Decision procedures for a deontic logic modeling temporal inheritance of obligations

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Abstract

In nowadays applications of temporal deontic logic to the verification of security policies, an issue arises concerning the temporal inheritance of future directed obligations that have net yet been met. We investigate decision procedures for temporal deontic logics that account for this particular interaction between time and obligation.

Keywords: deontic logic, temporal logic, tableaux method, axiomatization

1 Introduction

At least since the eighties, deontic logicians have been interested in the relation between deontic modalities and time [23,2,11,24,18,22,1,5]. The addition of a temporal dimension to deontic operators was expected to shed new light on the problems concerning the so called deontic 'paradoxes'. And indeed, several authors have shown that at least part of the confusion raised by such examples as Chisholm's 'helping the neighbors' paradox [8], or Forrester's 'gentle murder paradox' [14] can be disambiguated by making the implicit temporal connotations of deontic modalities explicit.

But, also recent developments in logics for dynamical phenomena such as announcement and update logics, have sparked new interest in the relation between epistemic and preferential modalities on the one hand, and time, action, or update modalities, on the other hand. For instance, in Van Benthem and Liu's upgrade

This paper is electronically published in
Electronic Notes in Theoretical Computer Science
URL: `www.elsevier.nl/locate/entcs`

logics [26], preference relations, which can also be used to model dispositions in *deontic* ideality, are updated through announcements, raising new questions about the dynamic properties of obligations.

A seemingly rather important, though not extensively studied aspect of the interaction of time and obligation, is the issue of preservation. In particular, a sensible and often desirable property, is that if a future directed obligation is not met currently, it propagates to the next moment of time. Another way of putting it is to say that moments *inherit* obligations from the past, if they have not been complied to. In earlier papers [7,6] we have mainly tried to give a formal semantics to this intuition concerning the preservation of obligations. The present paper investigates inference mechanisms that are complete with respect to the earlier proposed semantics.

2 Expressiveness needs

Temporal deontic frameworks have many applications. An example is the specification of security policies which go beyond usual access control. For instance, if we want to specify that it is obligatory for a user to release a resource if this user has been idle for more than one minute. More specification examples involving temporal and deontic concepts are given below.

- *$user_i$ has to release resource r after 5 time units of utilization*
- *$user_i$ has the permission to use resource r continuously for 5 time units, and he must be able to access it 15 time units after asking, at the latest*
- *If $user_i$ is asking for resource r and he has the permission to use it, then the system has the obligation to give it to him before 5 time units*
- *If $user_i$ uses the resource without permission, he must not ask for it during 10 time units*

Such sentences can be easily formalized in a logical language with standard temporal and deontic operators. For instance, the first rule is formally expressed by

$$G\ (access_i \Rightarrow \boldsymbol{O}(\ F_{\leqslant 5}\ release_i))$$

where $G(\varphi)$ means *it is henceforth true that* φ, $\boldsymbol{O}(\varphi)$ means that *it is obligatory to satisfy* φ, and $F_{\leqslant k}\ (\varphi)$ means that φ *will be satisfied before* k *time units.*

For the pure temporal and deontic fragments of our logic, we consider *Linear Temporal Logic* (*LTL*) [21] and *Standard Deontic Logic* (*SDL*) [27]. Of course, the more interesting questions concern the possible and desirable interactions of the temporal and deontic modalities. The interaction property we focus on concerns the *propagation of obligations*: if a future-directed obligation is not fulfilled yet, then it propagates to the next moment. For instance, if it is obligatory to release the resource before 5 time units and it is not released now, then in one time unit from now it will be obligatory to release the resource before 4 time units.

Other combinations of temporal and deontic logics have been studied. For instance, in [1,3,11] branching-time temporal and deontic logics are proposed. However, the temporal operators in these frameworks differ considerably from the more standard operators we use in our work. Essentially, these operators have a semantics that nowadays would be called 'hybrid'. In this semantics, each operator refers to a specific time point. Also, the interaction of the temporal and deontic dimensions in the above mentioned approaches has the following drawback: if φ is a propositional formula then $\varphi \Rightarrow \boldsymbol{O}(\varphi)$ is valid, i.e., every proposition which is true in the current state is obligatory.

In [5,10], the propagation of obligations is studied for deadline obligations. A reductionist approach is used, i.e., the deontic dimension is embedded in the temporal dimension. So it is not possible to talk about both dimensions independently.

Taking the modal product of linear temporal logic and standard deontic logic as the starting point, in [6] we developed a logic in which obligations are preserved depending on what is true in the current history. (Product based approaches without such interactions are studied in [9,7].) In [6] we suggested to incorporate the preservation property by designing a logic that validates:

$$\boldsymbol{O}(\varphi \vee X\psi) \wedge \neg\varphi \wedge \neg\boldsymbol{O}(\varphi) \;\Rightarrow\; X\,\boldsymbol{O}(\psi)$$

with φ any propositional formula, and ψ any formula. The initial idea was just to take sets of deontically ideal histories to give semantics to obligation.

One of our suggestions is then that for assessing deontic ideality of histories, we should *only* consider the histories whose past is identical to the past of the current history. The reason for this is that we assume obligations do not apply to the past, but only to the present and the future. Then, clearly, we do not want to consider histories whose past, as seen from the current state, is more or less deontically ideal than the 'current' past. We thus only assess ideality for the histories that share their past with the current history. The collective past of the set of histories thus obtained then represents what actually has happened. And what actually has happened, is going to influence what is obliged currently, according to the preservation property we aim at.

A second aspect of our semantics is that the future directed obligations are not 'forgotten' when time advances. This means that the number of ideal histories can only shrink as time advances. The principle of propagation then should point us to what subset of ideal histories to take when moving to a next moment in time.

An interesting question concerns what happens if the set of ideal histories shrinks to the empty set. Of course, this could happen easily if p is obliged currently while at the same time $\neg p$ holds. Then, at the next moment, the intersection of, on the one hand, the histories with pasts identical to the past of the actual history and, on the other hand, the ideal histories, is empty. This yields a situation where everything is obliged. This happens whenever an obligation whose compliance can no longer be deferred, is violated anyway. Different reactions are possible to this peculiarity of the semantics. One way out, though not an attractive one, is to suggest that the models should be such that this is excluded. But, then we would have a logic where the obligations are never violated, which means that the logic cannot count as 'deontic'. But, in [6] we discussed that a natural way to solve this problem is to adopt a generalization to *levels* of ideality. And that is also the setting in the present paper. We use an ordering on histories to define obligation as follows:

$\boldsymbol{O}(\varphi)$ is satisfied at an instant i of an history w if : *there is an history w' whose past coincides with w's until i, such that every history with the same past, which is at least as good as w', satisfies φ at instant i.* This semantics seems rather complex. But its naturalness is underlined by the fact that Horty, in his book on deontic logic [17], gives a basically identical semantics for 'ought-to-be' in his branching time STIT framework. The complexity of this semantics makes it difficult to establish logical results, such as an axiomatization or a decision procedure. In order to develop a decision procedure and an axiomatization, we propose to decompose the modality $\boldsymbol{O}$ into more primitive normal operators.

3 Syntax and semantics

In this section, we define the language of our logic and its semantics. Different quantifiers are hidden in the semantic definition of the obligation. Two accessibility relations are involved: the relation 'at least as good as', which will be denoted by $\preccurlyeq$ and models preference between histories, and the relation 'has the same past as', which will be denoted by $SamePast$. In order to define $\boldsymbol{O}$ in terms of more primitive modal operators, it is natural to introduce a modal operator $[SP]$ which corresponds to the relation 'has the same past' ($SamePast$), and another operator $[SP \cap \preccurlyeq]$ which corresponds to the intersection of both semantic relations. The obligation operator $\boldsymbol{O}$ will be defined in terms of the primitive operators $[SP]$ and $[SP \cap \preccurlyeq]$.

