and S_2 non standard):

$$\begin{cases} S_1 =]^{-}(-1); 1^{+}[\cup \{-2, {}^{-}2^{+}, 3\} \\ S_2 =]^{-}(-2); {}^{-}0[\cup \{-3, 1^{+}\} \end{cases} \Rightarrow S_1 \ominus S_2 =]^{-}(-2); {}^{-}5^{+}[\cup \{-3^{+}, 6\}$$

- Multiplication

$$S_1 \odot S_2 = S_2 \odot S_1 \triangleq \{x | x = s_1 \cdot s_2, \forall s_1 \in S_1, \forall s_2 \in S_2\} \tag{8}$$

For *real positive* subsets, one gets

$$\mathrm{Inf}[S_1 \odot S_2] = \mathrm{Inf}[S_1] \cdot \mathrm{Inf}[S_2] \quad \text{and}$$
$$\mathrm{Sup}[S_1 \odot S_2] = \mathrm{Sup}[S_1] \cdot \mathrm{Sup}[S_2]$$

Example 1 (S_1 and S_2 real standard):

$$\begin{cases} S_1 = [-1; 1] \cup \{-2, 2, 3\} \\ S_2 = [-2; 0] \cup \{-3, 1\} \end{cases} \Rightarrow S_1 \odot S_2 = [-6; 4] \cup \{-9, 6\}$$

Example 2 (S_1 and S_2 non standard):

$$\begin{cases} S_1 =]^{-}(-1); 1^{+}[\cup \{-2, {}^{-}2^{+}, 3\} \\ S_2 =]^{-}(-2); {}^{-}0[\cup \{-3, 1^{+}\} \end{cases} \Rightarrow S_1 \odot S_2 = [^{-}(-6); {}^{-}4] \cup \{-9, 6\}$$

- Division of a set by a non null standard number

Let $k \in \mathbb{R}^{*}$, then

$$S_1 \oslash k \triangleq \{x | x = s_1/k, \forall s_1 \in S_1\} \tag{9}$$

Example 1 (S_1 real standard and $k = 2$):

$$\text{if} \quad S_1 = [-1; 1] \cup \{-2, 2, 3\} \Rightarrow S_1 \oslash 2 = [-1/2; 1/2] \cup \{-1, 1, 3/2\}$$

Example 2 (S_1 non standard and $k = 2$):

$$\text{if} \quad S_1 =]^{-}(-1); 1^{+}[\cup \{-2, {}^{-}2^{+}, 3\} \Rightarrow S_1 \oslash 2$$
$$=]^{-}(-1/2); (1/2)^{+}[\cup \{-1, {}^{-}1^{+}, 3/2\}$$

4. NEUTROSOPHIC STATEMENT A

In the development of his new Neutrosophic Logic ($\mathfrak{N}$-Logic for short) [128–130], F. Smarandache introduces the notion of neutrosophic statement A which has the possibility to be at the same time T% true, F% false and I% indeterminate. This is an abuse of language since more precisely and more generally, T, F and I are not necessary real/classical percentage but rather standard or non-standard real subsets *included in non-standard unit interval* $]^{-}0;1^{+}[$. In general, T, I and F may be any real subsets: discrete or continuous, singletons, finite, or (either countably or uncountably) infinite; union or intersection of various subsets; *etc*. They may also overlap. The inferior and superior values of T, I and F are denoted

$$t_{\inf} \triangleq \inf(T) \;\; t_{\sup} \triangleq \sup(T) \quad i_{\inf} \triangleq \inf(I) \;\; i_{\sup} \triangleq \sup(I)$$
$$f_{\inf} \triangleq \inf(F) \;\; f_{\sup} \triangleq \sup(F)$$

Since T, I and F are included in $]^{-}0;1^{+}[$, one always has

$$t_{\inf} \geq {}^{-}0 \quad i_{\inf} \geq {}^{-}0 \quad f_{\inf} \geq {}^{-}0 \quad \text{and} \quad t_{\sup} \leq 1^{+} \quad i_{\sup} \leq 1^{+} \quad f_{\sup} \leq 1^{+}$$

Hence,

$$s_{\inf} \triangleq t_{\inf} + i_{\inf} + f_{\inf} \geq {}^{-}0 \quad \text{and} \quad s_{\sup} \triangleq t_{\sup} + i_{\sup} + f_{\sup} \leq 3^{+}$$

The *neutrosophic value* (or $\mathfrak{N}$-*value*) of a statement A in a given world W under consideration (see details in next section) is given by the knowledge of the three subsets T, I and F and we will be denoted by

$$\mathfrak{N}_{|W}(\mathrm{A}) = (\mathfrak{N}_t, \mathfrak{N}_i, \mathfrak{N}_f) \triangleq (T, I, F) \tag{10}$$

When there is no ambiguity about the knowledge of world W we are working in, indice $_{|W}$ in $\mathfrak{N}_{|W}$ notation will be removed for convenience notation. More rigourously, we have to write T(A), I(A) and F(A) to characterize the fact that subsets T, I and F refer only to the specific statement A. Any statement A represented by a triplet $\mathfrak{N}$(A) will be called a *neutrosophic event* or $\mathfrak{N}$-*event* in the sequel.

The subset $\mathfrak{N}_t \triangleq T(\mathrm{A})$ characterizes the truth part of statement A. $\mathfrak{N}_i \triangleq (\mathrm{A})$ and $\mathfrak{N}_f \triangleq F(\mathrm{A})$ represent the inderterminacy and the falsehood of A. This Smarandache's representation is closer to the human reasoning. It characterizes/catches the imprecision of knowledge or linguistic inexactitude received by various observers, uncertainty due incomplete knowledge of acquisition errors or stochasticity, and vagueness due to lack of clear contours or limits.

