Static Analysis of Control Command Systems: Synthesizing non Linear Invariants

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Digital Flight Commands

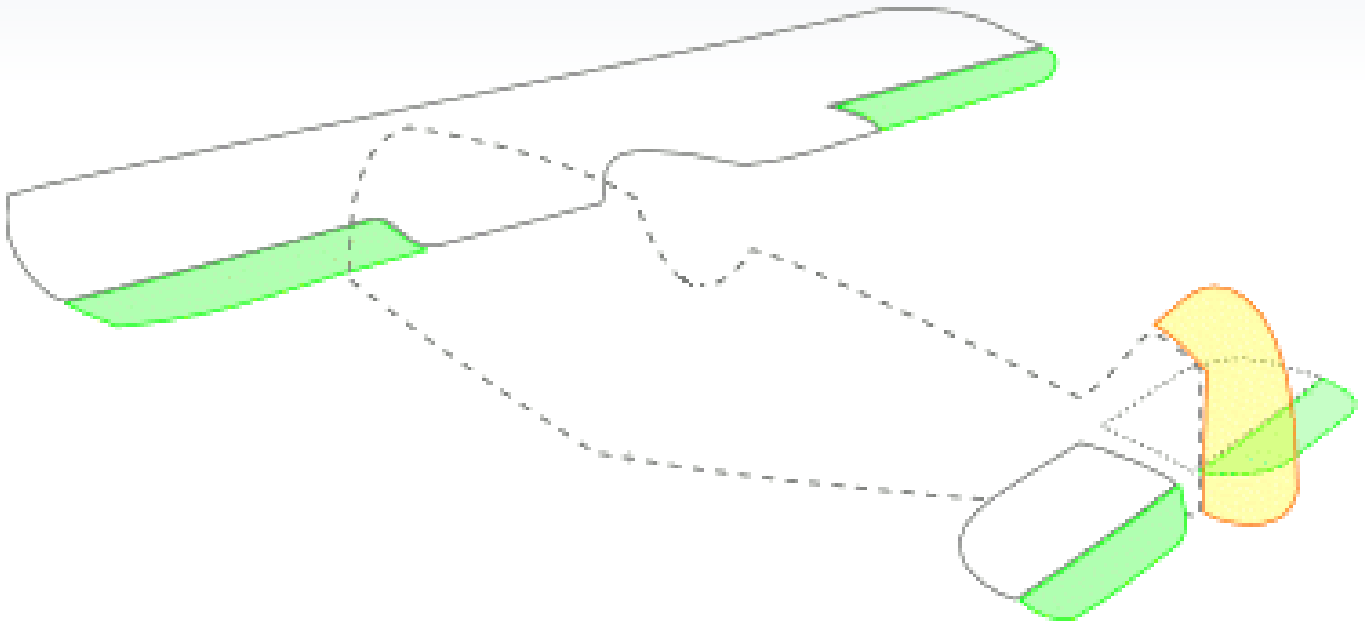

Image: Piotr Jaworski (GFDL)

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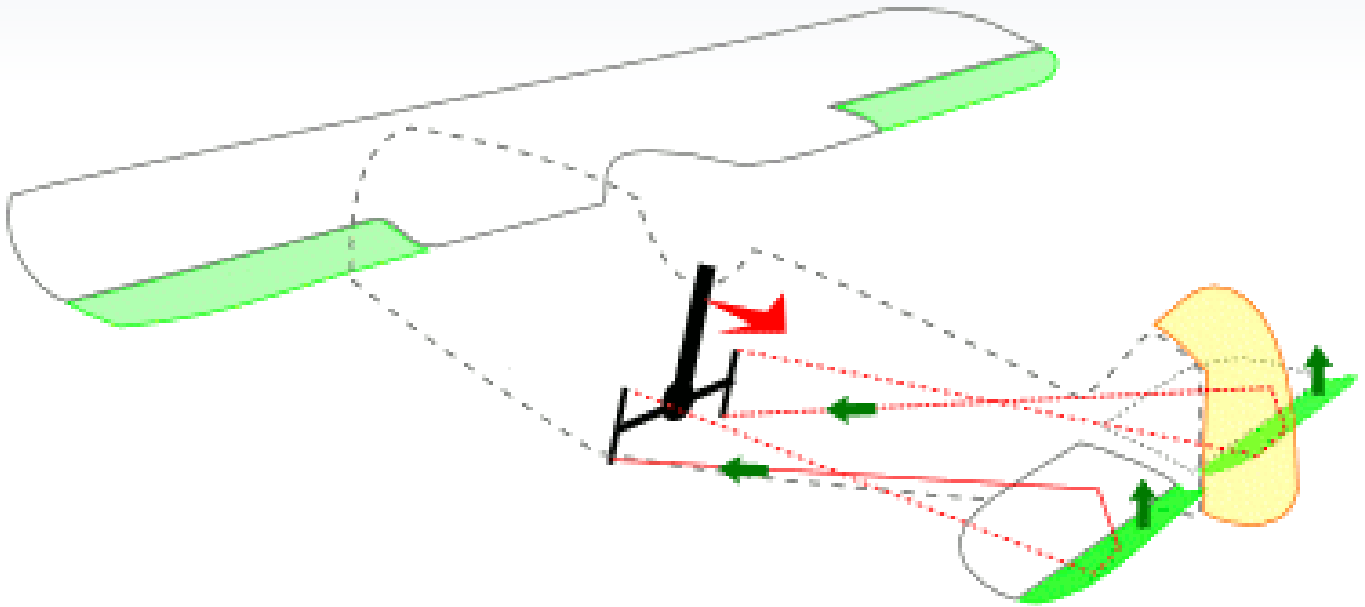

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Cables

Image: public domain

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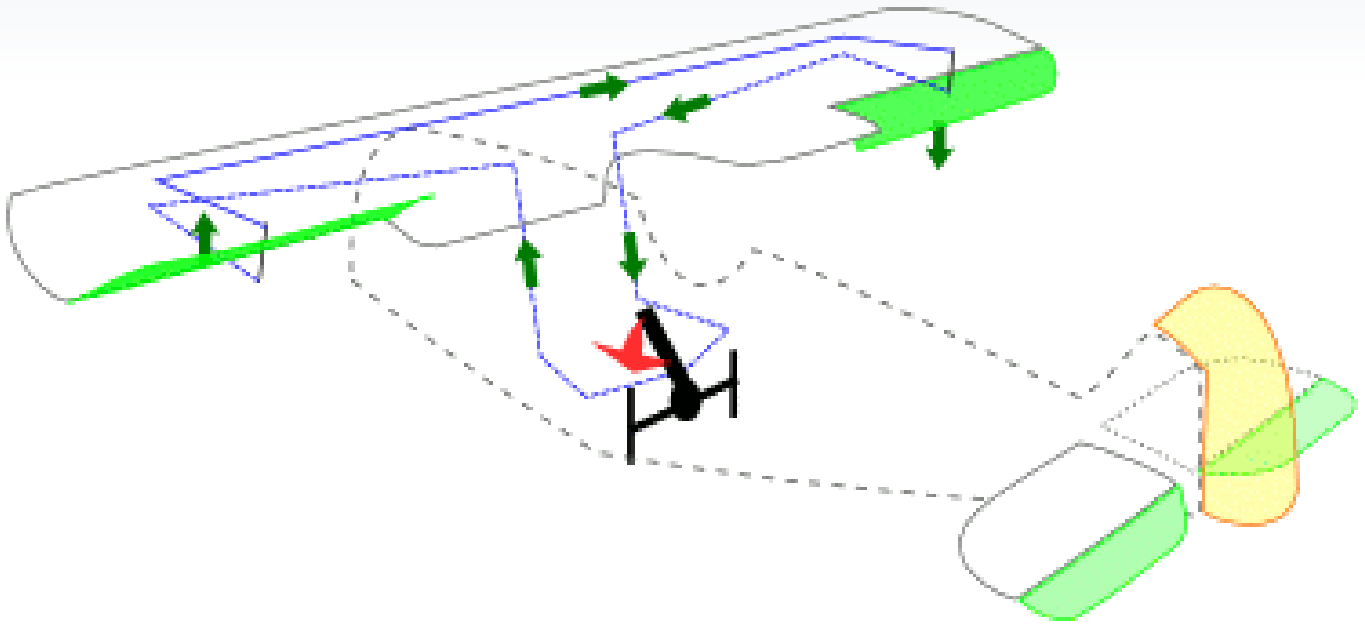

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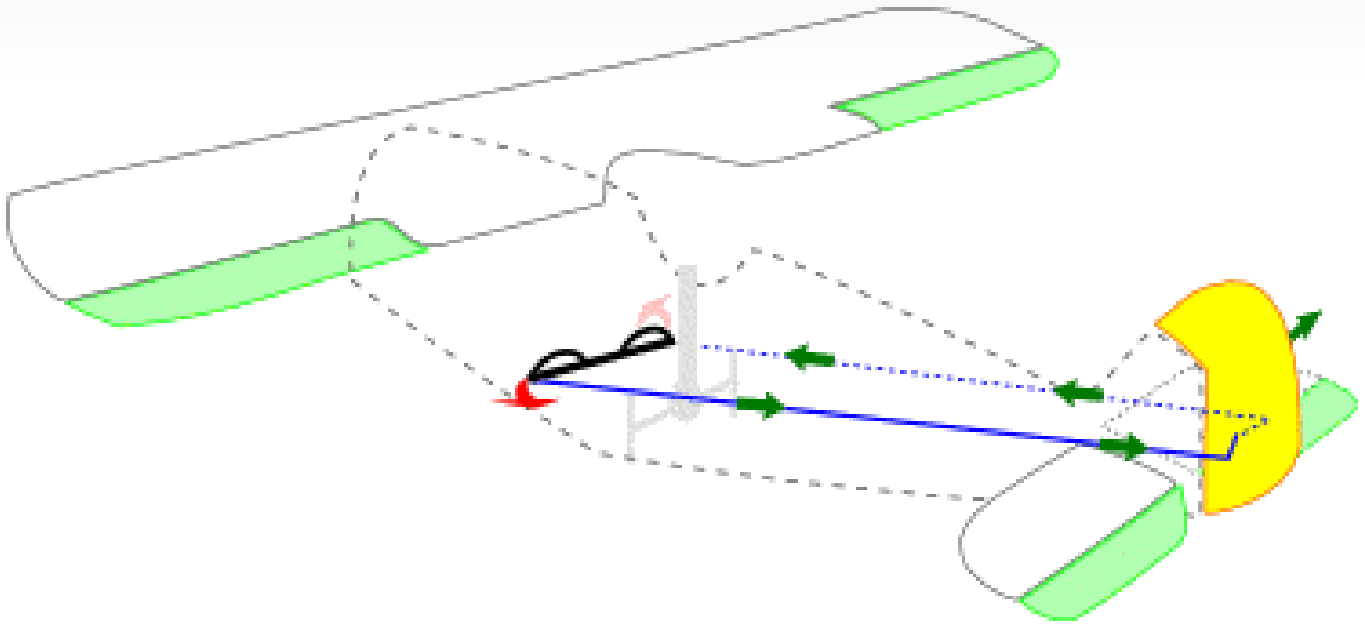