Definition 3.1 [Syntax] Given a finite set P of atomic propositions, the language of our logic is defined by the following syntax:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid X\varphi \mid \varphi U \varphi \mid [SP]\varphi \mid [SP \cap \preccurlyeq]\varphi$$

where $\top$ is a logical constant ('true'), and $p \in P$ is an atomic proposition. We define the following standard abbreviations:

$$\bot \stackrel{def}{=} \neg\top$$

$$\begin{array}{llllll} \varphi_1 \wedge \varphi_2 & \stackrel{def}{=} & \neg(\neg\varphi_1 \vee \neg\varphi_2) & <SP> \varphi & \stackrel{def}{=} & \neg[SP]\neg\varphi \\ \varphi_1 \Rightarrow \varphi_2 & \stackrel{def}{=} & \neg\varphi_1 \vee \varphi_2 & <SP \cap \preccurlyeq> \varphi & \stackrel{def}{=} & \neg[SP \cap \preccurlyeq]\neg\varphi \\ F\ \varphi & \stackrel{def}{=} & \top U \varphi & G\ \varphi & \stackrel{def}{=} & \neg F\ \neg\varphi \end{array}$$

- $X\varphi$ means *φ will hold in the next state.*
- $\varphi_1 U \varphi_2$ means *there is an instant i at which φ_2 will hold, and φ_1 holds from now until the instant before i.*
- $[SP]\varphi$ means *in every state that has the same past as the current state, φ holds.*
- $[SP \cap \preccurlyeq]$ means *in every state that has the same past and is at least as good as (or preferred to) the current state, φ holds.*

Definition 3.2 [Model] A model is a tuple $(W, \preccurlyeq, V)$, where

- W is a set of histories. Time is implicitly modeled by the set $\mathbb{N}$ of non-negative integers. A state is then a moment/history pair $(i, w) \in \mathbb{N} \times W$,
- $\preccurlyeq \subseteq W \times W$ is a total (or complete) quasi-ordering on W (reflexive, transitive, and total relation),
- $V : \mathbb{N} \times W \to 2^P$ is a valuation which associates each state with a set of atomic propositions.

Notice that the frames these models are based on can be suitably viewed as products [16], of the temporal frames $(\mathbb{N}, +1, <)$ and the deontic frames $(W, \preccurlyeq)$, where $+1$ and $\leqslant$ are the successor relation and the usual order on $\mathbb{N}$, respectively.

Definition 3.3 [Satisfaction]Given a model $\mathcal{M} = (W, \preccurlyeq, V)$, a state (i, w), and a formula φ, we define the satisfaction relation by induction on φ:

$$
\begin{array}{llllll}
\mathcal{M}, i, w \models \top & & & \mathcal{M}, i, w \models p \quad \text{iff} \quad p \in V(i, w) & & \\
\mathcal{M}, i, w \models \neg\varphi & \text{iff} & \mathcal{M}, i, w \not\models \varphi & & & \\
\mathcal{M}, i, w \models \varphi_1 \vee \varphi_2 & \text{iff} & \mathcal{M}, i, w \models \varphi_1 & \text{or} & \mathcal{M}, i, w \models \varphi_2 & \\
\mathcal{M}, i, w \models X\varphi & \text{iff} & \mathcal{M}, i+1, w \models \varphi & & & \\
\mathcal{M}, i, w \models \varphi_1 U \varphi_2 & \text{iff} & \exists i' \geqslant i & \text{such that} & \mathcal{M}, i', w \models \varphi_2 & \text{and} \\
 & & \forall i" \in \mathbb{N} & \text{if} \quad i \leqslant i" < i' & \text{then} \quad \mathcal{M}, i", w \models \varphi_1 & \\
\mathcal{M}, i, w \models [SP]\varphi & \text{iff} & \forall w' \in W & \text{if} \quad SamePast(i, w, w') & \text{then} \quad \mathcal{M}, i, w' \models \varphi & \\
\mathcal{M}, i, w \models [SP \cap \preccurlyeq]\varphi & \text{iff} & \forall w' \in W & \text{if} \quad w \preccurlyeq w' \text{ and } SamePast(i, w, w') & \text{then } \mathcal{M}, i, w' \models \varphi &
\end{array}
$$

where $SamePast(i, w, w') \overset{def}{=} \forall j < i \quad V(j, w) = V(j, w')$.

A model $(W, \preccurlyeq, V)$ satisfies φ if for every state $i, w \in \mathbb{N} \times W \quad i, w \models \varphi$. φ is said to be valid if every model satisfies it. φ is said to be satisfiable if there is a model $(W, \preccurlyeq, V)$, a history $w \in W$, and a number $i \in \mathbb{N}$, such that $i, w \models \varphi$.

Remark 3.4 Notice that satisfiability and validity can be evaluated at the beginning of time, and the resulting logic is then called the *anchored* version [19] of the logic. On the other hand, our definition corresponds to the so-called *floating* version. If the logic contains no past operators, then both notions define the same set of valid formulas. If the logic contains past operators, as will be the case in section 5, both versions are related to each other through an additional constant *init*, which is true in the first state of every model: φ is valid in the anchored version iff $(init \Rightarrow \varphi)$ is valid in the floating version, and φ is valid in the floating version iff $G\ \varphi$ is valid in the anchored version.

The modal operators $[SP]$ and $[SP \cap \preccurlyeq]$ that allow to quantify over the histories only consider histories which have the same past as the current one. So, from a given instant i of a given history w, they only 'access' histories w' such that $SamePast(i, w, w')$.

One might wonder if a branching time setting would not be more appropriate for our semantics. For one thing, in a branching time setting, ideal versions of the current history ‘automatically’ have the same past. However, in settings like CTL (though not in STIT-frameworks), ideal versions of the current history would also automatically have the same ‘present’. This means that if we apply the semantics to a CTL-setting, we cannot talk about what is obliged for the current moment. Another reason for not considering branching time in the current setting, is that it is not clear whether or not in a branching time setting we should actually look at deontic ideality of ‘trees’ instead of histories. And then, if we decide to look at ideal histories, in branching time structures two histories still may satisfy the same atomic propositions once they have split. Then it is unclear whether our criterion for ‘sameness’ still applies, or whether it should be made more precise.

Definition 3.5 [Obligation]

$$\boldsymbol{O}(\varphi) \stackrel{def}{=} < SP > [SP \cap \preccurlyeq]\varphi$$

Notice that the semantic characterization of $\boldsymbol{O}$ coincides with the semantic definition of obligation given in [6]:

$$\begin{aligned} i, w \models \boldsymbol{O}(\varphi) \quad \text{iff} \quad & \exists w' \in W \quad \text{such that} \quad SamePast(i, w, w') \quad \text{and} \\ & \forall w'' \in W \quad \text{if} \quad SamePast(i, w, w'') \quad \text{and} \quad w' \preccurlyeq w'' \quad \text{then} \quad i, w'' \models \varphi \end{aligned}$$

4 Decision procedure for satisfiability

As a first remark, our logic lacks the finite model property. Indeed, it can easily be shown that the following formula has only infinite models, i.e., models $(W, \preccurlyeq, V)$ where W is an infinite set of histories:

$$[SP]AtMostOnce(p) \quad \wedge \quad G \ < SP > p$$

where $AtMostOnce(p)$ stands for $G(p \Rightarrow XG\neg p)$. To establish the decidability of this logic would require complex techniques, such as quasi-model method [28,15]. In this section, we show the decidability of the until-free fragment of the logic, using a tableaux-like decision procedure.

We describe a tableaux method with explicit accessibility relations. We use the notation of prefixed formulas $i, w : \varphi$, where the prefix i, w intuitively represents a state that satisfies the formula φ. i is a non-negative integer, and w is a history. Contrary to usual prefixed tableaux [13,20], we do not encode accessibility relation into the node names. We represent explicitly the three distinct accessibility relations (’temporal successor’, ’at least as good as’, and ’same past’).

A tableau T is a structure we keep as close to a model as possible. It consists of a set of histories W, a set of moments $M \subseteq \mathbb{N}$, a labelling function L which associates each moment/history pair with a set of formulas, a quasi-ordering $R_{\preccurlyeq}$ on W, and a set of equivalence relations $(R_{SPi})_{i \in M}$ on W. Intuitively, $(w, w') \in R_{\preccurlyeq}$ means that history w' is at least as good as w; $(w, w') \in R_{SPi}$ means that histories w and w' have the same past until i (they satisfy the same propositions until the moment before i). Tableaux rules specify how, and under which conditions, T is updated.

We now formally describe tableau data structure and update operations.