5. WORLD AND UNIVERSE

The notion of "world" usually depends highly on many parameters. In most of cases, assertions depend mainly on our own source of knowledge (*i.e.*, our own world W of knowledge representation) about the problem under consideration. Usually our own world is imprecise, incomplete, vague and even sometimes can become intentionally (or unintentionally) subjective or biaised. It can moreover change with time because of environmental conditions and context variations, but our world of knowledge, relative to our body of evidence [119], can also be refined sometimes by adding extra information coming from exogeneous sources of information. The abstract set of all possible worlds W is refered as universe U and a finite set of n distinct worlds (like a pool of experts for example) will be denoted W^n in the sequel.

We can distinguish several levels of truths, indeterminacies and falsehoods of a statement A depending on the world(s) under consideration. Most important levels which are directly supported by Neutrosophic logic will be listed below. Neutrosophic logic is actually the only one general theory which attempts to unify all previous existing logics in the same global theoretical framework. Even if more advances are necessary to get and reinforce a strong and solid ground setting for Neutrosophy, we already believe in this new Logic and expect many potential benefits from it to solve important problems arising by example in distributed Artificial Intelligence and Multi-Sensor Data Fusion research areas. A very good detailed discussion about $\mathfrak{N}$-Logic within the Logic history can be found in [130].

- Absolute truth of A If statement A is true in the universe U (that is in all possible worlds) and for all conjunctures, we will say that A is absolutely (or universally) true which is represented by setting $\mathfrak{N}_t(\mathrm{A})$ to 1^+.

$$\boxed{\mathfrak{N}_{t|U}(\mathrm{A}) = 1^+ \Leftrightarrow \mathrm{A}\,\text{is absolutely true}} \tag{11}$$

- Relative truth of A If a statement A is true in *only one* possible world W and for a specific conjunctures, we will say that A is relatively true which is represented by having $\mathfrak{N}_t(\mathrm{A})$ to 1.

$$\boxed{\mathfrak{N}_{t|W}(\mathrm{A}) = 1 \Leftrightarrow \mathrm{A}\,\text{is relatively true}} \tag{12}$$

- Quantumly (n-level relative) truth of A If we consider jointly, several (say n, with $n > 1$) different worlds W_i, $i = 1, \ldots, n$ (*i.e.*, experts), an assertion about a given proposition A can take at same time k different several neutrosophic values depending on each world W_i. Moreover, each elementary "world" can also be interpreted as a dynamical system because of its possible variations of its own knowledge with different varying parameters and past observations, *etc*. One defines the "n-level relative truth" of the statement A (we will say that A is quantumly truth) if the statement is true in *at least n distinct worlds*, and similarly "countably-" or "uncountably-level relative truth" as gradual degrees between "first-level relative truth" (1) and "absolute truth" (1^+) in the monad $\mu(1^+)$.

$$\boxed{\mathfrak{N}_{t|W^n}(\mathrm{A}) \in]1; 1^+[\Leftrightarrow \mathrm{A}\text{ is quantumly true}} \tag{13}$$

We just point out that this definition remains actually only conceptual and quite abstract since there is for now no effective construction algorithm to evaluate such neutrosophic value $\mathfrak{N}_{t|W^n}(\mathrm{A})$ from each individual neutrosophic values $\mathfrak{N}_{t|W_i}(\mathrm{A})$, $i = 1, \ldots, n$. This is actually the main challenge we are faced now. Some issues will be proposed and discussed in the sequel.

With analogue definitions, one gets by substituting "truth" with "falsehood" or "indeterminacy" in the above the following characterizations of a statement A

- Absolute indeterminacy of A

$$\mathfrak{N}_{i|U}(\text{A}) = 1^{+} \Leftrightarrow \text{A is absolutely indeterminate}$$

- Relative indeterminacy of A

$$\mathfrak{N}_{i|W}(\text{A}) = 1^{+} \Leftrightarrow \text{A is relatively indeterminate}$$

- Quantumly indeterminacy of A

$$\mathfrak{N}_{i|W^n}(\text{A}) = \in]1;1^{+}[\Leftrightarrow \text{A is quantumly indeterminate}$$

- Absolute falsehood of A

$$\mathfrak{N}_{f|U}(\text{A}) = 1^{+} \Leftrightarrow \text{A is absolutely false}$$

- Relative falsehood of A

$$\mathfrak{N}_{f|W}(\text{A}) = 1 \Leftrightarrow \text{A is relatively false}$$

- Quantumly falsehood of A

$$\mathfrak{N}_{f|W^n}(\text{A}) = \in]1;1^{+}[\Leftrightarrow \text{A is quantumly false}$$

On the other hand, one also has

$$\mathfrak{N}_{t|U}(\text{A}) = {}^{-}0 \Leftrightarrow \text{A is false in all possible worlds}$$
$$\mathfrak{N}_{t|W}(\text{A}) = 0 \Leftrightarrow \text{A is false in at least one world}$$
$$\mathfrak{N}_{i|U}(\text{A}) = {}^{-}0 \Leftrightarrow \text{A is indeterminate in no possible world}$$
$$\mathfrak{N}_{i|W}(\text{A}) = 0 \Leftrightarrow \text{A is not indeterminate in at least one world}$$
$$\mathfrak{N}_{f|U}(\text{A}) = {}^{-}0 \Leftrightarrow \text{A is true in all possible worlds}$$
$$\mathfrak{N}_{f|W}(\text{A}) = 0 \Leftrightarrow \text{A is true in at least one world}$$

The ${}^{-}0$ and 1^{+} monads leave room for degrees of super-truth (truth whose values are greater than 1), as well as for degrees of super-falsehood, and super-indeterminacy. All kinds of statements can be described based on neutrosophic approach. Here are some typical ones:

- Absolute/universal Tautology A statement A is called an *absolute* or *universal* tautology (like for example A = "B is B"), if and only if in all possible worlds of the universe U, one has

$$\mathfrak{N}_{|U}(\mathrm{A}) = (1^+, {}^-0, {}^-0)$$

- Tautology A statement A is a *relative* tautology (tautology for short) in a given world W of universe U if and only if

$$\mathfrak{N}_{|W}(\mathrm{A}) = (1, 0, 0)$$

- Absolute Contradiction A statement A is an absolute contradiction (like A = "B is not B"), if and only if in all possible worlds of the universe U, one has

$$\mathfrak{N}_{|U}(\mathrm{A}) = ({}^-0, {}^-0, 1^+)$$

- Relative Contradiction A is a relative contradiction (contradiction for short) if and only if

$$\mathfrak{N}_{|W}(\mathrm{A}) = (0, 0, 1)$$

- Paradox and Anti-Paradox A statement A is a (relative) paradox in W if and only if one has either

$$\mathfrak{N}_{|W}(\mathrm{A}) = (1, 1, 1) \quad \text{or} \quad \mathfrak{N}_{|W}(\mathrm{A}) = (0, 0, 0)$$

  To distinguish $\mathfrak{N}_{|U}(A) = (1, 1, 1)$ from $\mathfrak{N}_{|U}(\mathrm{A}) = (0, 0, 0)$, we suggest to call the last statement an *anti-paradox*. This must not be confused with (relative) tautology corresponding to $\mathfrak{N}_{|w}(\mathrm{A}) = (1, 0, 0)$. The absolute/universal paradox and anti-paradox are in same way characterized by

$$\mathfrak{N}_{|U}(\mathrm{A}) = (1^+, 1^+, 1^+) \quad \text{or} \quad \mathfrak{N}_{|U}(\mathrm{A}) = ({}^-0, {}^-0, {}^-0)$$

- Absolute Full Ignorance

$$\mathfrak{N}_{|U}(\mathrm{A}) = (]^-0; 1^+[,]^-0; 1^+[,]^-0; 1^+[)$$

Note that Absolute Full Ignorance is not equivalent to Absolute Indeterminacy which is charaterized by $\mathfrak{N}_{|U}(\mathrm{A}) = ({}^{-}0, 1^{+}, {}^{-}0)$

- Relative Full Ignorance

$$\mathfrak{N}_{|W}(\mathrm{A}) = ([0;1],[0;1],[0;1])$$

6. UNIVERSAL NEUTROSOPHIC CUBE

Any neutrosophic statement can be easily interpreted graphically by using the following universal neutrosophic cube representation ($\mathfrak{N}$-Cube). The 3D space limited by vertices $A, B, \ldots, H$ corresponds to the representation of universe U (*i.e.*, the set of all possible worlds of knowledge). The small classical unit cube defined by vertices $a, b, \ldots, h$ represents a typical world of knowledge W related to a given source of information (body of evidence) under consideration. In general, a statement A will not correspond to a specific point of $\mathfrak{N}$-Cube but rather a set of non-connexe volumes included into it.

Each vertice of the $\mathfrak{N}$-Cube corresponds to a very special kind of logical formula (tautology, contradiction, paradox, *etc*.). The list of all these specific statements is given in Table I. By example, if statement A lies in G this means that A is an universal tautology. If the position of statement A is anywhere in the UNC, we have an absolute full ignorance about A and no (or all) conclusions/decision can be drawn from A.

TABLE I Types of specific logical statements in $\mathfrak{N}$-Cube

Location of statement	*Type of statement*
A	Anti Paradox
B	Universal Truth
C	Anti Falsehood
D	Universal Indeterminacy
E	Universal Contradiction
F	Anti Indeterminacy
G	Universal Paradox
H	Anti Truth

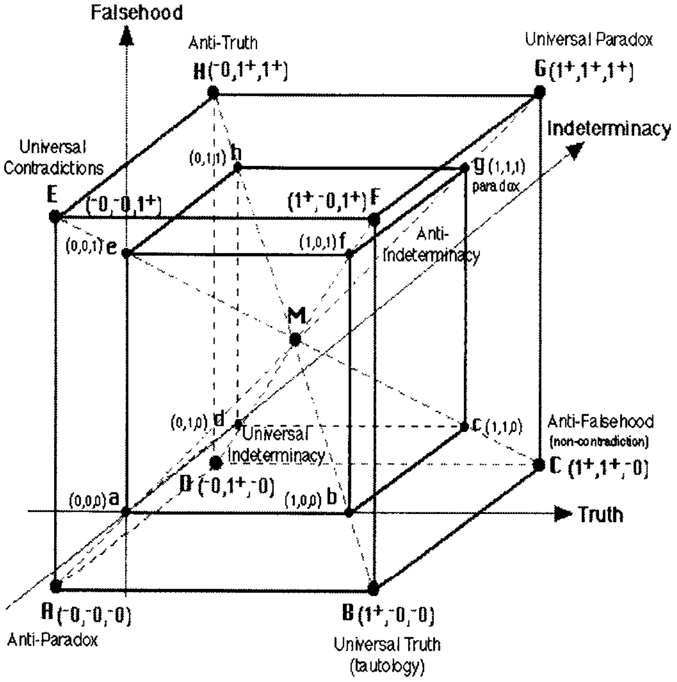

FIGURE 1 Universal $\mathfrak{N}$-Cube.