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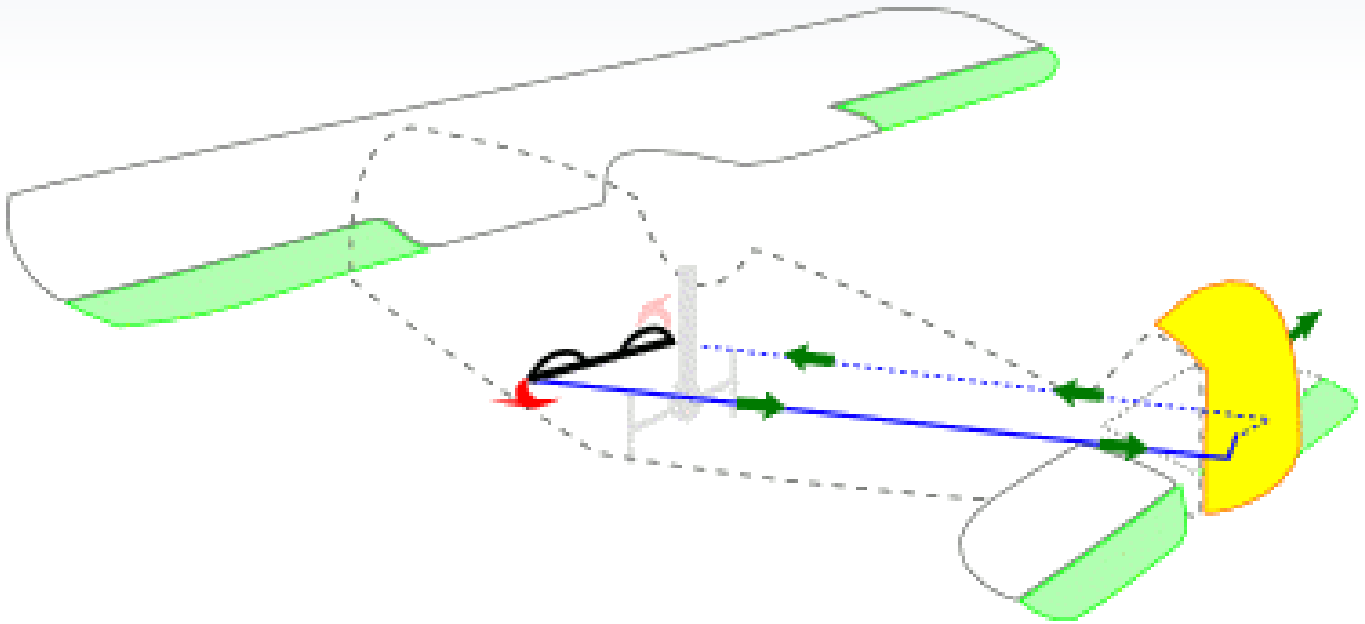

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Hydraulic Actuators

Image: Woodward

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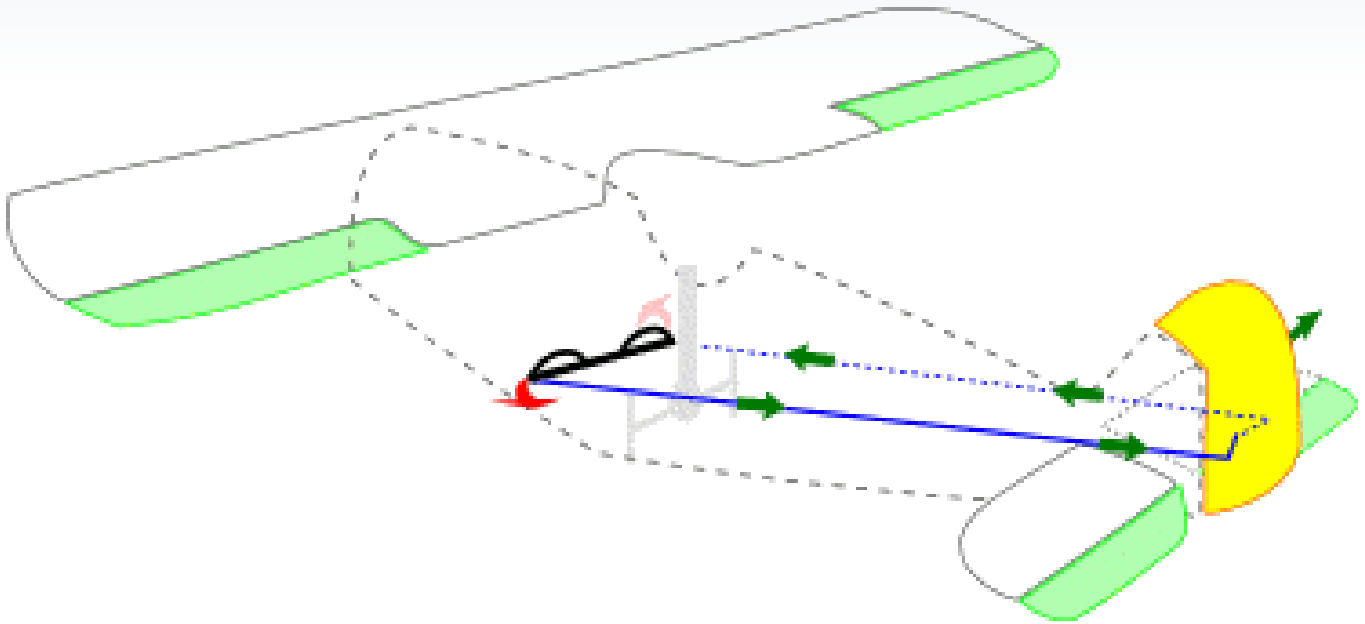

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Computers

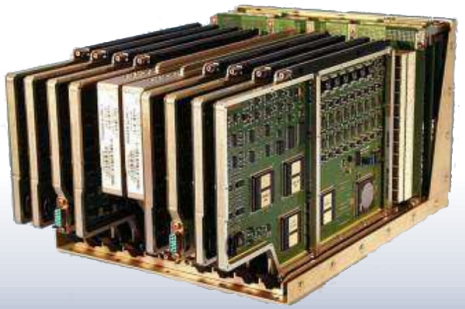

Image: Airbus

Control Command Systems

plant (physical system to control)

Image: public domain

command

Image: Thomasione (CC by-sa)

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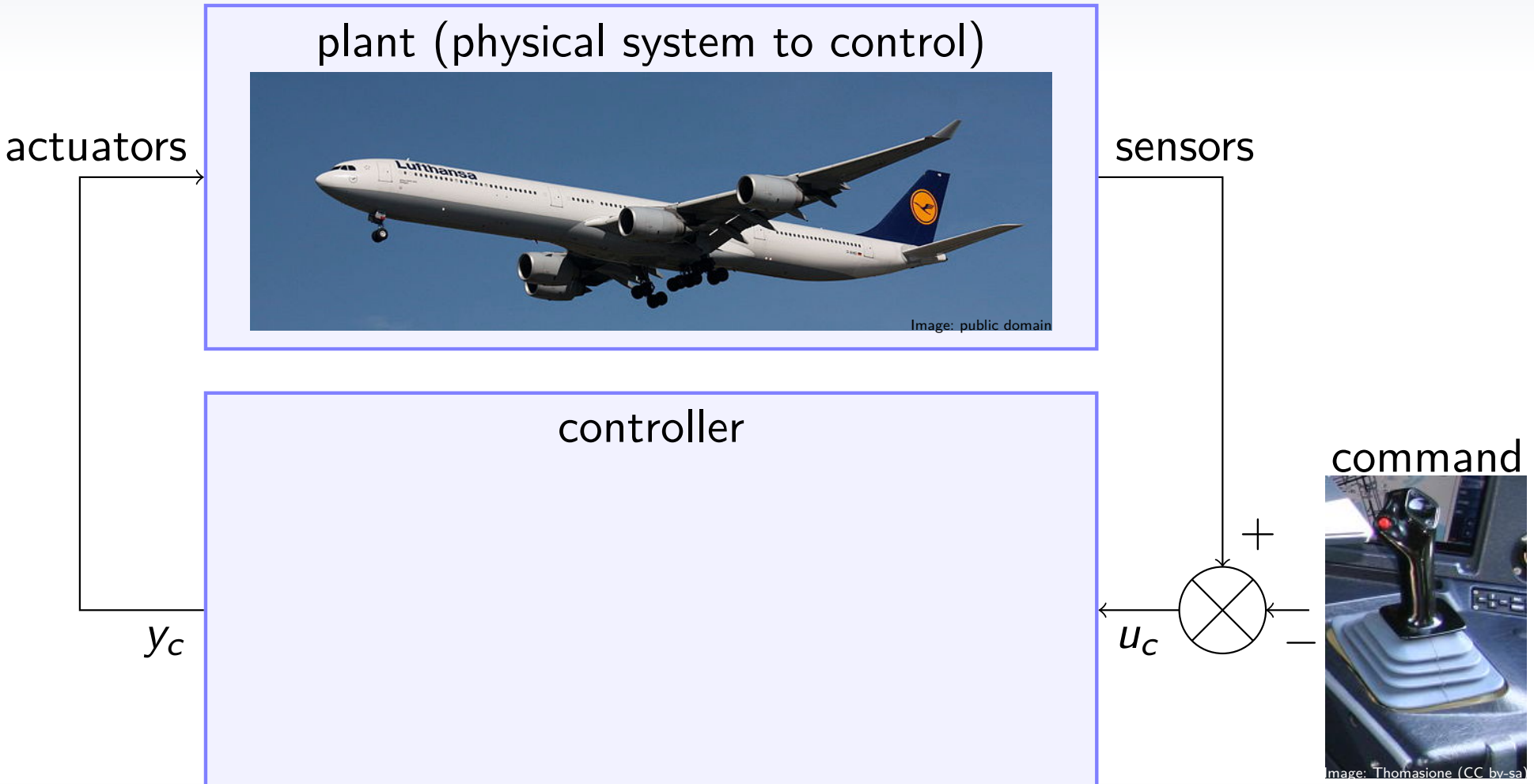

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actuators

sensors

controller

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double x[3] = {0, 0, 0};
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double in;
while (1) {
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  send_output(x);  // y_c
  wait_next_clock_tick();  // a tick every 10 ms
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y_c

u_c

command

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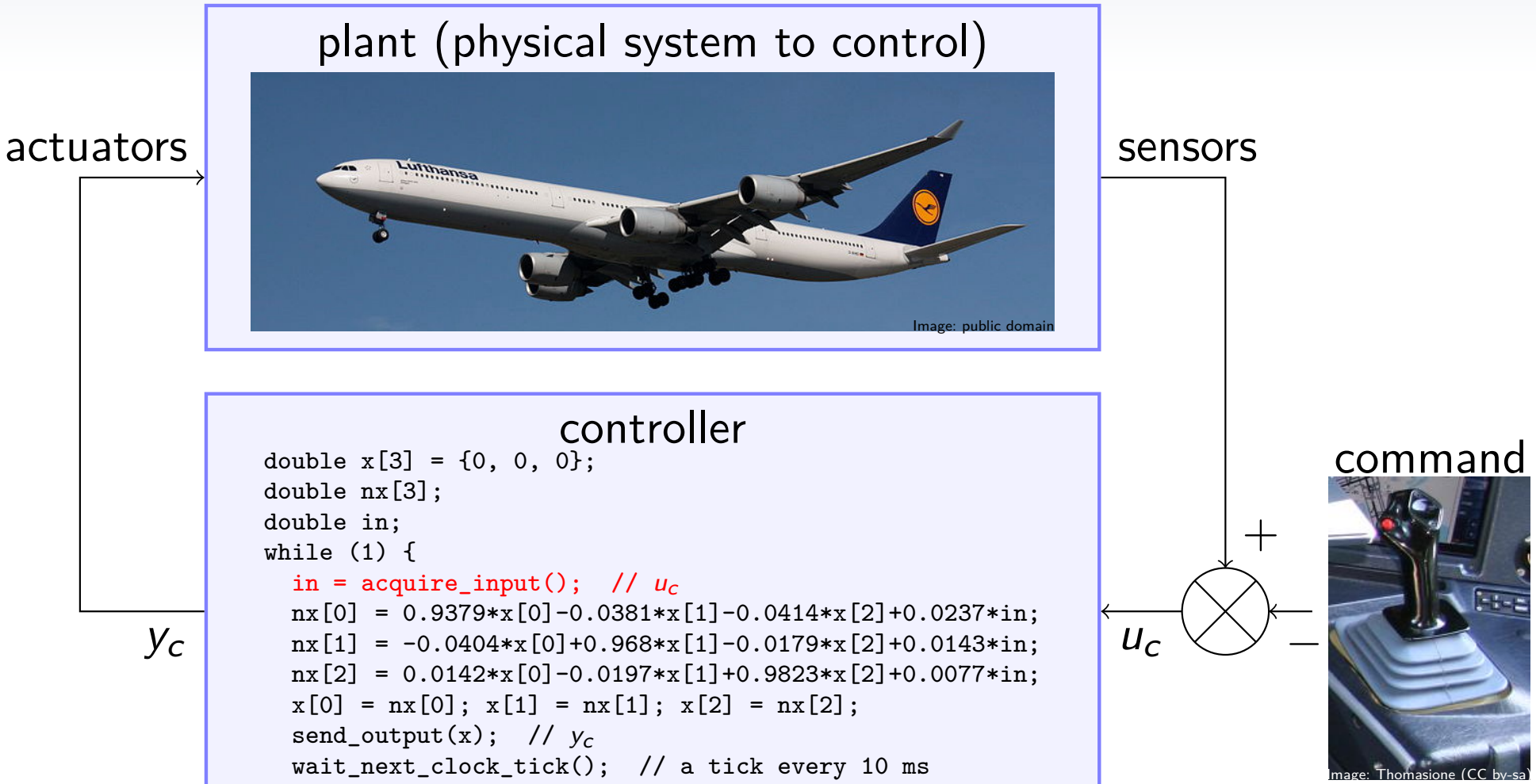