4.1 Tableau data structure and update operations

A tableau for a formula ϕ is a tuple $T = (W, M, v_0, L, R_{\preccurlyeq}, (R_{SPi})_{i \in M})$ where

- W is a set of histories
- $M \subseteq \mathbb{N}$ is a set of moments; a node of the tableau is then a moment/history pair $(i, w) \in M \times W$
- $v_0 \in M \times W$ is the root
- $L : M \times W \to sub(\phi)$ is a label function which associates each node with a set of sub-formulas of ϕ. In the remainder, we write $i, w : \varphi$ for $\varphi \in L(i, w)$. The label of the root contains ϕ: $\phi \subseteq L(v_0)$.
- $R_{\preccurlyeq} \subseteq W \times W$ is a reflexive and transitive relation on W
- $R_{SPi} \subseteq W \times W$ for each $i \in M$, is an equivalence relation on W. The following property between the different R_{SPi} is satisfied:

$$(*) \quad \forall w, w' \in W \ \forall i \in M \quad (w, w') \in R_{SPi} \ \Rightarrow \ \forall j \in M \quad j < i \Rightarrow (w, w') \in R_{SPj}$$

We now give the procedural semantics of our tableau operations *add_form*, *new_world*, *new_instant*, and *add_pair*, which update a data structure T.

- $add\_form(i, w, \varphi)$ adds the formula φ to the label $L(i, w)$.
- $add\_pair(R \ ; \ (w, w'))$, for $R = R_{\preccurlyeq}$ or $R = R_{SPi}$, adds pair (w, w') to relation R, and updates R with its reflexive and transitive closure in case $R = R_{\preccurlyeq}$, with its reflexive, transitive, and symmetric closure in case $R = R_{SPi}$. Moreover, if $R = R_{SPi}$, then for every $j \in M$ such that $j < i$, R_{SPj} is updated so that the constraint $(*)$ is satisfied.
- *new_history* adds a new history to W and returns the corresponding name.
- $add\_inst(i)$ adds instant i to the set M if $i \notin M$.

We can combine these atomic actions with the two following combinators: the sequential operator ';' and the nondeterministic choice operator '[]'.

4.2 Tableaux rules

In this section, we present our tableaux rules.

- double negation rule

$$\frac{i,w{:}\neg\neg\varphi}{add\_form(i,w,\varphi)}\neg\neg$$

- rule α (resp. β) is the usual rule for conjunction (resp. disjunction).

$$\frac{i, w : \neg(\varphi_1 \vee \varphi_2)}{add\_form(i, w, \neg\varphi_1); add\_form(i, w, \neg\varphi_2)}\alpha \qquad \frac{i, w : \varphi_1 \vee \varphi_2}{add\_form(i, w, \varphi_1) \ [] \ add\_form(i, w, \varphi_2)}\beta$$

This presentation of rule β corresponds to a depth-first computation, as in [12], whereas other presentations (equivalent to ours) compute both possibilities in parallel (width-first computation).

- rules X and $\neg X$ extend the label of the successor node as follows:

$$\frac{i, w : X\varphi}{add\_inst(i+1)\ ;\ add\_form(i+1, w, \varphi)}X \qquad \frac{i, w : \neg X\varphi}{add\_inst(i+1)\ ;\ add\_form(i+1, w, \neg\varphi)}\neg X$$

- rules Π add a new history if a node is labelled by a 'diamond' formula of the form $\neg[SP]\varphi$ or of the form $\neg[SP\cap \preccurlyeq]\varphi$.

$$\frac{i, w : \neg[SP]\varphi}{w' := new\_history\ ;\ add\_form(i, w', \neg\varphi)\ :\ add\_pair(R_{SPi}, (w, w'))}\Pi_{SP}$$

$$\frac{i, w : \neg[SP\cap \preccurlyeq]\varphi}{\begin{array}{l} w' := new\_history \quad ;\ add\_form(i, w', \neg\varphi)\ ; \\ add\_pair(R_{\preccurlyeq}\ ,\ (w, w'))\ ;\ add\_pair(R_{SPi}\ ,\ (w, w'))\end{array}}\Pi_{SP\cap\preccurlyeq}$$

- rules K adds formula φ to a node i, w' if node (i, w) is labeled by a 'box' formula of the form $[SP]\varphi$, or of the form $[SP\cap \preccurlyeq]\varphi$, and w' is an accessible history from w.

$$\frac{i, w : [SP]\varphi \quad \text{and} \quad (w, w') \in R_{SPi}}{add\_form(i, w', \varphi)} \quad K_{SP}$$

$$\frac{i, w : [SP\cap \preccurlyeq]\varphi \quad \text{and} \quad (w, w') \in R_{SPi} \quad \text{and} \quad (w, w') \in R_{\preccurlyeq}}{add\_form(i, w', \varphi)} \quad K_{SP\cap\preccurlyeq}$$

- rule $update\_SP$ applies if two states which share the same past until moment i still satisfy the same propositions at i. Besides, for each atomic proposition, either this proposition or its negation has to be satisfied in both states (i.e., the states have to be saturated). Then R_{SPi+1} is updated so that w and w' are considered as having the same past until $i + 1$.

$$\frac{(w, w') \in R_{SPi} \ \text{ and } \ \forall p \in P\ (i, w : p \text{ and } \ i, w' : p) \ \text{ or } \ (i, w : \neg p \text{ and } i, w' : \neg p)}{add\_pair(R_{SPi+1}\ ;\ (w, w'))}update\_SP$$

- rule $Saturation$ aims at saturating the states in atomic propositions so that rule $update\_SP$ can be applied.

$$\frac{p \in P \quad \text{and} \quad i \in M \quad \text{and} \quad w \in W}{add\_form(i, w, p \vee \neg p)}saturation$$

- rule $\preccurlyeq -totality$ aims at guaranteeing the totality of $R_{\preccurlyeq}$.

$$\frac{(w, w') \notin R_{\preccurlyeq} \quad \text{and} \quad (w', w) \notin R_{\preccurlyeq}}{add\_pair(R_{\preccurlyeq}, (w, w'))\ []\ add\_pair(R_{\preccurlyeq}, (w', w))} \preccurlyeq -totality$$

Definition 4.1 [Closed tableau] A tableau is said to be closed if

- φ and $\neg\varphi$ label some node i, w,
- or $\quad \exists w, w' \in W \quad \exists i \in M \quad \exists p \in P$ such that

$$i, w : p \quad \text{and} \quad i, w' : \neg p \quad \text{and} \quad (w, w') \in R_{SPi+1}$$

Definition 4.2 [Completed and open tableau] A tableau T is completed if for every rule r

- either r is not enabled, i.e., the premise of r is not satisfied
- or r is enabled, and the application of the consequent of r has no effect

We consider that $add\_inst(i)$ has no effect if $i \in M$, $add\_form(i, w, \varphi)$ has no effect if $\varphi \in L(i, w)$, and $add\_pair(R, (w, w'))$ has no effect if $(w, w') \in R$, with $R = R_{\preccurlyeq}$ or $R = R_{SPi}$.

A tableau is open if it is completed and not closed.

4.3 Soundness and completeness

Theorem 4.3 (Soundness) *If a formula φ is satisfiable then there is an open tableau whose root is labeled by φ.*

Definition 4.4 [Tableaux interpretation] Let $T = (W, M, v_0, L, R_{\preccurlyeq}, (R_{SPi})_{i \in M})$ be a tableau and $(\hat{W}, \hat{\preccurlyeq}, V)$ be a model. An interpretation of T in $(\hat{W}, \hat{\preccurlyeq}, V)$ is a mapping ι from W to $\hat{W}$ such that for every w_1, w_2 in W, and every nonnegative integer i

- $(w_1, w_2) \in R_{\preccurlyeq}$ implies $\iota(w_1) \hat{\preccurlyeq} \iota(w_2)$, and
- $(w_1, w_2) \in R_{SPi}$ implies $\forall j < i \; V(j, \iota(w_1)) = V(j, \iota(w_2))$

Definition 4.5 [Satisfiable tableau] A tableau T for a formula ϕ is satisfiable if there is a model $(W, \preccurlyeq, V)$ and a tableau interpretation ι of T in $(\hat{W}, \hat{\preccurlyeq}, V)$ such that for every node (i, w) and every formula $\varphi \in L(i, w)$, we have $i, \iota(w) \models \varphi$.