7. MANIPULATIONS ON NEUTROSOPHIC ASSERTIONS AND SETS

In the neutrosophic topology defined on $]^{-}0;1^{+}[$, the union and intersection of two any neutrosophic subsets A and B (corresponding to either the part of truth, indeterminacy or falsehood of a given assertion) are defined as follows

$$A \cup B = (A \oplus B) \ominus (A \odot B) \quad \text{and} \quad A \cap B = A \odot B \tag{14}$$

The neutrosophic complement of A is defined as $\bar{A} = \{1^{+}\} \ominus A$

The neutrosophic logical (neutrosophical) value of an assertion A conditioned on a given world of knowledge W is characterized by a mapping function $\mathfrak{N}_{|W}(\cdot)$ such that

$$\mathfrak{N}_{|W} : A \mapsto \mathfrak{N}_{|W}(\mathsf{A}) = (T(\mathsf{A}), I(\mathsf{A}), F(\mathsf{A})) \subset]^{-}0;1^{+}[^{3} \tag{15}$$

Consider now two statements A_1 and A_2 conditioned on the same world of knowledge W. The following basic operators are then defined (indexe $_{|W}$ has been omitted for simplicity and $T_i = T(A_i)$, $I_i = I(A_i)$, $F_i = F(A_i)$ for $i = 1, 2$)

$$\mathfrak{N}(\mathsf{A}_1) \boxplus \mathfrak{N}(\mathsf{A}_2) = (T_1 \oplus T_2, I_1 \oplus I_2, F_1 \oplus F_2) \tag{16}$$

$$\mathfrak{N}(\mathsf{A}_1) \boxminus \mathfrak{N}(\mathsf{A}_2) = (T_1 \ominus T_2, I_1 \ominus I_2, F_1 \ominus F_2) \tag{17}$$

$$\mathfrak{N}(\mathsf{A}_1) \boxdot \mathfrak{N}(\mathsf{A}_2) = (T_1 \odot T_2, I_1 \odot I_2, F_1 \odot F_2) \tag{18}$$

Since the truth, falsehood and indeterminacy of any statement must belong to $]^-0; 1^+[$, the result of each previous operator $\boxplus$, $\boxminus$ and $\boxdot$ must be in $]^-0; 1^+[^3$. Therefore upper and lower bounds of $T_1 \oplus T_2$ must be set respectively to $^-0$ and 1^+ whenever $inf(T_1 \oplus T_2) < 0$ or $\sup(T_1 \oplus T_2) > 1$. The same remark applies for $\boxminus$ and $\boxdot$ operators and for falsehood and inderterminacy part of compounded statement.

All classical logical operators and connectors can be extented in the $\mathfrak{N}$-Logic. For notation convenience, we will identify logical operators with their classical counterpart in set theory as pointed out in [92] (hence the following equivalences will be used $\neg\mathsf{A} \equiv \mathsf{A}, \mathsf{A}_1 \wedge \mathsf{A}_2 \equiv \mathsf{A}_1 \cap \mathsf{A}_2$ and $\mathsf{A}_1 \vee \mathsf{A}_2 \equiv \mathsf{A}_1 \cup \mathsf{A}_2$ throughout this paper). We recall here only important operators used in the sequel. Additional neutrosophic logical operators can be found in [130].

- Negation

$$\mathfrak{N}(\mathsf{A}) = (\{1\} \ominus T(\mathsf{A}), \{1\} \ominus I(\mathsf{A}), \{1\} \ominus F(\mathsf{A})) \tag{19}$$

- Conjunction

$$\mathfrak{N}(\mathsf{A}_1 \cap \mathsf{A}_2) = \mathfrak{N}(\mathsf{A}_1) \boxdot \mathfrak{N}(\mathsf{A}_2) = (T_1 \odot T_2, I_1 \odot I_2, F_1 \odot F_2) \tag{20}$$

- Disjunction

$$\begin{aligned}\mathfrak{N}(\mathsf{A}_1 \cup \mathsf{A}_2) &= (T_1 \cup T_2, I_1 \cup I_2, F_1 \cup F_2)\\ &= ((T_1 \oplus T_2) \ominus (T_1 \odot T_2),\\ &\quad\ (I_1 \oplus I_2) \ominus (I_1 \odot I_2), (I_1 \oplus I_2) \ominus (I_1 \odot I_2))\\ &= [\mathfrak{N}(\mathsf{A}_1) \boxplus \mathfrak{N}(\mathsf{A}_2)] \boxminus [\mathfrak{N}(\mathsf{A}_1) \boxdot \mathfrak{N}(\mathsf{A}_2)]\end{aligned} \tag{21}$$

7.1. $\mathfrak{N}$-Membership Function Over a Neutrosophic Set

We denote by Θ a world of discourse for a given world of knowledge W, called also frame of discernment in DS (Dempster-Shafer) theory. Each "neutrosophical" element x of Θ is characterized by its own neutrosophical basic assignment ($\mathfrak{N}$-value)$\mathfrak{N}_{|W}(x) \triangleq (T(x), I(x), F(x))$ with $T(x)$, $I(x)$ and $F(x) \subset]^{-}0; 1^{+}[$. The $\mathfrak{N}$-membership function of any neutrosophical element x with any subset $M \subset \Theta$ is defined in similar way by

$$\mathfrak{N}_{|W}(x|M) \triangleq (T_M(x), I_M(x), F_M(x)) \tag{22}$$

with $T_M(x)$, $I_M(x)$ and $F_M(x) \subset]^{-}0; 1^{+}[$. The $\mathfrak{N}$-value of x over M can be interpreted, by abuse of language, as its membership function to M in the following sense: x is t% true in the set M, i% indeterminate (unknown if it is) in M, and f% false in M, where t varies in T, i varies in I, f varies in F. The standard notation $x \in M$ will be used in the sequel to denote the neutrosophical membership of x to M. One can say actually that any element x of a given frame of discernment supported by a body of evidence neutrosophically belongs to any set, due to the percentages of truth/indeterminacy/falsity involved, which varies between 0 and 1 or even less than 0 or greater than 1. From this definition and previous neutrosophic rules, one gets directly following basic neutrosophical set operations:

- Complement of M
  If $x \in M$ with $\mathfrak{N}_{|W}(x|M) \triangleq (T_M(x), I_M(x), F_M(x))$, then $x \notin M$ with

$$\mathfrak{N}_{|W}(x|\bar{M}) = (\{1\} \ominus T_M(x), \{1\} \ominus I_M(x), \{1\} \ominus F_M(x)) \tag{23}$$