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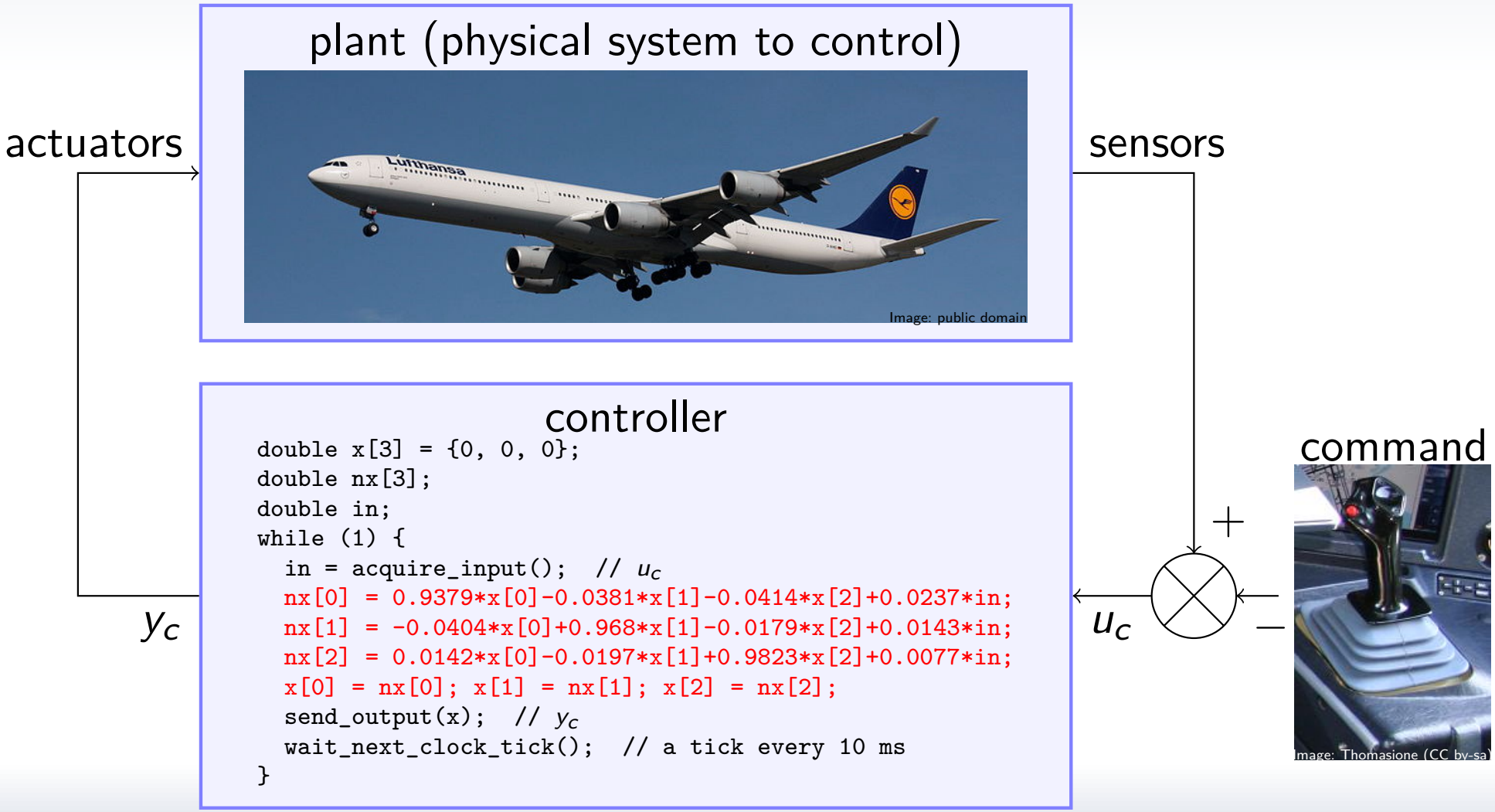

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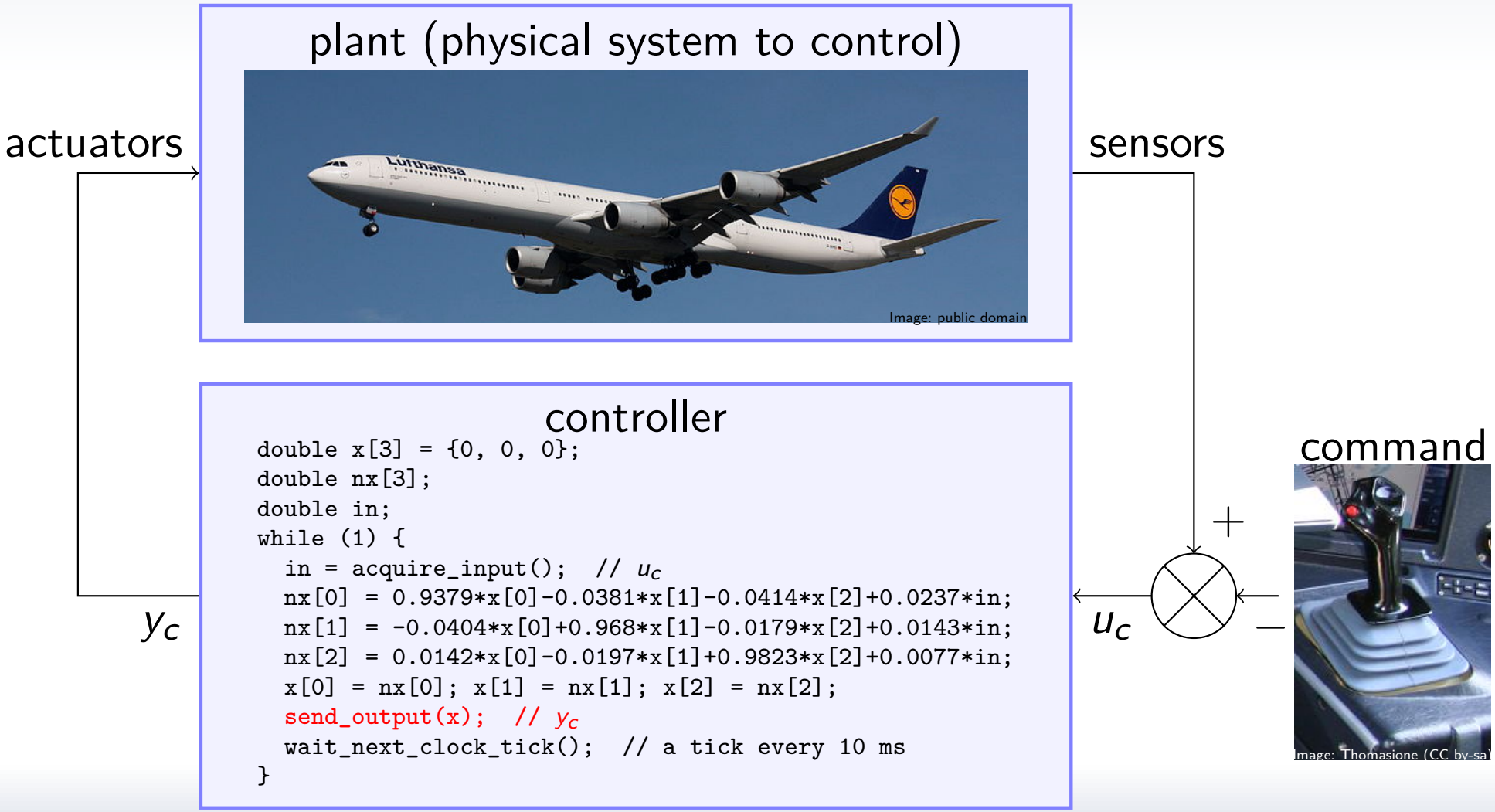

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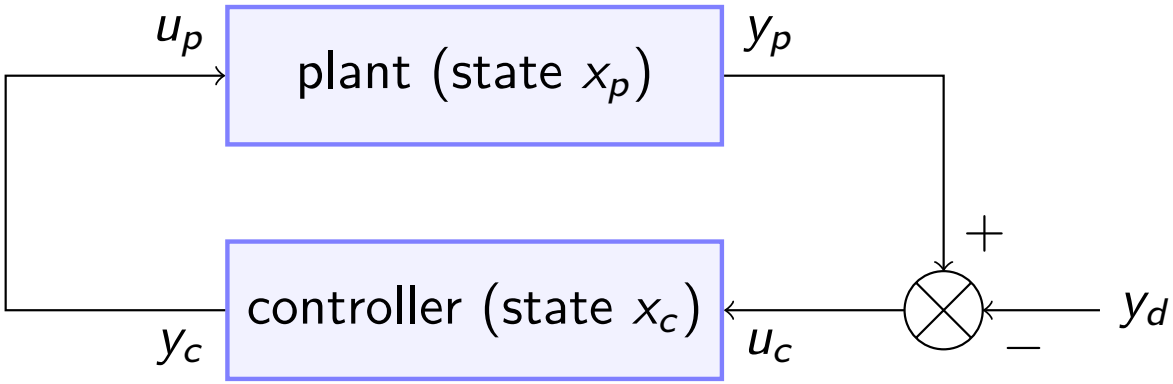
Stability Proofs

- Closed loop stability



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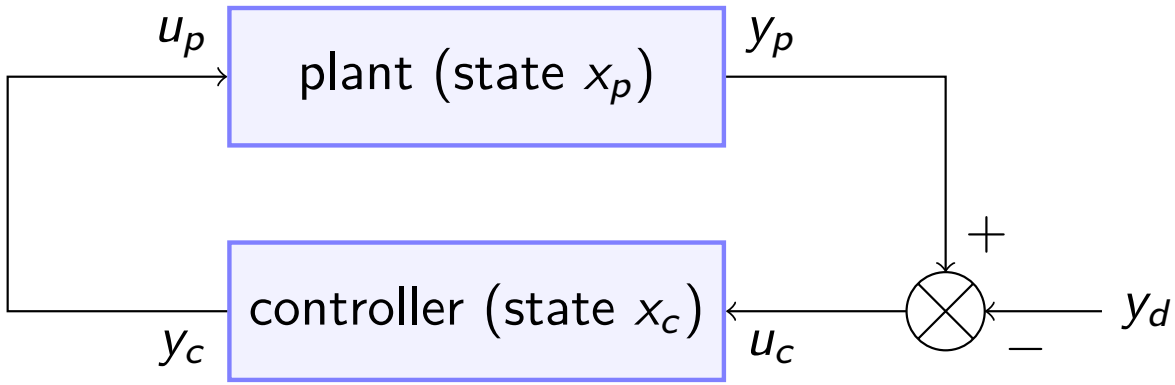

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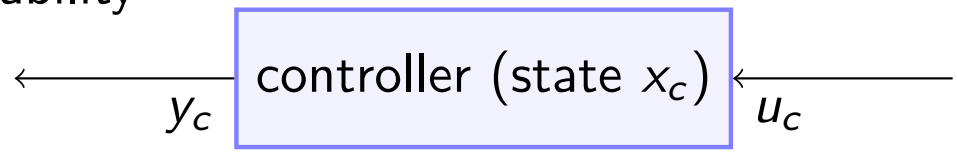