Lemma 4.6 *Let T be a satisfiable tableau. The tableau T' (or one of the two tableaux T', T", in case the nondeterministic choice operator is used) obtained by the application of some rule is also satisfiable.*

Proof Let T be a satisfiable tableau. There is a model $(\hat{W}, \hat{\preccurlyeq}, \hat{V})$ and a tableau interpretation ι such that for every node (i, w) and every formula $\varphi \in L(i, w)$, we have $i, \iota(w) \models \varphi$. We have to consider each rule and prove that the application of this rule preserves the tableau satisfiability.

- $\neg\neg, \alpha, \beta, X, \neg X, saturation, \preccurlyeq -totality$: the proof is left to the reader.
- rule K_{SP} (the proof for $K_{SP \cap \preccurlyeq}$ is similar). Suppose that $i, w : [SP]\varphi$. Then rule K_{SP} adds φ to any node (i, w') such that $(w, w') \in R_{SPi}$. By hypothesis, $i, \iota(w) \models [SP]\varphi$, and, since $(w, w') \in R_{SPi}$, we have $SamePast(i, \iota(w), \iota(w'))$. Thus, $i, \iota(w') \models \varphi$, and so T' is still satisfiable.
- rule Π_{SP} (the proof for $\Pi_{SP \cap \preccurlyeq}$ is similar):
  Suppose that $i, w : \neg[SP]\varphi$. Then the application of Π_{SP} creates a new history w', labels it with $\neg\varphi$, and adds (w, w') to R_{SPi}. We have to extend the mapping ι so that it associates a history with the new prefix w'. By hypothesis, $i, \iota(w) \models \neg[SP]\varphi$. Then there is some $\hat{w'} \in \hat{W}$ such that $SamePast(i, \iota(w), \hat{w'})$ and $i, \hat{w'} \models \neg\varphi$. Then we define $\bar{\iota}$ by:

  $$\bar{\iota}(s) = \begin{cases} \hat{w'} & \text{if } s = w' \\ \iota(s) & \text{else} \end{cases}$$

  $\bar{\iota}$ is a tableau interpretation, and T' is satisfiable.

- rule *update_SP*:
  Let i, w and i, w' be two nodes of T such that $(w, w') \in R_{SPi}$ and $\forall p \in P$ $(i, w : p$ and $i, w' : p)$ or $(i, w : \neg p$ and $i, w : \neg p)$. Then the pair (w, w') is added to R_{SPi+1}. We must show that $\forall j < i+1$ $\hat{V}(j, \iota(w)) = \hat{V}(j, \iota(w'))$. Since ι is a T-interpretation, then $\forall j < i$ $\hat{V}(j, \iota(w)) = \hat{V}(j, \iota(w'))$. Besides, $\forall p \in P$ $(i, w : p$ and $i, w' : p)$ or $(i, w : \neg p$ and $i, w : \neg p)$. So $\forall p \in P$ $i, \iota(w) \models p$ iff $i, \iota(w') \models p$. So $\hat{V}(i, \iota(w)) = \hat{V}(i, \iota(w'))$.

■

Proof [**Proof of the soundness theorem (4.3)**]
Suppose φ is a satisfiable formula. then there is a model $(W, \preccurlyeq, V)$, a nonnegative integer $i \in \mathbb{N}$ and a history $w \in W$ such that $i, w \models \varphi$. Then the tableau whose only node (i, w) is labelled by φ, and whose relations (R_{SPi}) and $R_{\preccurlyeq}$ are reduced to singleton $\{(w, w)\}$, is satisfiable (with the identity function as a tableau interpretation). Then, by lemma 4.6, the application of any rule provides a satisfiable tableau. Since a closed tableau is obviously unsatisfiable, we can generate a (possibly infinite) open tableau whose root is labelled by φ. ■

Theorem 4.7 (Completeness) *If there is an open tableau whose root (i, w) is labeled by φ, then φ is satisfiable.*

Proof Let $T = (W, M, v_0, L, R_{\preccurlyeq}, (R_{SPi})_{i \in M})$ be an open tableau whose root v_0 is labeled by ϕ. We build a model $(\hat{W}, \hat{\preccurlyeq}, V)$ from T such that for every $w \in W$ and $i \in M$, $i, w : \varphi$ iff $i, \hat{w} \models \varphi$. We define

$$\hat{W} \stackrel{def}{=} W \qquad \text{and} \qquad \hat{\preccurlyeq} \stackrel{def}{=} R_{\preccurlyeq}$$

We can now define the valuation V as follows, for any $i \in \mathbb{N}, w \in \hat{W}$:

- if $i \in M$ then $V(i, w) \stackrel{def}{=} \{p \in P \; / \; (i, w : p)\}$
- if $i \notin M$ then $V(i, w) \stackrel{def}{=} \emptyset$

We now prove by induction on the structure of φ that for every $i \in M$, $w \in W$, if $i, w : \varphi$ then $i, w \models \varphi$ (in the model $(\hat{W}, \hat{\preccurlyeq}, V)$). Cases $\varphi_1 \vee \varphi_2$, $\neg(\varphi_1 \vee \varphi_2)$, $X\varphi$, $\neg X\varphi$, are obvious.

- Suppose $i, w :< SP > \varphi$. Rule Π_{SP} ensures the existence of a node i, w' labeled by φ such that (w, w') in R_{SPi}. Then, since T is open, $\forall p \in P$, $\forall j < i$, $j, w : p$ iff $j, w' : p$. So, $\forall j < i$ $V(j, w) = V(j, w')$, and $i, w' \models \varphi$ (by induction hypothesis). So $i, w \models < SP > \varphi$.
  The proof for $< SP \cap \preccurlyeq > \varphi$ is similar.
- Suppose $i, w : [SP]\varphi$. Let $w' \in \hat{W}$ be a history such that $\forall j < i$, $V(j, w) = V(j, w')$. Thanks to rule *update_SP*, $(w, w') \in R_{SPi}$. By rule K_{SP}, we have that $i, w' : \varphi$. By the induction hypothesis, $i, w' \models \varphi$, and thus $i, w \models [SP]\varphi$.
  The proof for $[SP \cap \preccurlyeq]\varphi$ is similar.

■

4.4 Termination

We now define a terminating strategy which is still sound and complete. Termination is based on loop detection. Although it is clear that the number of created instants is bounded by the modal depth of φ with respect to X, the tableau construction may create an infinite number of histories. We have to block the creation of new histories when a loop is detected. Since we have two modal operators that can create new histories, we define looping histories with respect to each one.

- a history w is looping with respect to $< SP >$ if
  - · rule Π_{SP} is applicable in (i, w), for some $i \in M$
  - · w has been created by rule Π_{SP} at instant i and there exists an older history w' such that $(w', w) \in R_{SPi}$ and $L(i, w) \subseteq L(i, w')$ (such a history w' is denoted by $loop_{SP}(w)$)
- a history w is looping with respect to $< SP\cap \preccurlyeq>$ if
  - · rule $\Pi_{SP\cap\preccurlyeq}$ is applicable in (i, w), for some $i \in M$
  - · w has been created by rule $\Pi_{SP\cap\preccurlyeq}$ at instant i and there exists an older history w' such that $(w', w) \in R_{SPi}$, $(w', w) \in R_{\preccurlyeq}$ and $L(i, w) \subseteq L(i, w')$ (such a history w' is denoted by $loop_{SP\cap\preccurlyeq}(w)$)

Definition 4.8 [Strategy] Let us consider the algorithm which consists in applying successively the following steps while the tableau is not closed, starting from the tableau such that $W = \{w\}$, $M = \{0\}$, $v_0 = (0, w)$, $L(v_0) = \{\phi\}$, $R_{\preccurlyeq} = \{(w, w)\}$, and $R_{SP0} = \{(w, w)\}$.