- Intersection $M \cap N$ If $x \in M$ with $\mathfrak{N}_{|W}(x|M) \triangleq (T_M(x), I_M(x), F_M(x))$ and $x \in N$ with $\mathfrak{N}_{|W}(x|N) \triangleq (T_N(x), I_N(x), F_N(x))$ then $x \in M \cap N$ with

$$\mathfrak{N}_{|W}(x|M \cap N) = (T_M(x) \odot T_N(x), I_M(x) \odot I_N(x), F_M(x) \odot F_N(x)) \tag{24}$$

- Union $M \cup N$
  If $x \in M$ with $\mathfrak{N}_{|W}(x|M) \triangleq (T_M(x), I_M(x), F_M(x))$ and $x \in N$ $\mathfrak{N}_{|W}(x|N) \triangleq (T_N(x), I_N(x), F_N(x))$ then $x \in M \cup N$ with

$$\mathfrak{N}_{|W}(x|M \cup N) = (T_{M\cup N}(x), I_{M\cup N}(x), F_{M\cup N}(x)) \tag{25}$$

where

$$T_{M\cup N}(x) \stackrel{\Delta}{=} [T_M(x) \oplus T_N(x)] \ominus [T_M(x) \odot T_N(x)] \tag{26}$$

$$I_{M\cup N}(x) \stackrel{\Delta}{=} [I_M(x) \oplus I_N(x)] \ominus [I_M(x) \odot I_N(x)] \tag{27}$$

$$F_{M\cup N}(x) \stackrel{\Delta}{=} [F_M(x) \oplus F_N(x)] \ominus [F_M(x) \odot F_N(x)] \tag{28}$$

- Difference $M - N$
  Since $M - N \stackrel{\Delta}{=} M - \bar{N}$ if $x \in M$ with $\mathfrak{N}_{|W}(x|M) \stackrel{\Delta}{=} (T_M(x), I_M(x), F_M(x))$ and $x \in N$ with $\mathfrak{N}_{|W}(x|N) \stackrel{\Delta}{=} (T_N(x), I_N(x), F_N(x))$ then $x \in M - N$ with

$$\mathfrak{N}_{|W}(x|M - N) = (T_{M-N}(x), I_{M-N}(x), F_{M-N}(x)) \tag{29}$$

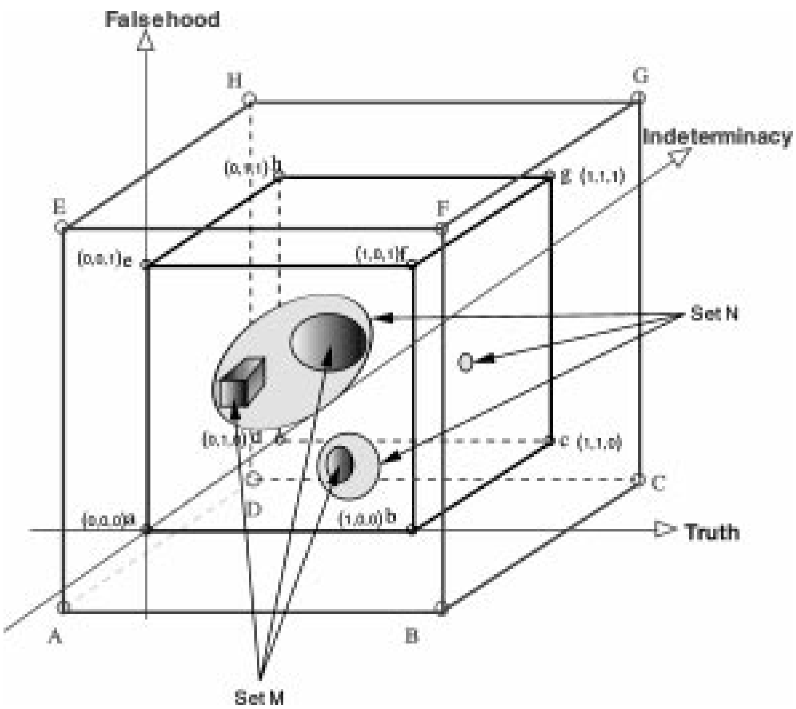

FIGURE 2 Example of neutrosophic inclusion of M in N.

where

$$T_{M-N}(x) \triangleq T_M(x) \ominus [T_M(x) \odot T_N(x)] \tag{30}$$

$$I_{M-N}(x) \triangleq I_M(x) \ominus [I_M(x) \odot I_N(x)] \tag{31}$$

$$F_{M-N}(x) \triangleq F_M(x) \ominus [F_M(x) \odot F_N(x)] \tag{32}$$

- Inclusion $M \subset N$
  We will said that $M \subset N$ if for all $x \in M$ with $\mathfrak{N}_{|W}(x|M) \triangleq (T_M(x), I_M(x), F_M(x))$ and $x \in N$ with $\mathfrak{N}_{|W}(x|N) \triangleq (T_N(x), I_N(x), F_N(x))$ one has jointly $T_M(x) \subset T_N(x)$, $I_M(x) \subset I_N(x)$ and $F_M(x) \subset F_N(x)$.

8. NEUTROSOPHIC INFERENCE

Before discussing about neutrosophy inference, it is important to remind the framework of the two main rules of inference on which are based most of modern data fusion algorithms: the Bayesian inference and Dempster-Shafer rule of combination.

- Bayesian Inference and Bayesian Fusion Rule
  To each event A of random experiment defined on frame of discernment (worl os discourse) Θ, we assign a real number $P\{\mathrm{A}\}$, called probability verifying some properties. Traditional definitions $P\{\mathrm{A}\}$ are [92]:

(1) within classical approach

$$P\{A\} = \frac{\text{Number of possible outcomes for event } A}{\text{Number of possible outcomes for space } \Theta}$$

For example, if we consider the fair Die-rolloing experiment and the event $A =$ "an event number shows up", then $P\{A\} = (1 + 1 + 1)/6 = 1/2$.