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Static Analysis

The controller

x0 := 0; x1 := 0; x2 := 0;
while $-1 \leq 0$ **do**
  in := **?(**-1, 1**)**;
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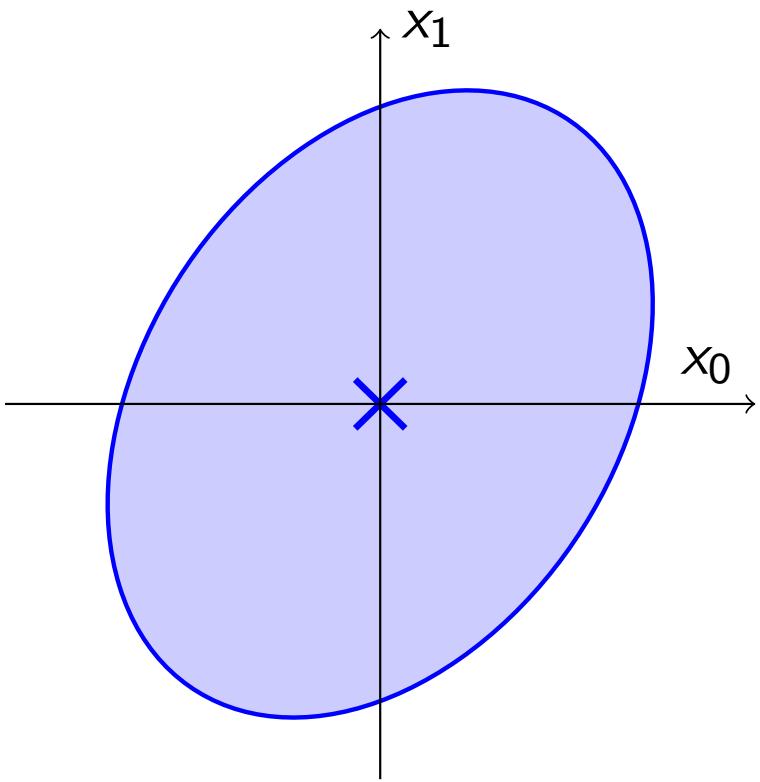
A set of points is an inductive invariant for a controller if it:



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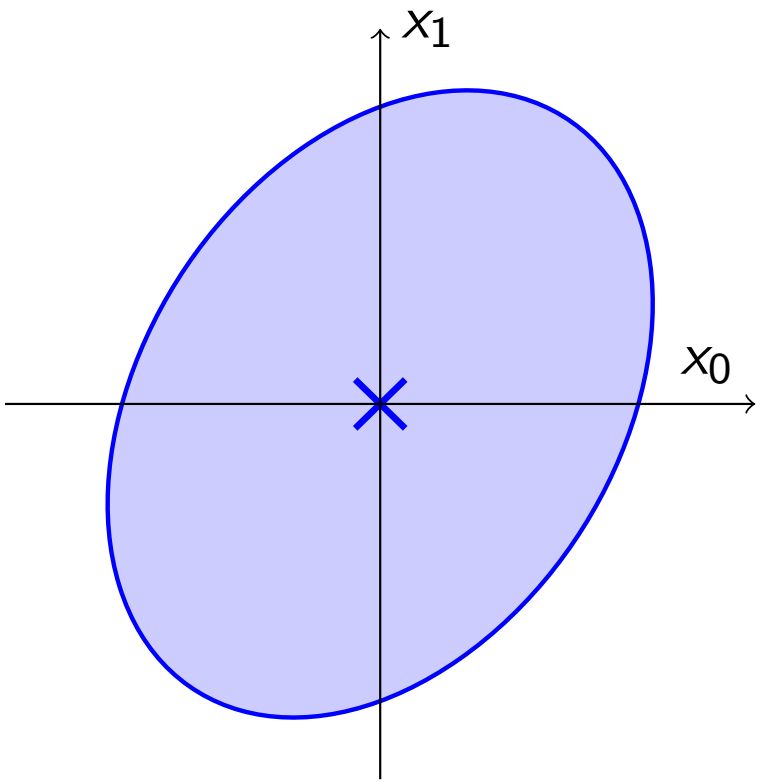

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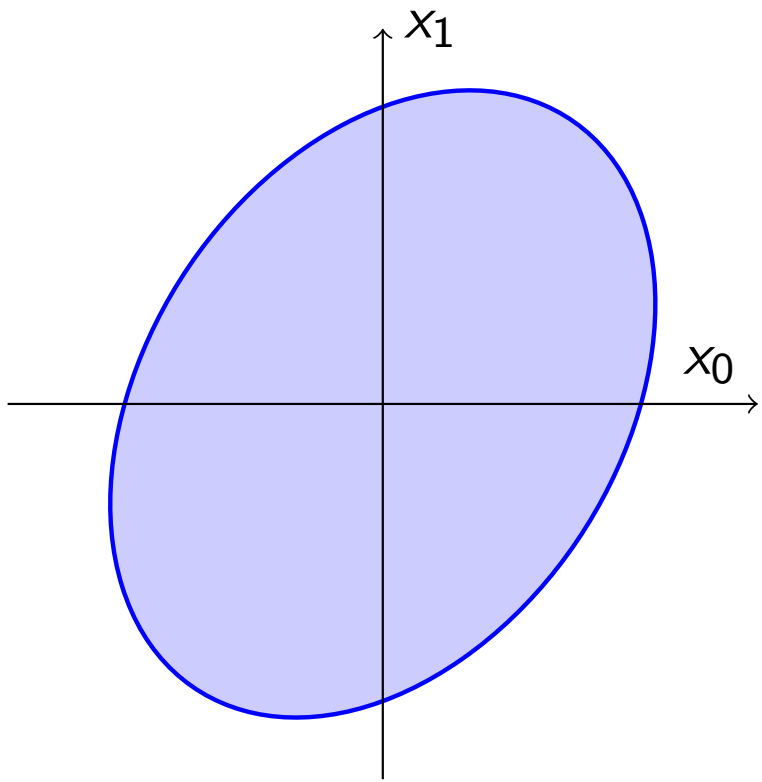

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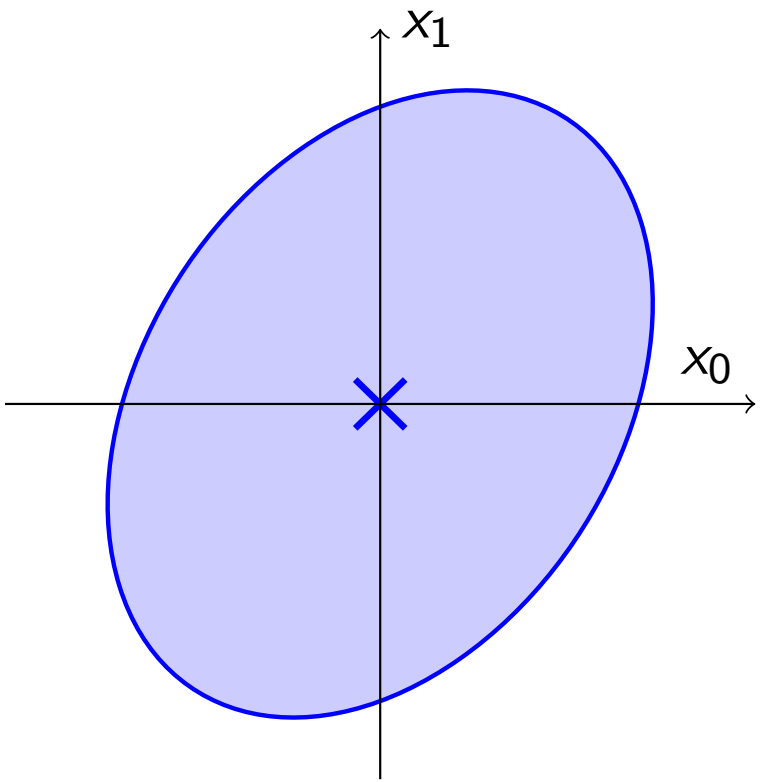

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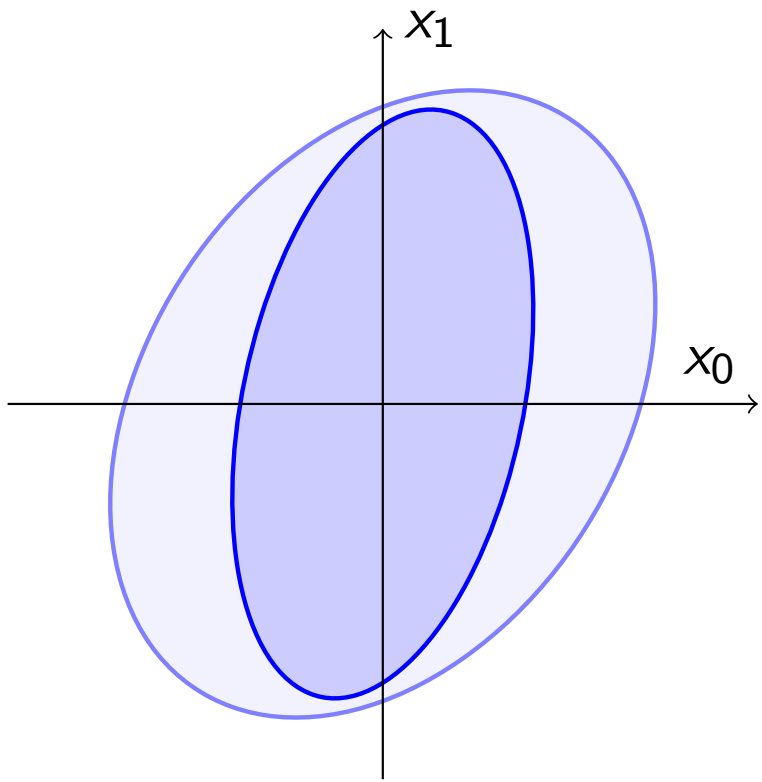

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1 Quadratic Invariants

2 Policy Iterations

3 Open Directions

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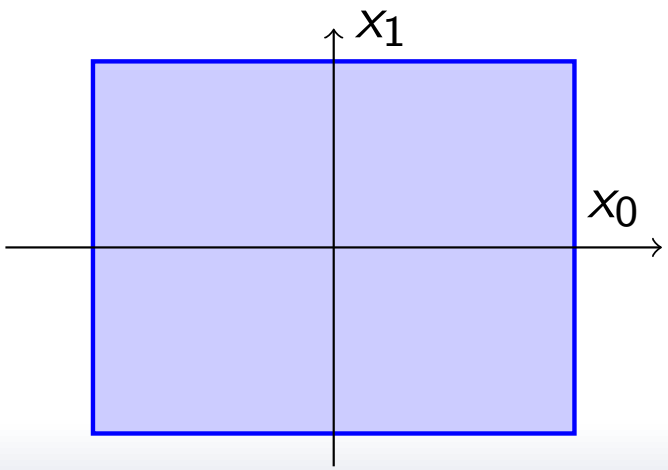

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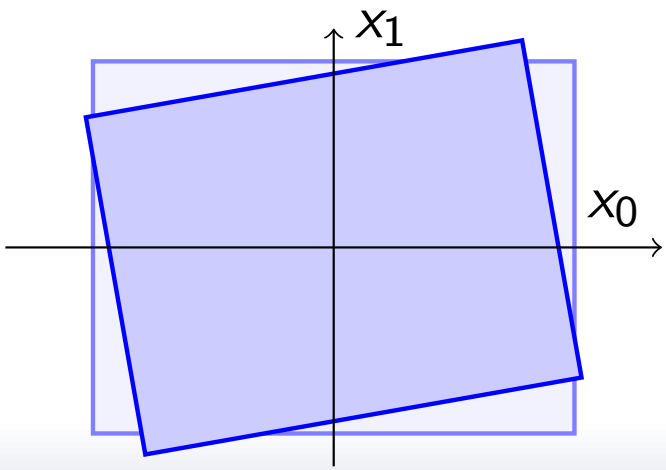