- Application of classical rules $\neg\neg$, α, β as much as possible.
- Loop detection step for $< SP >$: *mark* every looping history with respect to $< SP >$.
- Loop detection step for $< SP\cap \preccurlyeq>$: *mark* every looping history with respect to $< SP\cap \preccurlyeq>$.
- Application of rules Π_{SP} and $\Pi_{SP\cap\preccurlyeq}$ on every state on which they have not already been applied, and which is not marked with respect to $< SP >$ and $< SP\cap \preccurlyeq>$, respectively.
- Application of rule *saturation* and then rule *update_SP* as much as possible.
- Application of rule $\preccurlyeq$ *−totality* on every pair (w, w') on which it has not been applied.
- Application of rules X, $\neg X$, K_{SP}, and $K_{SP\cap\preccurlyeq}$ as much as possible.

Proposition 4.9 (Termination) *The strategy given above terminates.*

Proof First, remark that

(1) M is finite (bounded by the modal depth of the initial formula ϕ with respect to X)

(2) there are finitely many sets of sub-formulas of the initial formula ϕ

We show that there cannot be an infinite sequence of histories $(w_0, w_1, w_2, \ldots)$ such that each w_{k+1} is created by the application of rule Π_{SP} or $\Pi_{SP\cap\preccurlyeq}$ to some point of the history w_k. Indeed, suppose it is the case.

Suppose that there are infinitely many applications of rule Π_{SP}. Since there are finitely many sub-formulas of ϕ, there is a formula $\neg[SP]\varphi$ which triggers rule Π_{SP} infinitely often. Suppose that Π_{SP} is triggered infinitely often by $\neg[SP]\varphi$ at instant 0. Then, $\neg[SP]\varphi$ appears necessarily in the scope of $[SP]$ or $[SP\cap \preccurlyeq]$ in $L(0, w_k)$ for some w_k in the sequence, and we can prove that there exists k' from which $\neg[SP]\varphi$ labels every history of the sequence ($\forall k" > k'$ $\neg[SP]\varphi \in L(0, w_{k"})$). So, there is an application of Π_{SP} which creates a history w_{k_0} (at instant 0) such that

- $\neg[SP]\varphi \in L(0, w_{k_0})$
- $\exists k < k_0$ such that $L(0, w_{k_0}) \subseteq L(0, w_k)$ (because of remark (2)) and $(w_k, w_{k_0}) \in SP_0$

Therefore, w_{k_0} is looping with respect to Π_{SP}, and our strategy cannot generate such an infinite sequence. We then prove that Π_{SP} cannot be triggered infinitely often at instant 1, 2, ..., and $max(M)$. (Existence of $max(M)$ follows from remark (1).)

The same reasoning shows that there cannot be infinitely many applications of $\Pi_{SP\cap\preccurlyeq}$.

■

Proposition 4.10 *The strategy given above is sound and complete.*

Proof The soundness of the strategy obviously follows from the soundness of the tableaux system (theorem 4.3).

On the other hand, in order to prove the completeness of the strategy, the completeness proof of theorem 4.7 has to be adapted. Suppose that $T = (W, M, v_0, L, R_{\preccurlyeq}, (R_{SPi})_{i\in M})$ is an open tableau resulting from our strategy. We build a model $(\hat{W}, \hat{\preccurlyeq}, V)$ where $\hat{W}$ contains every history which is not marked as a looping history (at the last iteration of the strategy). Every pair (w, w_{loop}) in $R_{\preccurlyeq}$, where w_{loop} is a looping history with respect to $< SP >$ (resp. $< SP\cap \preccurlyeq>$), is replaced by the pair $(w, loop_{SP}(w_{loop}))$ (resp. $loop_{SP\cap\preccurlyeq}(w_{loop})$) in $\hat{\preccurlyeq}$. We then have to prove that $i, w : \varphi$ implies $i, w \models \varphi$ by induction on the structure of φ, for every non-looping history w. The proof for cases $\varphi_1 \vee \varphi_2$, $\neg(\varphi_1 \vee \varphi_2)$, $X\varphi$, $\neg X\varphi$, $[SP]\varphi$, and $[SP\cap \preccurlyeq]\varphi$ is similar to the proof of theorem 4.7. Suppose that $i, w :< SP > \varphi$. Rule Π_{SP} ensures the existence of a node i, w' labeled by φ such that $(w, w') \in R_{SPi}$. If w' is not looping, then we can conclude $i, w' \models \varphi$ as in the proof of theorem 4.7. If w' is looping with respect to $< SP >$, then we can prove that $i, loop_{SP}(w') \models \varphi$. Notice that w' cannot be lopping with respect to $< SP\cap \preccurlyeq>$ since we suppose it has been created by application of rule Π_{SP} in node i, w. The proof for case $< SP\cap \preccurlyeq>$ is similar.

■

5 Complete axiomatization

In this section, we propose an axiomatic system for our logic. For technical reasons, we enrich our language with three modal operators: X^{-1}, $[\preccurlyeq]$, and $[\not\succcurlyeq]$. X^{-1} is needed for the axiomatization of $[SP]$, and $[\preccurlyeq]$ and $[\not\succcurlyeq]$ are needed for the axiomatization of $[SP\cap \preccurlyeq]$. Our axiomatic system is complete with respect to a semantics which slightly differs from the one given in section 3. First, the time is implicitly modeled by the set $\mathbb{Z}$ of the integers instead of the set $\mathbb{N}$ of the non-negative integers. Second, we drop the constraint of totality of the quasi-ordering $\preccurlyeq$. We call L_{min} the logic defined by this semantics, and whose language contains the modal operators $X, X^{-1}, [SP], [SP\cap \preccurlyeq], [\preccurlyeq], [\succcurlyeq]$. Let us give the semantics of these new operators.

$$\begin{array}{lllllll} i, w \models X^{-1}\varphi & \text{iff} & i-1, w \models \varphi & & & & \\ i, w \models [\preccurlyeq]\varphi & \text{iff} & \forall w' \in W & \text{if} & w \preccurlyeq w' & \text{then} & i, w' \models \varphi \\ i, w \models [\succcurlyeq]\varphi & \text{iff} & \forall w' \in W & \text{if} & w' \preccurlyeq w & \text{then} & i, w' \models \varphi \end{array}$$

In this section we will propose an axiomatic system for L_{min}. For all formulas φ, we define $X^0\varphi \stackrel{def}{=} \varphi$, for each positive integer i, $X^i\varphi \stackrel{def}{=} X^{i-1}X\varphi$, and for each negative integer i, $X^i\varphi \stackrel{def}{=} X^{i+1}X^{-1}\varphi$.

5.1 Admissible forms

For the definition of the special rules of inference, we will need expressions of a special form, called admissible forms, denoted by capital Latin letters A, B, etc. Let the language of L_{min} be extended with a new atomic proposition $\sharp$. Admissible forms are defined by the following rule:

$$A ::= \sharp \mid (\varphi \Rightarrow A) \mid XA \mid X^{-1}A \mid [SP]A \mid [\preccurlyeq]A \mid [\succcurlyeq]A \mid [SP\cap \preccurlyeq]A.$$

Note that in each admissible form, $\sharp$ has a unique occurrence. Let A be an admissible form and φ be a formula. The result of the replacement of the unique occurrence of $\sharp$ in its place in A with φ will be denoted by $A(\varphi)$.

5.2 Axiomatization

Our axiomatic system for L_{min} is based on the following set of axioms and rules of inference:

Axioms

(A0) Classical tautologies are axioms.

(K) For all $L \in \{X, X^{-1}, [SP], [\preccurlyeq], [\succcurlyeq], [SP\cap \preccurlyeq]\}$,
$L(\varphi_1 \Rightarrow \varphi_2) \Rightarrow (L\varphi_1 \Rightarrow L\varphi_2)$.

(A1) $\neg X\varphi \Leftrightarrow X\neg\varphi$, $\varphi \Rightarrow XX^{-1}\varphi$.

(A2) $\neg X^{-1}\varphi \Leftrightarrow X^{-1}\neg\varphi$, $\varphi \Rightarrow X^{-1}X\varphi$.

(A3) $[SP]\varphi \Rightarrow \varphi$, $[SP]\varphi \Rightarrow [SP][SP]\varphi$, $\varphi \Rightarrow [SP]\langle SP\rangle\varphi$.

(A4) $[\preccurlyeq]\varphi \Rightarrow \varphi$, $[\preccurlyeq]\varphi \Rightarrow [\preccurlyeq][\preccurlyeq]\varphi$, $\varphi \Rightarrow [\preccurlyeq]\langle\succcurlyeq\rangle\varphi$.