(2) within geometrical approach

$$P\{A\} = \frac{\text{Geometric measure of set } A}{\text{Geometric measure of space } \Theta}$$

(3) within relative frequency approach

$$P\{A\} = \lim_{N\to\infty} \frac{\text{Number of occurrences of set } A}{\text{Total number of trials } N}$$

In this approach, we implicitly assume that each elementary (focal) element of Θ is equally probable. This fundamental assumption is called principle of sufficient reason or principle of indifference. This principle has been strongly critized especially for cases involving infinitely many possible outcomes because this can lead to confusing paradoxes. That is why since the work of A. Kolmogorov in 1933, the axiomatic of the probability theory based on (σ)algebraes and measure theory has been definitely adopted.

(4) within axiomatic approach The probability measure P is defined by the following axioms

$A1$: (Nonnegativity)

$$0 \leq P\{A\} \leq 1 \tag{33}$$

$A2$: (Unity) Any sure event (the sample space) has unity probability

$$P\{\Theta\} = 1 \tag{34}$$

$A3$: (finite additivity) If $A_1, \ldots, A_n$ are disjoint events, then

$$P\{A_1 \cup \cdots \cup A_n\} = P\{A_1\} + \cdots + P\{A_n\} \tag{35}$$

$A4$: (countable additivity) If $A_1, A_2, \ldots$ are disjoint events

$$P\left\{\bigcup_{i=1}^{\infty} A_i\right\} = \sum_{i=1}^{\infty} P\{A_i\} \tag{36}$$

From these axioms, all other probability laws (specially Total Probability Theorem and Bayes's rule as it will be reminded) can be derived. In particular,

$$P\{\emptyset\} = 0 \quad \text{and} \quad P\{A^c\} = 1 - P\{A\} \tag{37}$$

$$A \subset B \Rightarrow P\{A\} \leq P\{B\} \tag{38}$$

$$\forall A, B \subset \Theta, \quad P\{A \cup B\} = P\{A\} + P\{B\} - P\{A \cap B\} \tag{39}$$

$$\forall A_1, \ldots, A_n \subset \Theta, \quad P\{A_1 \cup \cdots \cup A_n\} \leq \sum_{i=1}^{n} P\{A_i\} \tag{40}$$

More precisely, in the general case one has

$$P\{A_1 \cup \cdots \cup A_n\} = \sum_{i=1}^{n} P\{A_i\} - \sum_{i<j} P\{A_i \cap A_j\} + \cdots + (-1)^{k-1} \sum_{i_1 < \ldots < i_k} P\{A_{i_1} \cap \ldots \cap A_{i_k}\} + \cdots + (-1)^n P\left\{\bigcap_{i=1,n} A_i\right\} \tag{41}$$

which can be also written as

$$P\{A_1 \cup \cdots \cup A_n\} = \sum_{\substack{I \subset \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|+1} P\left\{\bigcap_{i \in I} A_i\right\} \tag{42}$$

The probability of an event A under the condition that event B has occured (with probability $P\{B\} \neq 0$) is called the conditional (or a posteriori) probability of A given B and is defined as

$$P\{A|B\} = \frac{P\{A \cap B\}}{P\{B\}} \tag{43}$$

Events A and B are said to be independent if $P\{A \cap B\} = P\{A\}P\{B\}$ or equivalently $P\{A|B\} = P\{A\}$ and $P\{B|A\} = P\{B\}$.

Total Probability Theorem: The probability of any event B can be recovered from any partition (*i.e.*, a set of exhaustive and disjoint events) $A_1, \ldots, A_n$ of sample space Θ by the following relationship

$$P\{B\} = \sum_{i=1}^{n} P\{B|A_i\}P\{A_i\} \tag{44}$$

Baye's rule: $P\{A|B\}P\{B\} = P\{A \cap B\} = P\{B|A\}P\{A\}$, one gets the famous Baye's formula also called **Bayesian inference**

$$P\{A|B\} = \frac{P\{B|A\}P\{A\}}{P\{B\}} \tag{45}$$

8.1. Bayesian Fusion Rule

Let have a given set of hypothesis $\Theta = \{\theta_1, \ldots, \theta_m\}$ corresponding to a world of discourse (frame of discernment) about the true nature of a parameter θ. If we now consider N body of evidence (experts) providing an unreliable assertion/disjonction $W_i^{\delta_i}, i = 1, \ldots, N$ (where $\delta_i = 1$ if $\theta \in W_i$ or $\delta_i = 0$ otherwise) about the truth in Θ, then general optimal Bayesian fusion rule for these N sources of knowledge is given by [36]

$$P\{\theta = \theta_i | W_1^{\delta_1}, \ldots, W_N^{\delta_N}\} = \frac{p_i^{1-N} \prod_{n=1,N} P\{\theta_i | W_n^{\delta_n}\}}{K_N} \tag{46}$$

where the normalization constant K_N is given by

$$K_N = \sum_{i=1,M} p_i^{1-N} \prod_{n=1,N} P\{\theta_i | W_n^{\delta_n}\} \tag{47}$$

If we assume the so-called principle of sufficient reason, *i.e.*, we consider uniform prior $p_i \triangleq P\{\theta = \theta_i\}$, then $P\{\theta = \theta_i | W_1^{\delta_1}, \ldots, W_N^{\delta_N}\}$ will reduce to

$$P\{\theta = \theta_i | W_1^{\delta_1}, \ldots, W_N^{\delta_N}\} = \frac{\prod_{n=1,N} P\{\theta_i | W_n^{\delta_n}\}}{\sum_{i=1,M} \prod_{n=1,N} P\{\theta_i | W_n^{\delta_n}\}} \tag{48}$$