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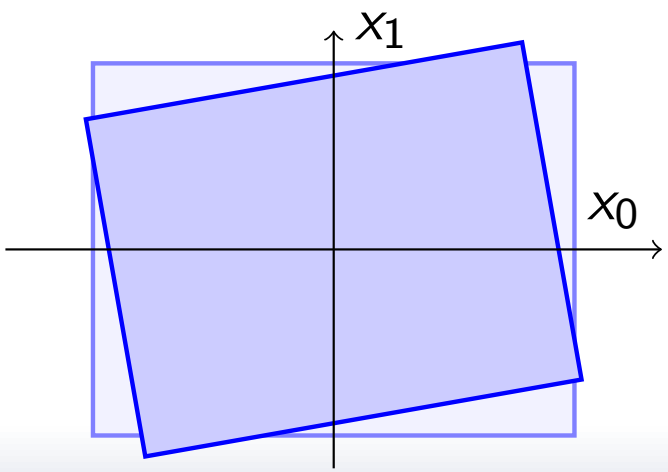

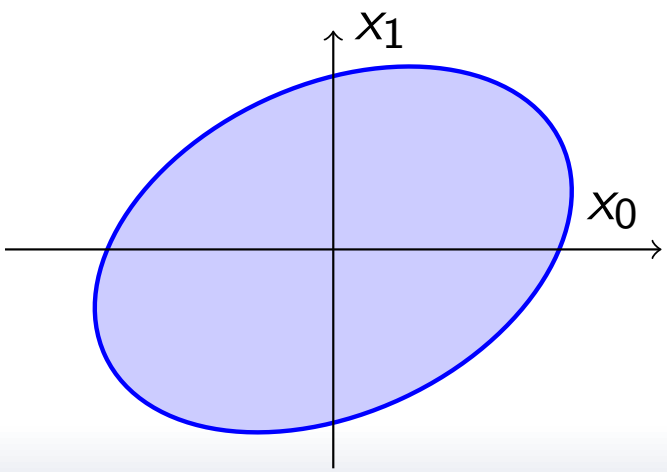

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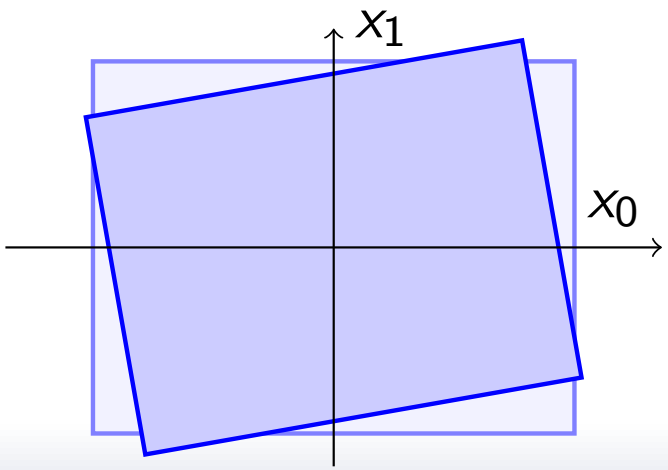

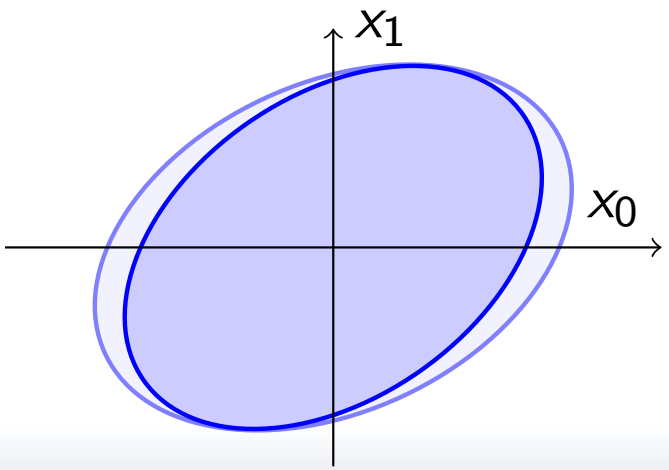

Quadratic Invariants

Remark

Reachable state space is usually **not** an ellipsoid.

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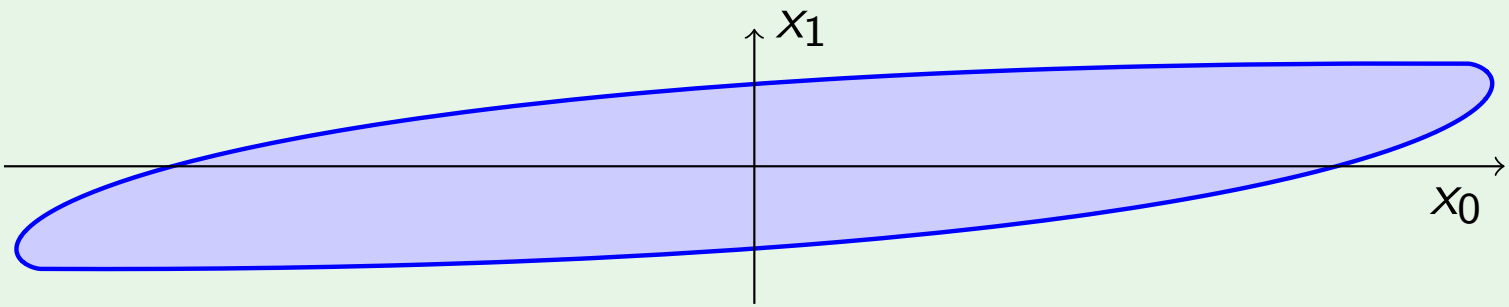
Reachable state space is usually **not** an ellipsoid.

Example

$x_0 := 0$ and $x_{k+1} := Ax_k + Bu_k$ where $\|u_k\|_\infty \leq 1$ and

$$A := \begin{bmatrix} 0.92565 & -0.0935 \\ 0.00935 & 0.935 \end{bmatrix} \qquad B := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



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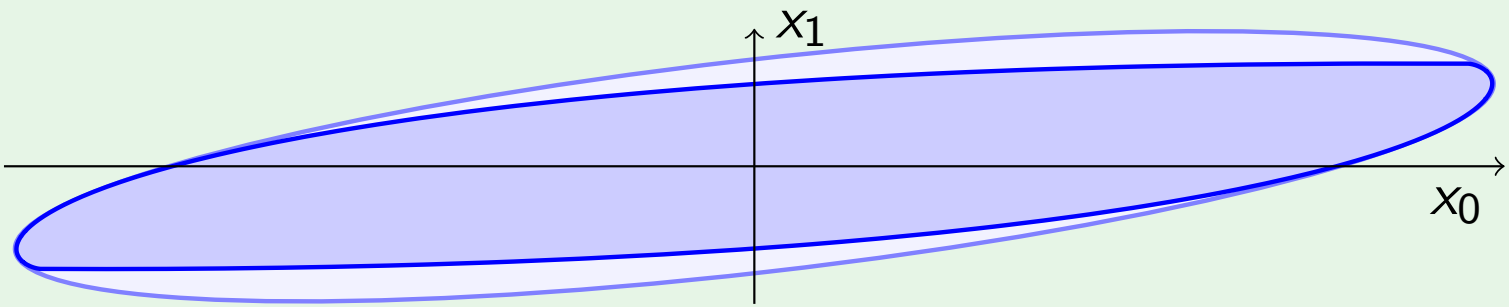


But remains reasonably close.

Lyapunov Stability [Lyapunov47]

Theorem

For $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, the sequence

$$\begin{cases} x_0 \in \mathbb{R}^n \\ x_{k+1} = Ax_k + Bu_k \end{cases}$$

is bounded for all $u \in (\mathbb{R}^p)^{\mathbb{N}}$ such that for all $k \in \mathbb{N}$, $||u_k||_\infty \leq 1$ if and only if there exists $P \in \mathbb{R}^{n \times n}$ positive definite such that

$$P - A^T P A \succ 0$$

where $M \succ 0$ means that for all $x \in \mathbb{R}^n$: $x \neq 0 \Rightarrow x^T M x > 0$.

Lyapunov Stability, Invariant

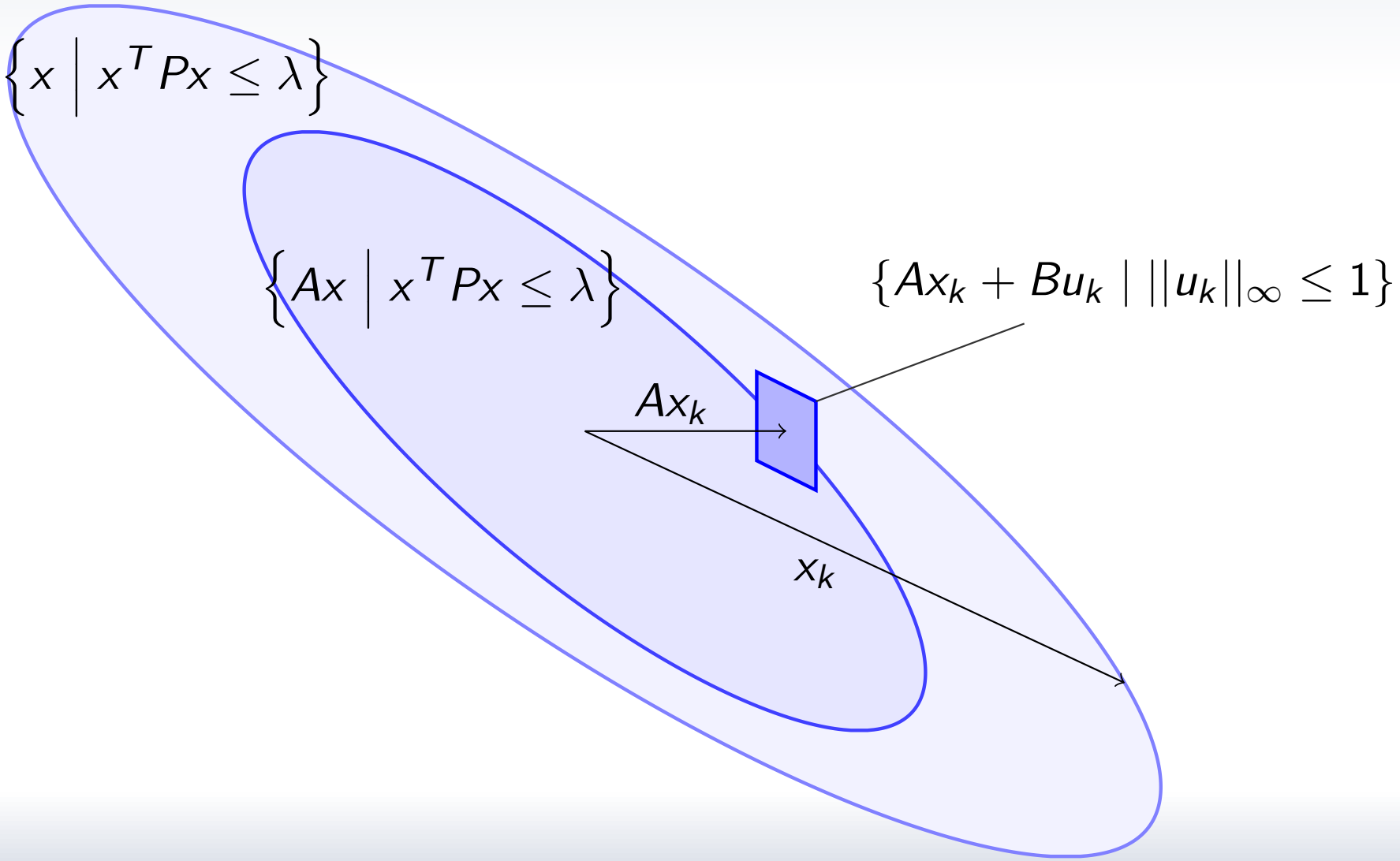
Invariant ellipsoid

Moreover, there exists a $\lambda > 0$ such that
x remains in the ellipsoid $\left\{ x \in \mathbb{R}^n \;\middle|\; x^T P x \leq \lambda \right\}$.

In Computer Science Language

The property "$x^T P x \leq \lambda$" is a **loop invariant**.

Lyapunov Stability, Illustration



Tools

To solve the Lyapunov equation $P - A^T P A \succ 0$:

Semidefinite Programming [VandenbergheB96]

Minimize a linear objective function of variables y_i
under constraint

$$A_0 + \sum_{i=1}^{k} y_i A_i \succeq 0$$

where the A_i are known matrices
and $M \succeq 0$ means $x^T M x \geq 0$ for all vector x.