(A5) $[\succcurlyeq]\varphi \Rightarrow \varphi$, $[\succcurlyeq]\varphi \Rightarrow [\succcurlyeq][\succcurlyeq]\varphi$, $\varphi \Rightarrow [\succcurlyeq]\langle\preccurlyeq\rangle\varphi$.

(A6) if $i < j$ then for all $p \in P$, the following formulas are axioms:
$X^ip \Rightarrow X^j[SP]X^{i-j}p$, $X^i\neg p \Rightarrow X^j[SP]X^{i-j}\neg p$.

(A7) $X[\preccurlyeq]\varphi \Leftrightarrow [\preccurlyeq]X\varphi$, $X^{-1}[\preccurlyeq]\varphi \Leftrightarrow [\preccurlyeq]X^{-1}\varphi$.

(A8) $X[\succcurlyeq]\varphi \Leftrightarrow [\succcurlyeq]X\varphi$, $X^{-1}[\succcurlyeq]\varphi \Leftrightarrow [\succcurlyeq]X^{-1}\varphi$.

(A9) $[SP]\varphi \vee [\preccurlyeq]\varphi \Rightarrow [SP\cap \preccurlyeq]\varphi$.

Rules of inference

Modus ponens: From φ_1 and $\varphi_1 \Rightarrow \varphi_2$ infer φ_2.

necessitation: For all $L \in \{X, X^{-1}, [SP], [\preccurlyeq], [\succcurlyeq], [SP\cap \preccurlyeq]\}$, from φ infer $L\varphi$.

$[SP]$ **special rule:** If $\Box \in \{[\preccurlyeq], [\succcurlyeq]\}^\star$ and $i < 0$, then from $\{A(\neg\Box(\varphi \vee X^i p) \vee X^i p) \ : \ p \in P\}$ infer $A(\neg[SP]\varphi)$.

$[SP\cap \preccurlyeq]$ **special rule:**
From $\{A(\langle SP\rangle(\varphi \wedge p) \vee \langle\preccurlyeq\rangle(\varphi \wedge \neg p)) \ : \ p \in P\}$ infer $A(\langle SP\cap \preccurlyeq\rangle\varphi)$.

Special rules are needed because of two non-standard aspects of our logic:

- the semantic relation associated with $[SP]$ refers to the valuation of a given model
- operator $[SP\cap \preccurlyeq]$ corresponds to the intersection of two semantic relations

Their origin is more technical that intuitive: they have been exhibited so that the truth lemma (lemma 5.5) can be proved for formulas of the form $[SP]\varphi$ and $[SP\cap \preccurlyeq]\varphi$. Special rule $[SP\cap \preccurlyeq]$ follows the idea already developed in [25,4] to give a complete axiomatization for the intersection of some semantic relations. Although intersection is not modally definable in ordinary quantifier-free modal languages, it becomes definable in languages with propositional quantifiers. Indeed, the following quantified axiom modally defines semantic intersection.

$$< R_1 \cap R_2 > \varphi \ \Leftrightarrow \ \forall p \ (< R_1 > (\varphi \wedge p) \vee < R_2 > (\varphi \wedge \neg p)))$$

Rule $[SP\cap \preccurlyeq]$ 'simulates' right to left direction while axiom (A9) corresponds to the left to right direction.

A formula φ is a theorem of L_{min} if it belongs to the least set of formulas containing all axioms and closed under the rules of inference.

5.3 Soundness and completeness

Theorem 5.1 (Soundness of L_{min}) *Let φ be a formula. If φ is a theorem of L_{min} then φ is valid in every model.*

Proof

By induction on the length of a deduction of φ in L_{min}, we that φ is valid in every model. We only develop the special rule cases .

We treat the case where admissible form is ♮.

$[SP]$ special rule: Let $\Box \in \{[\preccurlyeq], [\succcurlyeq]\}^*$ and i¡0. Let φ be a formula such that $\forall p \in P \ \Box(\varphi \vee X^i p) \Rightarrow X^i p$ is a valid. We show that $\neg[SP]\varphi$ is valid. Suppose that it is not the case: there is a model $(W, \preccurlyeq, V)$, and a state $j, w \in \mathbb{Z} \times W$ such that $j, w \models [SP]\varphi$. Let p be an atomic proposition which does not appear in φ. Let V' a valuation such that $V^{-1}(p) = \{(j+i, w') \ / \ \neg\text{SamePast}((j, w), (j, w'))\}$. Considering the model $(W, \preccurlyeq, V')$, we have $j, w \models \Box(\varphi \vee X^i p)$. Indeed, let w' a history accessible from w by the composition of relations corresponding to $\Box$. Either (j, w') has the same past as (j, w) and $j, w \models \varphi$, or (j, w') has not the same past as (j, w), and

$j, w \models X^i p$. Thus, we deduce that $j, w \models X^i p$. This is in contradiction with the definition of V' since (j, w) has the same past has itself.

$[SP\cap \preccurlyeq]$ special rule: Suppose that there is a model $M = (W, \preccurlyeq, V)$, and a state (i, w) in M such that $i, w \models [SP\cap \preccurlyeq]\varphi$. We have to show that $\exists p \in P$ and $\exists M', (i', w')$ such that $i', w' \models [SP](\varphi \vee p) \wedge [\preccurlyeq](\varphi \vee \neg p)$.

Consider an atom p which does not appear in φ. Let us define a valuation V' such that $V'^{-1}(p) = \{(i, w') \ / \ \text{SamePast}((i, w), (i, w')) \text{ and } \neg(w \preccurlyeq w')\}$, and $V'^{-1}(q) = V^{-1}(q) \forall q \neq p$. Then, in the model $(W, \preccurlyeq, V')$, $i, w \models [SP](\varphi \vee p) \wedge [\preccurlyeq](\varphi \vee \neg p)$. ■

Theorem 5.2 (Completeness of L_{min}) *Let φ be a formula. If φ is valid in every model then φ is a theorem of L_{min}.*

The completeness of L_{min} is more difficult to establish than its soundness and we defer proving that L_{min} is complete with respect to the class of all models till section 5.5.

5.4 Theories

In this section we introduce the notions of theories and maximal theories, the latter having a key role in the proof of the completeness theorem. A set x of formulas is called a theory if it satisfies the following conditions:

(th 1) x contains the set of all theorems of L_{min}.

(th 2) x is closed under modus ponens.

(th 3) x is closed under the $[SP]$ special rule.

(th 4) x is closed under the $[SP\cap \preccurlyeq]$ special rule.

Obviously the smallest theory is the set TH_{min} of all theorems and the greatest theory is the set of all formulas. The later theory is called trivial theory. A theory x is called consistent if $\bot \notin x$, otherwise it is called inconsistent. It is a well-known fact that a theory x is consistent iff it is not trivial and that x is inconsistent if it contains a formula φ together with its negation $\neg\varphi$. A theory x is called a maximal theory if it is consistent and for any formula φ: $\varphi \in x$ or $\neg\varphi \in x$. A set Σ of formulas is called consistent if it is contained in a consistent theory. It can be shown that a single formula φ is consistent (considered as a singleton $\{\varphi\}$), iff it is not equivalent to $\bot$. In the literature, instead of maximal theory, the notion of a maximal consistent set is used, where consistency is defined without using the notion of theory. It can be proved that each maximal theory is a maximal consistent set in the classical sense, and each maximal consistent set which is closed under the special rules for $[SP]$ and $[SP\cap \preccurlyeq]$ is a maximal theory. We will use the following properties of maximal theories without explicit reference (x is a maximal theory):

- $\top \in x$
- $\neg\varphi \in x$ iff $\varphi \notin x$,
- $\varphi_1 \vee \varphi_2 \in x$ iff $\varphi_1 \in x$ or $\varphi_2 \in x$,
- $\varphi_1 \wedge \varphi_2 \in x$ iff $\varphi_1 \in x$ and $\varphi_2 \in x$.