Because of exhaustive list of focal/atomic hypotheses θ_i, any disjunction can be evaluated by additivity property of probabilities from (46). Hence for example, if we consider $A = \theta_1 \cup \theta_2 \cup \theta_3$, then $P\{A | W_1^{\delta_1}, \ldots, W_N^{\delta_N}\}$ will be easily obtained by

$$P\{A | W_1^{\delta_1}, \ldots, W_N^{\delta_N}\} = \sum_{\theta_i \subset A} \left[\frac{p_i^{1-N}}{K_N} \prod_{n=1,N} P\{\theta_i | W_n^{\delta_n}\} \right] \tag{49}$$

The optimal Bayesian fusion rule always exists if the sources are unreliable even if they appear to be incompatible, that is to say when $W_1^{\delta_1} \cap, \ldots, \cap W_N^{\delta_N} = \emptyset$ when $\delta_1 = \cdots = \delta_N$. When the sources are fully reliable and if for $\delta_1 = \cdots = \delta_N$, one has $W_1^{\delta_1} \cap W_2^{\delta_2}, \ldots, \cap W_N^{\delta_N} = \emptyset$ then no theoretical optimal fusion rule exists because this would mean that these two sources of information assign different values for the unique absolute truth which yields to a physical paradox. Actually, such case must not appear in theory. Such case can however occur in some practical problem of multi-sensor data fusion only because the sources of information (experts) do not support the same world of knowledge. They may be actually only fully reliable with respect to their own world of knowledge but not in the absolute universe defined in Section 5.

- Dempster-Shafer's inference and fusion rule

In his theory of evidence (called sometimes theory of probable reasoning), Glenn Shafer has introduced in [119] the concept of belief functions to quantify the impact of an evidence on a finite and countably frame of discernment Θ provided by a given world of knowledge W. Formally, belief functions $Bel : 2^{\Theta} \rightarrow [0; 1]$ obey the following three axioms:

Axiom 1: The belief over all the frame of discernment is set to one

$$Bel(\Theta) = 1 \tag{50}$$

Axiom 2: The belief of impossible event is set to zero

$$Bel(\emptyset) = 0 \tag{51}$$

Axiom 3: For every positive integer n and every collection $A_1, \ldots, A_n$ of subset of Θ,

$$Bel(A_1 \cup \cdots \cup A_n) \geq \sum_{\substack{I \subset \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|+1} Bel\left(\bigcap_{i \in I} A_i\right) \tag{52}$$

Any belief function satisfying $Bel(\emptyset) = 0$, $Bel(\Theta) = 1$ and $Bel(A \cup B) = Bel(A) + Bel(B)$ whenever A, $B \subset \Theta$ and $A \cap B = 0$ is called a Bayesian belief function. In such case, relation (52) coincides exactly with (42).

The basic probability assignment (mass) $m: 2^{\Theta} \rightarrow [0; 1]$ defined by

$$m(\emptyset) = 0 \quad \text{and} \quad \sum_{A \subset \Theta} m(A) = 1 \tag{53}$$

are bi-uniquely related to the belief function *Bel* by

$$Bel(A) = \sum_{B \subset A} m(B) \tag{54}$$

In return, $m(A)$ for all $A \subset \Theta$ can be recovered from the belief function by

$$m(A) = \sum_{B \subset A} (-1)^{|A-B|} Bel(B) \tag{55}$$

The mass $m(\cdot)$ is sometimes called the Möbius inverse of belief function [113]. The quantity $m(A)$ is the measure of belief that is exactly committed to A. A subset A of a frame of discernment Θ is called focal element of a belief function *Bel* over θ if $m(A) > 0$. The union of all the focal elements of a belief function is called its core $\mathcal{C}$. A vacuous belief function is a belief function that allocates a null belief to any strict subset of Θ, *i.e.*, $Bel(A) = 0$ for all $A \neq \Theta$. It represents a state of full ignorance on Θ. It has been shown in [119], that *Bel* is a Bayesian belief function if and only if there exists a mass assignment $m(\cdot)$ such that $\sum_{\theta \in \Theta} m(\theta) = 1$ and $\forall A \subseteq \Theta$, $Bel(A) = \sum_{\theta \in A} m(\theta)$ which is equivalent to say that all *Bel*'s focal elements are singletons or $\forall A \subset \Theta$, $Bel(A) + Bel(A^c) = 1$.

8.2. Dempster's Rule of Combination

G. Shafer has proposed the ad-hoc Dempster's rule of combination (orthogonal summation), symbolized by the operator $\oplus$, to combine two so called distinct bodies of evidences over the same frame of discernment Θ. Let $Bel^1(\cdot)$ and $Bel^2(\cdot)$ be two belief functions over the same frame of discernment Θ and $m^1(\cdot)$ and $m^2(\cdot)$ their corresponding basic probability masses. The combined (*i.e.*, merged or fusionned) global belief function $Bel(\cdot) = Bel^2(\cdot) \oplus Bel^2(\cdot)$ is obtained from the combination of its basic mass assignments $m^1(\cdot)$ and $m^2(\cdot)$ as follows: $m(\emptyset) = 0$ and for any $C \neq \emptyset$ and $C \subset \Theta$,

$$m(C) = \frac{\sum_{A \cap B = C} m^1(A) m^2(B)}{\sum_{A \cap B \neq \emptyset} m^1(A) m^2(B)} = \frac{\sum_{A \cap B = C} m^1(A) m^2(B)}{1 - \sum_{A \cap B = \emptyset} m^1(A) m^2(B)} \tag{56}$$

The orthogonal sum $m(\cdot)$ is a basic probability assignment if $K \triangleq 1 - \sum_{A \cap B = \emptyset} m^1(A) m^2(B) \neq 0$. If $K = 0$, which means $\sum_{A \cap B = \emptyset} m^1(A)$ $m^2(B) = 1$ then orthogonal sum $m(\cdot)$ does not exist and $m^1(\cdot)$ and

$m^2(\cdot)$ are said to be totally or flatly contradictory. Such case arises whenever the cores of $Bel^1(\cdot)$ and $Bel^2(\cdot)$ are disjoint or equivalently when there exists $A \subset \Theta$ such that $Bel^1(A) = 1$ and $Bel^2(A^c) = 1$. The same problem has already been pointed out previously in the development of optimal Bayesian fusion rule.