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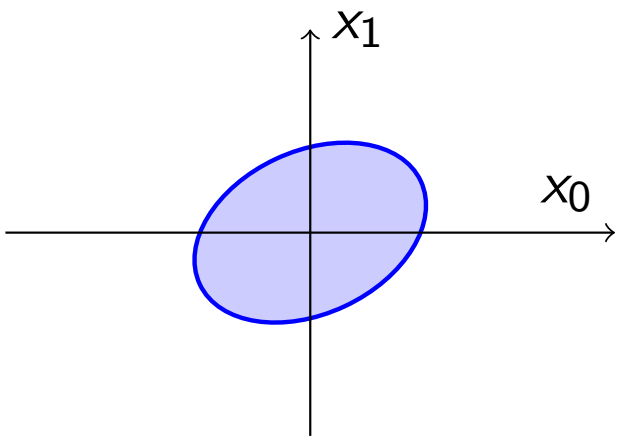
- "Efficient" solvers exist;
- Unknown matrices: Y can be decomposed as $\sum_{i,j} y_{i,j} E^{i,j}$;
- Multiple constraints: $A \succeq 0 \wedge B \succeq 0$ is equivalent to $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \succeq 0$.

Shape of the Ellipsoid

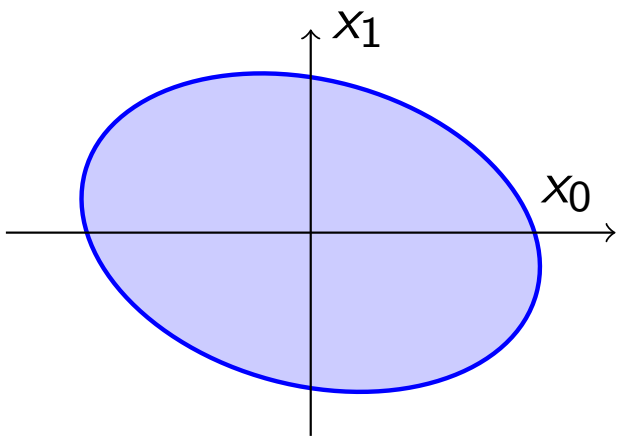
- We look for a positive definite matrix P such that $P - A^T P A \succ 0$.

Shape of the Ellipsoid

- We look for a positive definite matrix P such that $P - A^T P A \succ 0$.
- We will then look for a $\lambda > 0$ such that $x^T P x \leq \lambda$ is an invariant and the ellipsoid $\left\{ x \;\middle|\; x^T P x \leq \lambda \right\}$ should be "small".



is better than

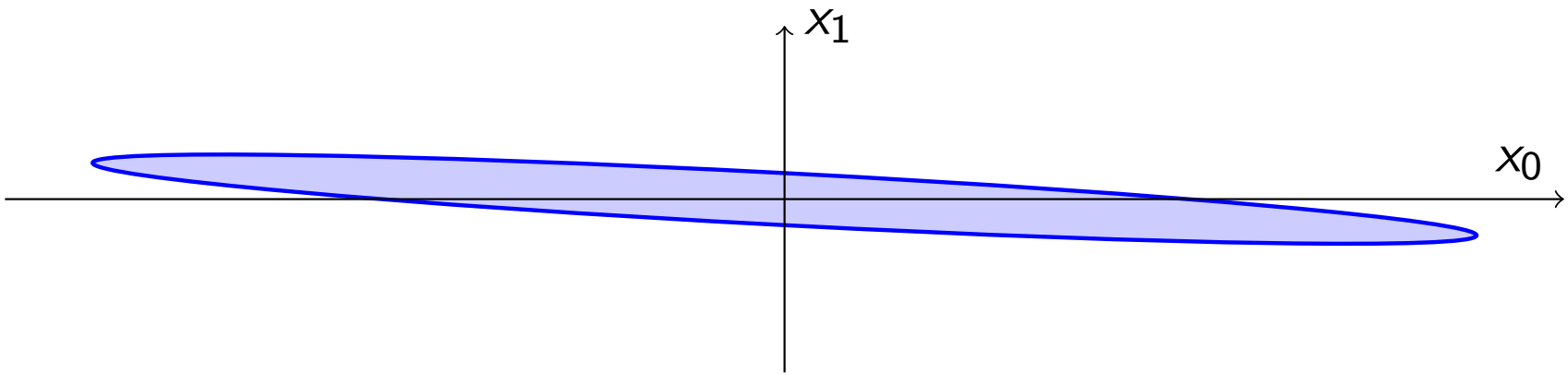


Heuristic 1: Minimizing Condition Number

We can look for the roundest possible ellipsoid
by minimizing a new variable r in the extra constraint

$$I \preceq P \preceq rI.$$

Rationale: "flat" ellipsoids can yield a very bad bound
on one of the variables.

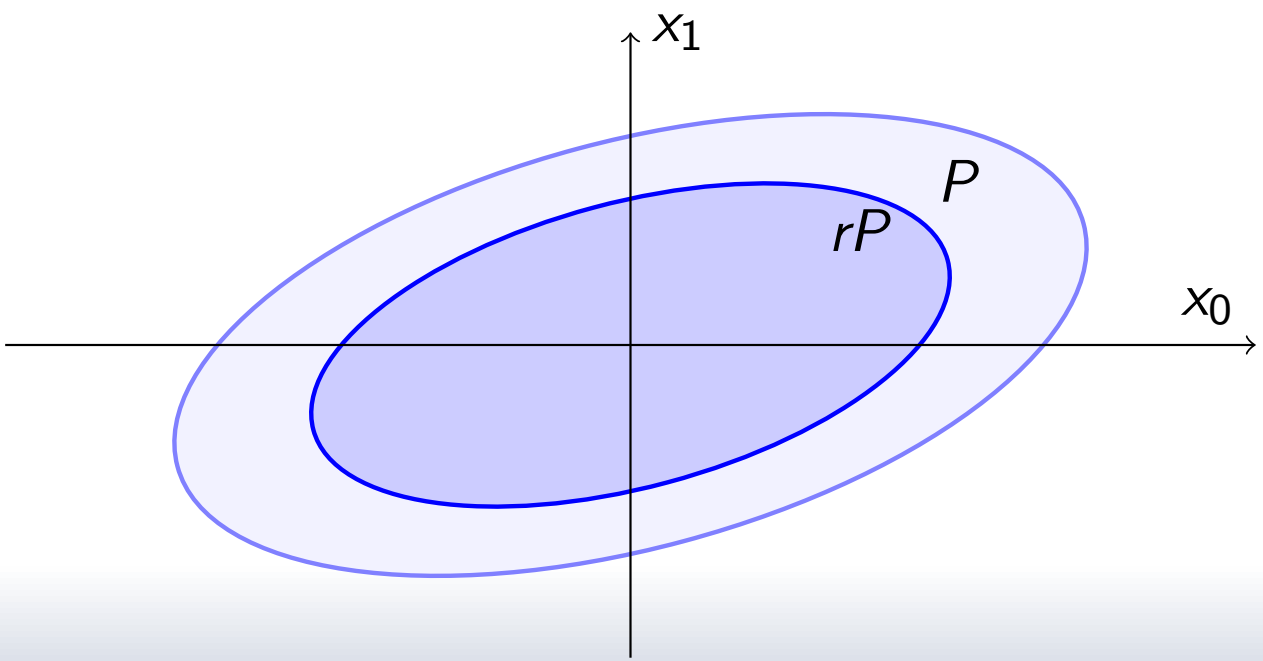


Heuristic 2: Preserving the Shape

Another heuristic is to minimize $r \in (0, 1)$ in

$$rP - A^T P A \succeq 0.$$

Intuitively, the shape of ellipsoid that gets "preserved" the best through the update $x_{k+1} = Ax_k$.

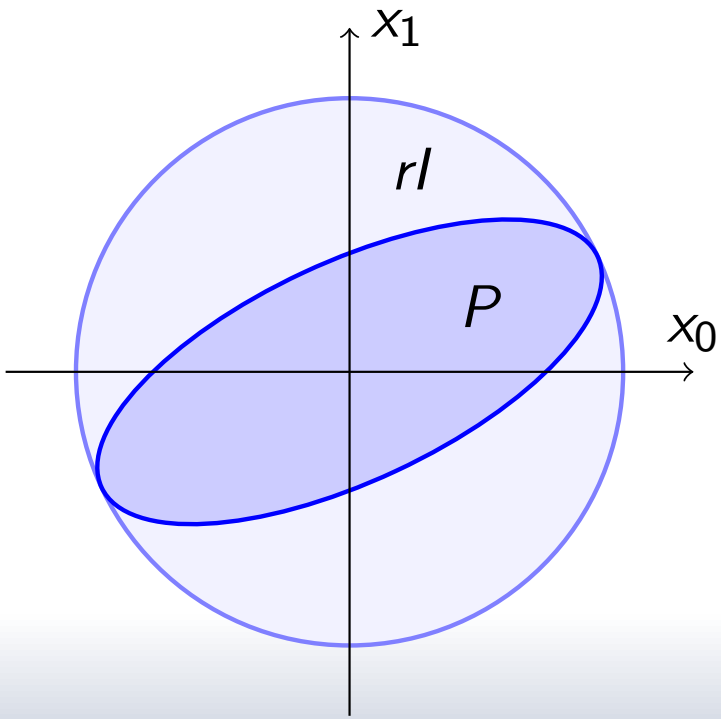


Heuristic 3: All in One

The two previous heuristics did not take B into account.
We could look for P maximizing r in $P \succeq rI$ such that

$$\forall x, \forall u, ||u||_\infty \leq 1 \Rightarrow x^T P x \leq 1 \Rightarrow (Ax + Bu)^T P (Ax + Bu) \leq 1.$$

Intuitively, invariant ellipsoid included in smallest possible sphere.

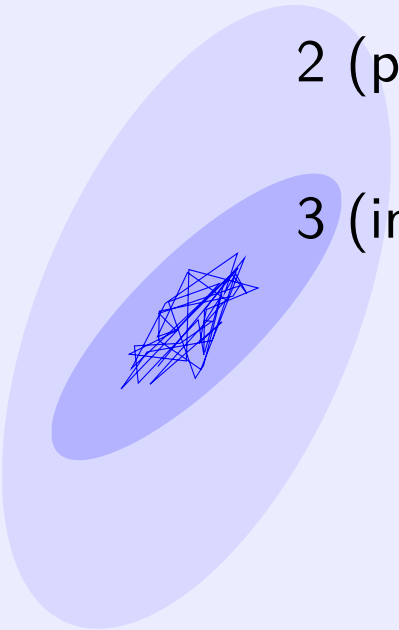


Example

1 (condition number)

2 (preserving shape)

3 (in smallest sphere)



Experimental Results

	H.	Bounds	Reachable
Ex. 2 n=4, 1 input	1	1661, 1795, 843, 1288	1.42, 1.42, 1.00, 1.00
	2	5.50, 6.71, 2.20, 3.44	
	3	1.69, 1.92, 2.13, 2.42	
Ex. 3 Lead-lag controller n=2, 1 input	1	60.93, 60.52	3.97, 20.00
	2	36.55, 35.50	
	3	28.83, 25.85	
Ex. 4 LQG regulator n=3, 1 input	1	1.26, 1.26, 1.26	0.32, 0.24, 0.22
	2	1.21, 1.23, 1.06	
	3	0.68, 0.41, 0.28	
Ex. 5 Coupled mass system n=4, 2 inputs	1	4980, 5061, 4768, 5693	2.79, 2.73, 3.50, 3.30
	2	6.37, 6.20, 6.07, 9.57	
	3	4.97, 4.90, 4.77, 4.63	
Ex. 6 Low-pass filter n=5, 1 input	1	253, 261, 251, 280, 286	1.42, 0.91, 1.44, 1.52, 2.14
	2	3.11, 4.30, 4.15, 8.16, 8.81	
	3	2.21, 1.10, 1.87, 1.98, 2.83	

Experimental Results, Analysis Times

	Heuristic	t_P (s)	t_λ (s)	$t_{valid.}$ (s)	t_{total} (s)
Ex. 2 n=4, 1 input	1	0.10	0.63	0.02	0.75
	2	0.21	0.37	0.01	0.59
	3	0.33	0.22	0.01	0.56
Ex. 3 Discretized lead-lag controller n=2, 1 input	1	0.08	0.47	0.03	0.60
	2	0.13	0.45	0.02	0.60
	3	0.16	0.21	0.02	0.39
Ex. 4 Linear quadratic gaussian regulator n=3, 1 input	1	0.08	0.33	0.02	0.43
	2	0.14	0.29	0.02	0.45
	3	0.16	0.22	0.02	0.40
Ex. 5 Controller for a coupled mass system n=4, 2 inputs	1	0.09	0.76	0.03	0.88
	2	0.17	0.43	0.03	0.63
	3	0.27	0.23	0.03	0.53
Ex. 6 Butterworth low-pass filter n=5, 1 input	1	0.11	0.65	0.03	0.79
	2	0.22	0.37	0.02	0.61
	3	0.56	0.25	0.02	0.83

On an Intel Core2 @ 2.66GHz.