Let x be a set of formulas. If $L \in \{X, X^{-1}, [SP], [\preccurlyeq], [\succcurlyeq], [SP\cap \preccurlyeq]\}$ then define $Lx = \{\varphi : \ L\varphi \in x\}$. If φ is a formula then define $x + \varphi = \{\varphi' : \ \varphi \Rightarrow \varphi' \in x\}$. For all sets x of formulas, we define $X^0 x = x$,for each positive integer i, $X^i\varphi \stackrel{def}{=} X^{i-1}X\varphi$, and for each negative integer i, $X^i\varphi \stackrel{def}{=} X^{i+1}X^{-1}\varphi$. In the next lemma we summarize some properties of theories.

Lemma 5.3 *Let x be a theory. The following statements hold.*

(i) *Lx is a theory too.*

(ii) *$x + \varphi$ is the smallest theory containing x and φ.*

(iii) *$x + \varphi$ is inconsistent iff $\neg\varphi \in x$.*

(iv) *If x is consistent and $\neg A(\neg[SP]\varphi) \in x$ then for all $\Box \in \{[\preccurlyeq], [\succcurlyeq]\}^\star$, and for all $i < 0$, there exists $p \in P$ such that $x + \neg A(\neg\Box(\varphi \vee X^i p) \vee X^i p)$ is consistent.*

(v) *If x is consistent and $\neg A(\langle SP\cap \preccurlyeq\rangle\varphi) \in x$ then there exists $p \in P$ such that $x + \neg A(\langle SP\rangle(\varphi \wedge p) \vee \langle\preccurlyeq\rangle(\varphi \wedge \neg p))$ is consistent.*

Proof We show statements (i) and (iv).

Statement 1. Let φ be a theorem. Then by the necessitation rules, $L\varphi$ is a theorem too. Hence, $L\varphi \in x$, so $\varphi \in Lx$. Thus, Lx contains the set of all theorems. Let $\varphi_1 \in Lx$ and $\varphi_1 \Rightarrow \varphi_2 \in Lx$. Then $L\varphi_1 \in x$ and $L(\varphi_1 \Rightarrow \varphi_2) \in x$. By the axiom (K), $L(\varphi_1 \Rightarrow \varphi_2) \Rightarrow (L\varphi_1 \Rightarrow L\varphi_2) \in x$. Applying modus ponens twice, we obtain that $L\varphi_2 \in x$, so $\varphi_2 \in Lx$. Thus Lx is closed under modus ponens.
To show that Lx is closed under the $[SP]$ special rule, let $\Box \in \{[\preccurlyeq], [\succcurlyeq]\}^\star$ and $i < 0$. Suppose that we have $A(\neg\Box(\varphi \vee X^i p) \vee X^i p) \in Lx$. Then, for all $p \in P$, we obtain $LA(\neg\Box(\varphi \vee X^i p) \vee X^i p) \in x$. Notice that $LA(\neg\Box(\varphi \vee X^i p) \vee X^i p)$ is an admissible form. Since x is closed under the $[SP]$ special rule, we obtain $LA(\neg[SP]\varphi) \in x$. Hence, $A(\neg[SP]\varphi) \in Lx$. Thus, Lx is closed under the $[SP]$ special rule.
Similarly, one can prove that Lx is closed under the $[SP\cap \preccurlyeq]$ special rule.

Statement 4. Suppose that $\neg A(\neg[SP]\varphi) \in x$. Since x is consistent, then $A(\neg[SP]\varphi) \notin x$. Thus, since x is closed under the $[SP]$ special rule, then for all $\Box \in \{[\preccurlyeq], [\succcurlyeq]\}^*$ and for all $i < 0$, there exists $p \in P$ such that $A(\neg\Box(\varphi \vee X^i p) \vee X^i p) \notin x$. (Otherwise, $A(\neg[SP]\varphi)$ would necessarily be in x.) Since x is a theory, $\neg A(\neg\Box(\varphi \vee X^i p) \wedge \neg X^i p) \in x$. From statement 3, we deduce that $x + \neg A(\neg\Box(\varphi \vee X^i p) \vee X^i p)$ is consistent.

The proof of statement (v) is similar.

■

Now we are ready for the main lemma in this section:

Lemma 5.4 *Each consistent theory can be extended to a maximal theory.*

Proof Suppose x is a consistent theory and let φ_0, φ_1, ... be an enumeration of all formulas. We define an increasing sequence of consistent theories x_0, x_1, ... by induction as follows. Let $x_0 = x$ and suppose that for some integer n, the consistent theory x_n has already been defined. For the definition of x_{n+1} we consider two cases.
Case 1: $x_n + \varphi_n$ is consistent. Then define $x_{n+1} = x_n + \varphi_n$.
Case 2: $x_n + \varphi_n$ is not consistent. Then $\neg\varphi_n \in x$. In this case we consider two sub-cases:

Sub-case 2.1: φ_n is neither in the form of a conclusion of the $[SP]$ special rule nor in the form of a conclusion of the $[SP\cap \preccurlyeq]$ special rule. Then let $x_{n+1} = x_n$.
Sub-case 2.2: φ_n is in the form of a conclusion of the $[SP]$ special rule or in the form of a conclusion of the $[SP\cap \preccurlyeq]$ special rule. We only consider the case where φ_n is in the form of a conclusion of the $[SP\cap \preccurlyeq]$ special rule, i.e. φ_n is in the following form $A(\langle SP\cap \preccurlyeq\rangle\varphi)$ where A is an admissible form. Therefore, there are finitely many such representations for φ_n: $A_i(\langle SP\cap \preccurlyeq\rangle\varphi_i)$ for $i = 1,\ldots,k$. We define inductively an increasing sequence of consistent theories x_n^i for $i = 0,\ldots,k$, as follows. Let $x_n^0 = x_n$. Suppose x_n^i is defined and consistent. Then it contains $\neg\varphi_n = \neg A_i(\langle SP\cap \preccurlyeq\rangle\varphi_i)$ and, by the properties of theories mentioned above, there exists a propositional variable $p_i \in P$ such that $x_n^i + \neg A_i(\langle SP\rangle(\varphi_i \wedge p) \vee \langle\preccurlyeq\rangle(\varphi_i \wedge \neg p))$ is consistent. We define x_n^{i+1} as follows: $x_n^{i+1} = x_n^i + \neg A_i(\langle SP\rangle(\varphi_i \wedge p) \vee \langle\preccurlyeq\rangle(\varphi_i \wedge \neg p))$. Now, we put $x_{n+1} = x_n^k$.
Finally, we define $y = \bigcup_{i=0}^{\infty} x_i$. It is straightforward to demonstrate that y is a maximal theory which extends x. ■

5.5 Canonical model construction

The canonical model of L_{min} is the structure $\mathcal{M}_c = (W_c, \preccurlyeq_c, V_c)$ defined as follows:

- W_c is the set of all maximal theories,
- $\preccurlyeq_c$ is the binary relation on W_c defined by $x \preccurlyeq_c y$ iff $[\preccurlyeq]x \subseteq y$,
- V_c is the function which associates each pair $(i,x) \in \mathbb{Z} \times W_c$ with the set $V_c(i,x) = \{p :\ X^i p \in x\}$ of atomic propositions.

To prove the completeness of our axiomatic system, it suffices to demonstrate the following lemma.