The quantity $\log 1/K$ is called the weight of conflict between the sources of evidence $Bel^1(\cdot)$ and $Bel^2(\cdot)$. It is easy to show that Dempster's rule of combination is commutative ($m^1 \oplus m^2 = m^2 \oplus m^1$) and associative ($[m^1 \oplus m^2] \oplus m^3 = m^1 \oplus [m^2 \oplus m^3]$). The vacuous belief function such that $m^v(\Theta) = 1$ and $m^v(A) = 0$ for $A \neq \Theta$ is the identity element for $\oplus$, *i.e.*, $m^v \oplus m = m \oplus m^v = m$. If $Bel^1(\cdot)$ and $Bel^2(\cdot)$ are two combinable belief functions and if $Bel^1(\cdot)$ is Bayesian, then $Bel^1 \oplus Bel^2$ is a bayesian belief functions.

This rule of combination, initially proposed by Shafer without strong theoretical justification ("ad-hoc justification"), has been criticized in the past decades by many disparagers of this theory. Nowadays, this rule of combination has however been fully justified by the axiomatic of the transfer belief model developed by Smets in
[136–138, 140]. We mention the fact that such theoretical justification had been already attempted by Cheng and Kashyap in [18].

We can see a very close similarity between Dempster's rule and optimal bayesian fusion (46). Actually these two rules coincides exactly when m^1 and m^2 become probability assignments and if we accept the principle of indifference within the optimal Bayesian fusion rule.

8.2.1. Conditional Belief Functions

Let $m_B(B) = 1$ and $m_B(A) = 0$ for all $A \neq B$. Then Bel_B is a belief function that focuses all of the belief on B (note that Bel_B is not in general a Bayesian belief function unless $|B| = 1$). If we now consider another belief function Bel over Θ combinable with Bel_B, then the orthogonal sum of Bel with Bel_B denoted as $Bel(\cdot|B) = Bel \oplus Bel_b$ is defined for all $A \subset \Theta$ by [119]

$$Bel(A|B) = \frac{Bel(A \cup B^c) - Bel(B^c)}{1 - Bel(B^c)} \tag{57}$$

If *Bel* is a Bayesian belief function, then

$$Bel(A|B) = \frac{Bel(A \cap B)}{Bel(B)} \tag{58}$$

which coincides with the classical conditional probability defined in (43).

- Neutrosophic inference

The major difficulty for the development of neutrosophic inference is to deal with 3 dimensions of the neutrosophic space and the freedom on each dimension with respect to the others.

We propose to use the following generalized neutrosophic mass assignment $m(\cdot) = (m_t(\cdot),\ m_i(\cdot),\ m_f(\cdot)){:}2^{\Theta} \to]^{-}0; 1^{+}[^3$ satisfying the following constraints for each dimension of the neutrosophic space

$$\sum_{A \subset \Theta} \sup(m_t(A)) \geq 1 \tag{59}$$

$$\sum_{A \subset \Theta} \inf(m_f(A)) \geq |\Theta| - 1 \tag{60}$$

where $|\Theta|$ represents the cardinality of frame of discernment Θ. For notation convenience we prefer to indroduce here the notation $(m_t(\cdot), m_i(\cdot), m_f(\cdot))$ rather than $(T(\cdot), I(\cdot), F(\cdot))$ to explicitly specify that these neutrosophic components have to follow the previous constraints.

We don't impose constraint on indeterminate part of neutrosophic statements but to belong to $]^{-}0; 1^{+}[$. This generalization coincides with the basic mass assignment whenever we impose the classical constraint $T(\cdot) = 1 - F(\cdot)$ with $T(\cdot)$ being a real number belonging to [0; 1]. Neither, we impose constraint on empty set $\emptyset$.

From this basic neutrosophic "mass" assignment, we would like to associate bi-uniquely a new neutrosophic belief function having some mathematical properties. It can be shown that classical belief functions are ∞-monotone capacities, form a convex subset of the space of non-additive measures on a given algebra (the algebra of events). A direct extension of Dempster Shafer's theory suggest us to take, for all $A \subset \Theta$, $Bel(\cdot) = (Bel_T(\cdot),\ Bel_I(\cdot),\ Bel_F(\cdot))$ as

$$Bel_T(A) = \boxplus_{B \subset A} m_t(B) \tag{61}$$

$$Bel_I(A) = \boxplus_{B \subset A} m_i(B) \tag{62}$$

$$Bel_F(A) = \boxplus_{B \subset A} m_f(B) \tag{63}$$

Such first tentative for a "neutrosophic belief function" construction must however be examined in details to be validated or rejected. Definitely, the mathematical properties of such kind functions have to be clarified. We expect that such function should extend in some way the classical belief functions proposed by Shafer. Some caution must be taken for now about the indeterminate part of neutrosophic belief function. Since we have introduced no constraint about indeterminate part (on $m_i(\cdot)$) of neutrosophic events, we can't expect some nice mathematical properties for $Bel_I(\cdot)$ (like super additivity, *etc.*). This is a miss and more deeper investigations are needed to develop a new neutrosophic belief formalism. We have just wanted, through this paper, to point out major open questions about neutrosophic inference. When neutrosophic belief formalism and axiomatic will be achieved, one should be able to develop a new neutrosophical rule of combination. Such rule (if any) will have to be commutative, associative and allow to reduce indeterminacy with the introduction of (better) sources of information and possibly eliminate antinomies of the world of discourse. The development of the theory of neutrosophic belief function and neutrosophic combination rule is the forthcoming theoretical challenge to overcome to bring the full usefulness of neutrosophy for solving many important problems involved mainly in data fusion and expert systems.

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