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For efficiency, computations made with floating point, hence rounding errors:

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- In the analyzed controller:

  Using classic ϵ methods [Higham96]:

  Theorem [Roux-PhD13]

  If $x^T P x \leq \lambda$, $\|u\|_\infty \leq 1$ and $(Ax+Bu)^T P (Ax+Bu) \leq \lambda'$, then

  $$\mathrm{fl}(Ax+Bu)^T P\, \mathrm{fl}(Ax+Bu) \leq \left(\sqrt{\lambda'} + \sqrt{\lambda} a + b\right)^2$$

  with a and b (very small) constants computed from A, B and P.

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Floating Point Issues

- In the analyzer:
  - Checking the invariant $\left\{x \;\middle|\; x^T P x \leq \lambda\right\}$ basically amounts to **checking positive definiteness** of following matrix (with appropriate τ and λ_i non negatives):

$$\begin{bmatrix} -A^T PA & -A^T PB & 0 \\ -B^T PA & -B^T PB & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \tau \begin{bmatrix} -P & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \sum_{i=0}^{p-1} \lambda_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & -E^{i,i} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

  - This is done by carefully bounding the rounding errors in a **Cholesky decomposition** [Rump06].
  - Hence an efficient soundness check (in $\mathrm{O}\left(n^3\right)$ floating point operations for an $n \times n$ matrix).

Summary

State of the Art (Static Analysis of Controllers)

- Linear Domains [Halbwachs78,...]
- Ellipsoids of dimension 2 [Feret04]
- Unrolling (no inductive invariant) [Feret04, Khalil09,...]

Contributions [RouxJGF-HSCC12]

Automatic computation of Lyapunov functions:

- Heuristics to optimize precision of invariants.
- Overapproximation of floating point computations.
- Soundness check.

1 Quadratic Invariants

2 Policy Iterations

3 Open Directions

Linear Systems with Guards

Actual systems are not purely linear but contain **guards**: resets, saturations, anti-windups,. . .

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Example

$x := ?(0, 1)$; $y := ?(0, 1)$;
while $-1 \leq 0$ **do**
  $in := ?(0, 1)$;
  if $in \geq 0.9$ **then**
    $x := 10 \times in - 9$;
    $y := 10 \times in - 9$
  else
    $t := x$;
    $x := 0.2 \times t - 0.7 \times y + 0.5 \times in$;
    $y := 0.7 \times t + 0.2 \times y + 0.5 \times in$
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Kleene Iterations

Classic abstract interpretation with interval domain:

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x := ?(0, 1); y := ?(0, 1);
while −1 ≤ 0 do
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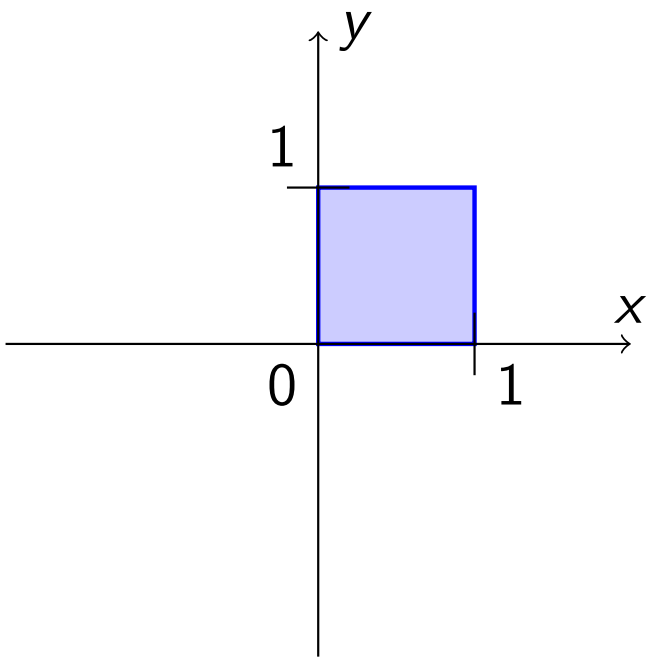

before entering the loop

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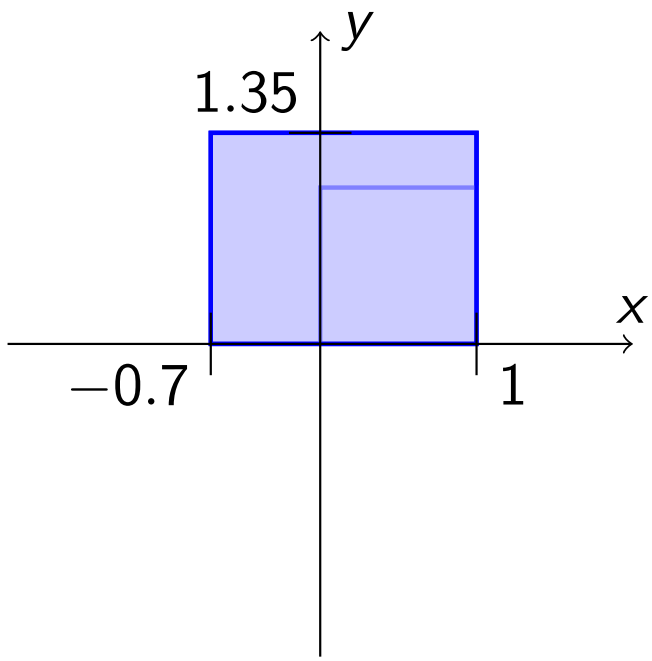

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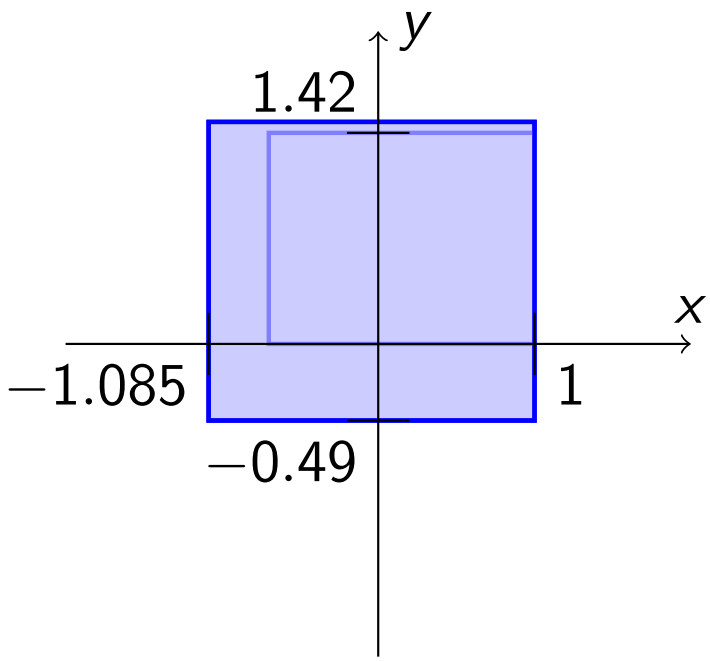

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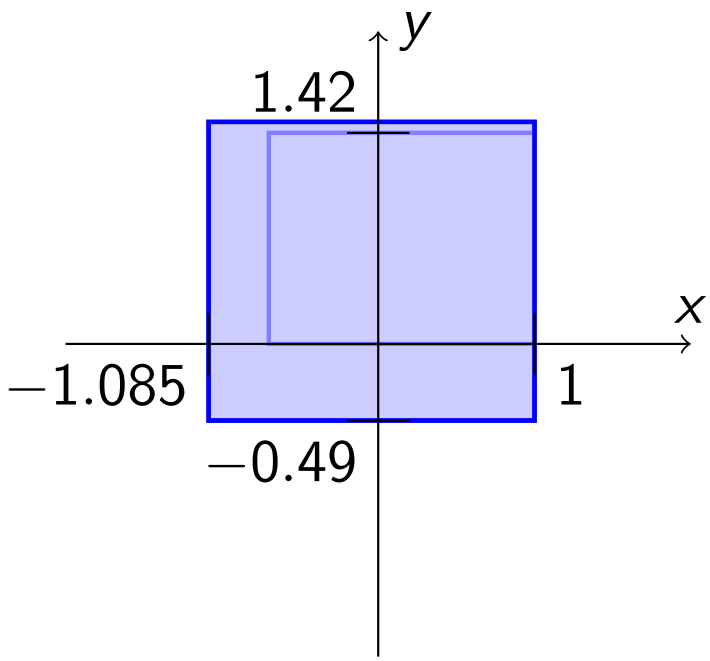

. . . does not converge

Widening for Kleene Iterations

Basic Idea

Jump to ensure convergence in **finitely many** iterations.

Example (Simplest Widening)

Jump to $(-\infty, +\infty)$ as soon as bounds are not stable.

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Example (Widening with Thresholds)

Stop on a finite number of thresholds T on the way to infinity.
For instance $T = \{ -5, 5 \}$.