Lemma 5.5 *Let φ be a formula. For all integers $i \in \mathbb{Z}$ and for all maximal theories $x \in W_c$, $\mathcal{M}_c,(i,x) \models \varphi$ iff $X^i\varphi \in x$.*

Proof The proof is done by induction on the complexity of φ. We only consider the cases $\varphi = L\phi$ for $L \in \{X, X^{-1}, [SP], [\preccurlyeq], [\succcurlyeq], [SP\cap \preccurlyeq]\}$.
Case $\varphi = X\phi$. Assume $\mathcal{M}_c,(i,x) \models X\varphi$. Consequently, $\mathcal{M}_c,(i+1,x) \models \varphi$. By induction hypothesis, $X^{i+1}\varphi \in x$. Hence, $X^i X\varphi \in x$. Reciprocally, assume $X^i X\varphi \in x$. Therefore, $X^{i+1}\varphi \in x$ and, by induction hypothesis, $\mathcal{M}_c,(i+1,x) \models \varphi$. Thus, $\mathcal{M}_c,(i,x) \models X\varphi$.
Case $\varphi = X^{-1}\phi$. Similar to the previous case.
Case $\varphi = [SP]\phi$. Assume $\mathcal{M}_c,(i,x) \models [SP]\varphi$. For the sake of the contradiction, assume $X^i[SP]\varphi \notin x$. Consequently, $[SP]\varphi \notin X^i x$ and $\varphi \notin [SP]X^i x$. Hence, the theory $[SP]X^i x + \neg\varphi$ is consistent. By Lindenbaum's lemma, there exists a maximal theory y such that $[SP]X^i x + \neg\varphi \subseteq y$. Remark that $[SP]X^i x \subseteq y$ and $\neg\varphi \in y$. Let $z = X^{-i}y$. Remark that $X^i z = y$. Since $\neg\varphi \in y$, then $X^{-i}X^i\neg\varphi \in y$ and $X^i\neg\varphi \in z$. Therefore, $X^i\varphi \notin z$ and, by induction hypothesis, $\mathcal{M}_c,(i,z) \not\models \varphi$. Since $\mathcal{M}_c,(i,x) \models [SP]\varphi$, then x and z do not have the same past with respect to i. Thus, there exists an integer $j \in \mathbb{Z}$ such that $i > j$ and for some atomic proposition p, either $X^j p \in x$ and $X^j p \notin z$ or $X^j p \notin x$ and $X^j p \in z$. Without loss of generality, let us suppose that $X^j p \in x$ and $X^j p \notin z$. Remark that $[SP]X^i x \subseteq X^i z$. Since $X^j p \notin z$, then $X^{j-i}p \notin X^i z$. Since $[SP]X^i x \subseteq X^i z$, then $[SP]X^{j-i}p \notin X^i x$.

Consequently, we have $X^jp \in x$ and $X^i[SP]X^{j-i}p \notin x$: a contradiction with $i > j$ and axiom $(A6)$. Reciprocally, assume that $X^i[SP]\varphi \in x$ and let us show that $\mathcal{M}_c, (i, x) \models [SP]\varphi$. For the sake of the contradiction, assume that $\mathcal{M}_c, (i, x) \not\models [SP]\varphi$. Consequently, there exists $y \in W_c$ such that x and y have the same past with respect to i and $\mathcal{M}_c, (i, y) \not\models \varphi$. By induction hypothesis, $X^i\varphi \notin y$ and $\varphi \notin X^iy$. Since $X^i[SP]\varphi \in x$, then $\neg[SP]\varphi \notin X^ix$. Let $\Box \in \{[\preccurlyeq], [\succcurlyeq]\}^\star$ be such that $\Box x \subseteq y$ and $j \in \mathbb{Z}$ be such that $i > j$. Remark that $j - i < 0$. Since X^ix is a theory, then X^ix is closed under the $[SP]$ special rule. Since $\neg[SP]\varphi \notin X^ix$, then there exists an atomic proposition p such that $\neg\Box(\varphi \vee X^{j-i}p) \vee X^{j-i}p \notin X^ix$. Therefore, $X^i\Box(\varphi \vee X^{j-i}p) \in x$ and $X^j\neg p \in x$. Thus, $\Box(X^i\varphi \vee X^jp) \in x$. Since $\Box x \subseteq y$, then $X^i\varphi \in y$ or $X^jp \in y$. If $X^i\varphi \in y$ then $\varphi \in X^iy$: a contradiction. If $X^jp \in y$ then $X^jp \in x$, seeing that x and y have the same past with respect to i and $i > j$. This contradicts the fact that $X^j\neg p \in x$.
Case $\varphi = [SP\cap \preccurlyeq]\phi$. Similar to the previous case (use the special rule for $[SP\cap \preccurlyeq]$ and the axiom $(A9)$ instead of the special rule for $[SP]$ and the axiom $(A6)$.
Case $\varphi = [\preccurlyeq]\phi$. Assume $\mathcal{M}_c, (i, x) \models [\preccurlyeq]\varphi$. For the sake of the contradiction, assume $X^i[\preccurlyeq]\varphi \notin x$. Consequently, $[\preccurlyeq]\varphi \notin X^ix$ and $\varphi \notin [\preccurlyeq]X^ix$. Hence, the theory $[\preccurlyeq]X^ix + \neg\varphi$ is consistent. By Lindenbaum's lemma, there exists a maximal theory y such that $[\preccurlyeq]X^ix + \neg\varphi \subseteq y$. Remark that $[\preccurlyeq]X^ix \subseteq y$ and $\neg\varphi \in y$. Let $z = X^{-i}y$. Remark that $X^iz = y$. Since $\neg\varphi \in y$, then $X^{-i}X^i\neg\varphi \in y$ and $X^i\neg\varphi \in z$. Therefore, $X^i\varphi \notin z$ and, by induction hypothesis, $\mathcal{M}_c, (i, z) \not\models \varphi$. Since $\mathcal{M}_c, (i, x) \models [\preccurlyeq]\varphi$, then $x \not\preccurlyeq_c z$. Thus, there exists a formula ψ such that $[\preccurlyeq]\psi \in x$ and $\psi \notin z$. Hence, $X^{-i}\psi \notin y$, $X^{-i}\psi \notin [\preccurlyeq]X^ix + \neg\varphi$, $X^{-i}\psi \notin [\preccurlyeq]X^ix$ and $X^i[\preccurlyeq]X^{-i}\psi \notin x$. Thus, $[\preccurlyeq]X^iX^{-i}\psi \notin x$ and $[\preccurlyeq]\psi \notin x$: a contradiction. Reciprocally, assume that $X^i[\preccurlyeq]\varphi \in x$ and let us show that $\mathcal{M}_c, (i, x) \models [\preccurlyeq]\varphi$. For the sake of the contradiction, assume that $\mathcal{M}_c, (i, x) \not\models [\preccurlyeq]\varphi$. Consequently, there exists $y \in W_c$ such that $x \preccurlyeq_c y$ and $\mathcal{M}_c, (i, y) \not\models \varphi$. By induction hypothesis, $X^i\varphi \notin y$. Since $x \preccurlyeq_c y$, then $[\preccurlyeq]x \subseteq y$. Consequently, $X^i\varphi \notin [\preccurlyeq]x$ and $[\preccurlyeq]X^i\varphi \notin x$. Hence, $X^i[\preccurlyeq]\varphi \notin x$: a contradiction.
Case $\varphi = [\succcurlyeq]\phi$. Similar to the previous case. ■

Now, we are ready for proving the main theorem of this section.

Proof of theorem 5.2. Let φ be a formula. Assume φ is not a theorem of L_{min}. Consequently, $TH_{min} + \neg\varphi$ is a consistent theory. By Lindenbaum's lemma, there exists a maximal theory x such that $TH_{min} + \neg\varphi \subseteq x$. Hence, $\neg\varphi \in x$, $\varphi \notin x$ and $X^0\varphi \notin x$. By the lemma 5.5, $\mathcal{M}_c, (0, x) \not\models \varphi$. Thus, φ is not valid.

6 Conclusion

We have designed a tableaux method which can handle the fragment of our language restricted to until-free formulas. Our method is sound and complete. A termination strategy based on loop detection shows that our tableaux method can be implemented as a decision procedure. We have also given a sound and complete axiomatization of the set of formulas valid in the class of all models, without considering the $\preccurlyeq$-totality constraint. Remark that this set of valid formulas is not closed with respect to the rule of uniform substitution, seeing that the formula $X^{-1}p \to [SP]X^{-1}p$ is valid for any atomic proposition p whereas the formula $X^{-1}Xp \to [SP]X^{-1}Xp$ (or, equivalently, the formula $p \to [SP]p$) is not valid. To the best of our knowledge, this is the first attempt at developing a decision procedure and an axiom system for a modal logic with such an operator as $[SP]$.

Plans for future work include: developing a decision procedure for the language that includes the until operator ; designing an axiom system that is sound and complete with respect to validity in the class of all models $(W, \preceq, V)$ where $\preceq$ is total ; adapting our tableaux method and our axiom system to the case where the set P of all atomic propositions is infinite. Note that if P is infinite then the inference rules for the modalities $[SP]$ and $[SP \cap \preceq]$ become infinitary, i.e. they both need infinitely many preconditions before one can apply them. In this case, an open question is whether one can replace these infinitary inference rules by finitary ones.

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