Widening for Kleene Iterations

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x := ?(0, 1); y := ?(0, 1);
while −1 ≤ 0 do
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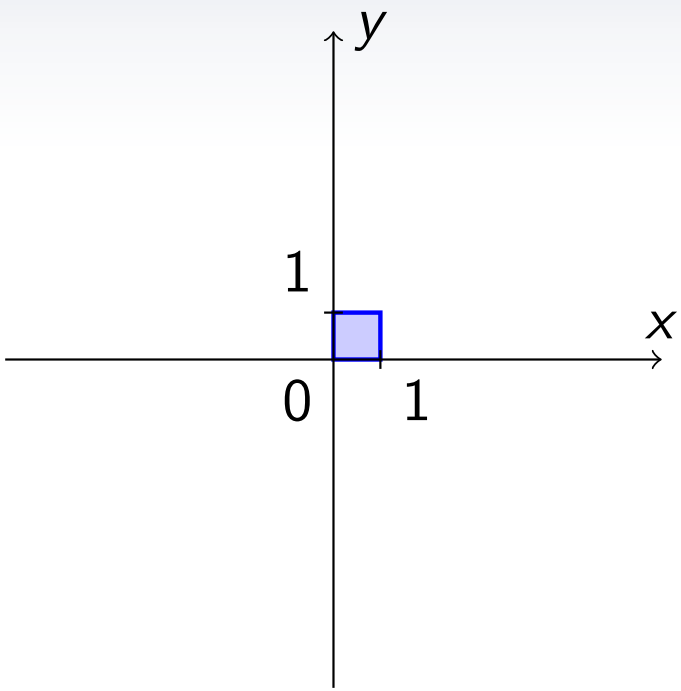
```



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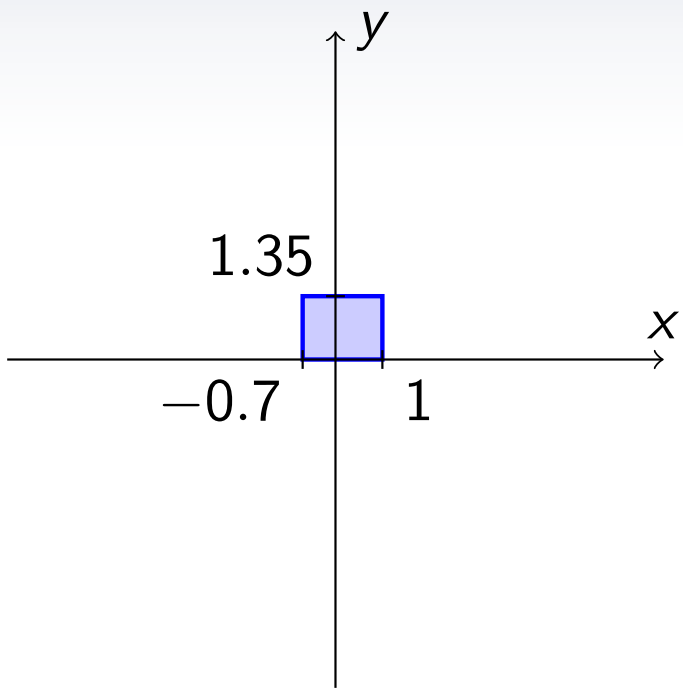

after a first iteration

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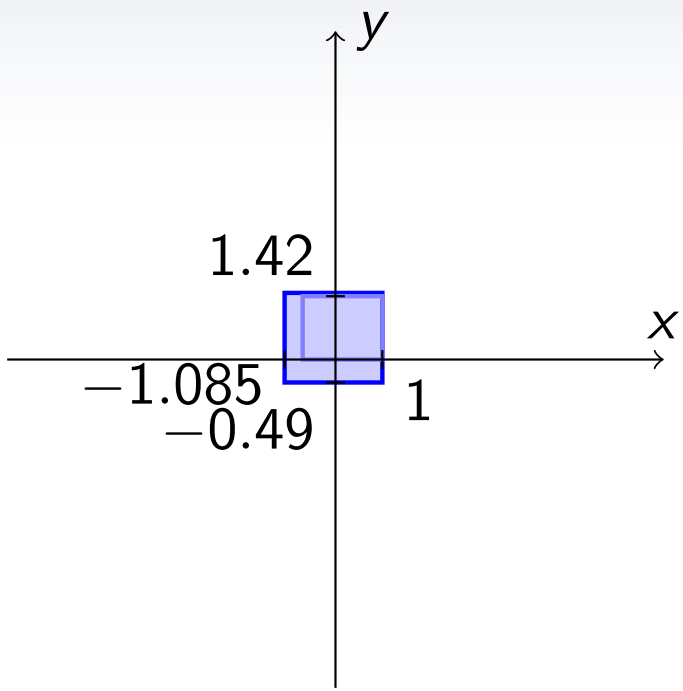

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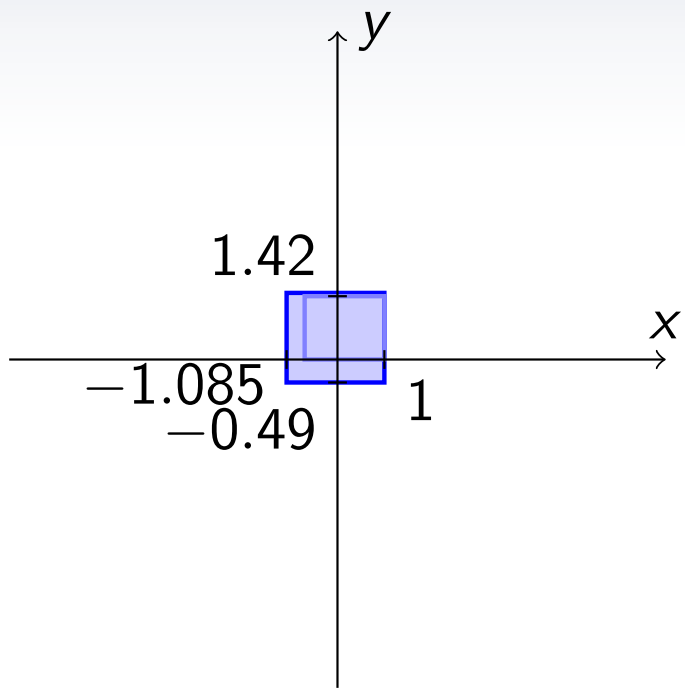

not stable → widening

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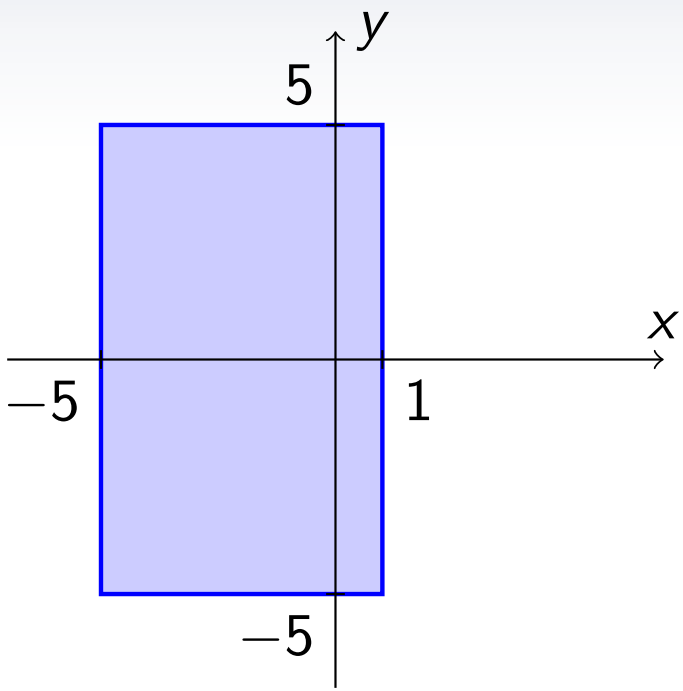

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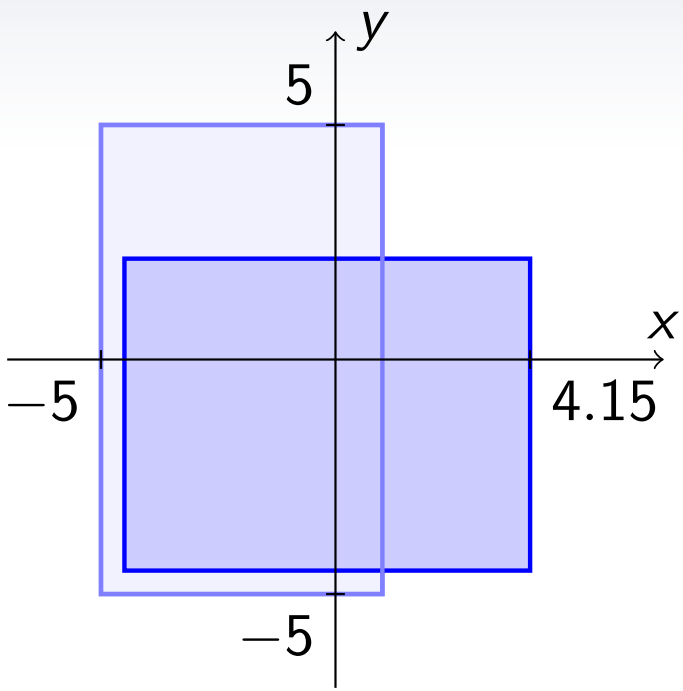

after another iteration

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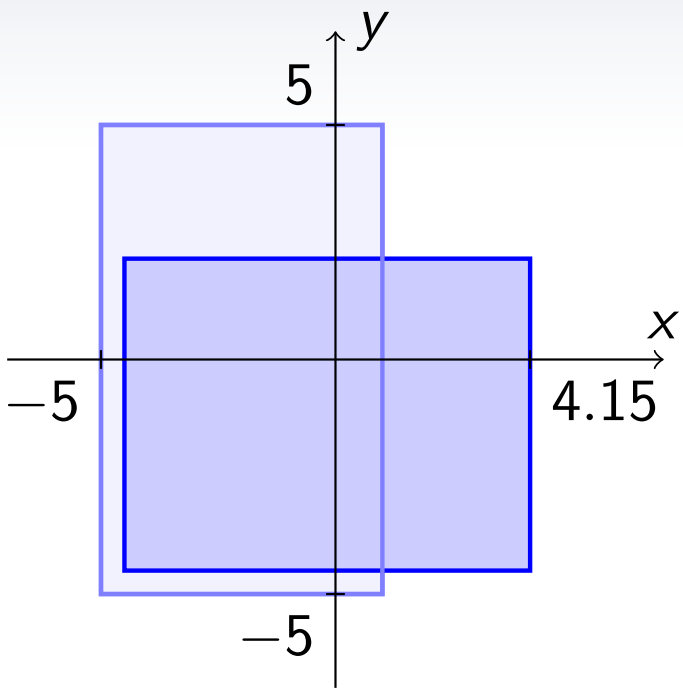

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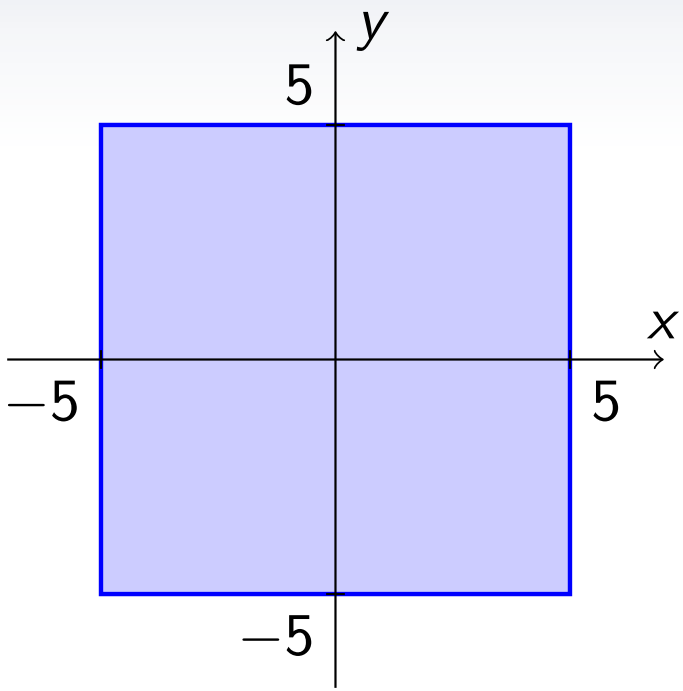

after widening

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$x := ?(0, 1)$; $y := ?(0, 1)$;
while $-1 \leq 0$ **do**
  in $:= ?(0, 1)$;
  if $0.9 - \text{in} \leq 0$ **then**
    $x := 10 \times \text{in} - 9$;
    $y := 10 \times \text{in} - 9$
  else
    $t := x$;
    $x := 0.2 \times t - 0.7 \times y + 0.5 \times \text{in}$;
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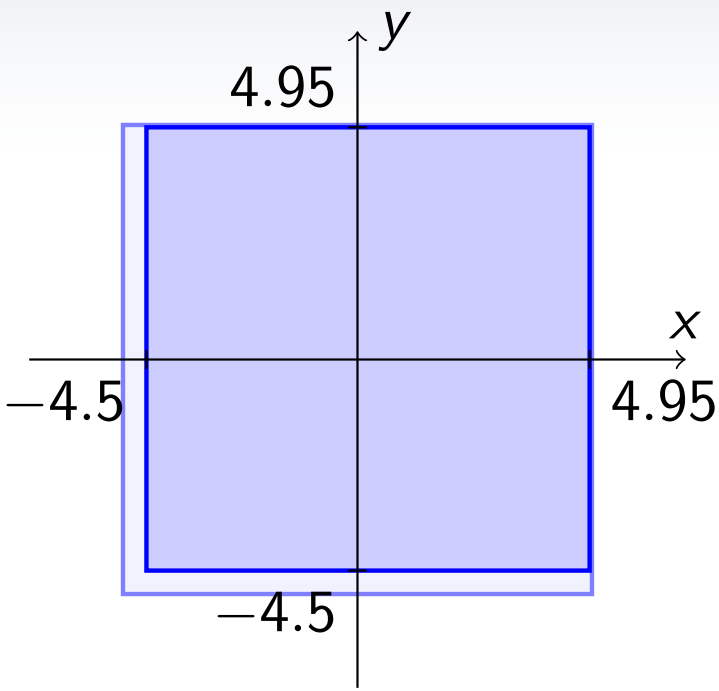

after another iteration

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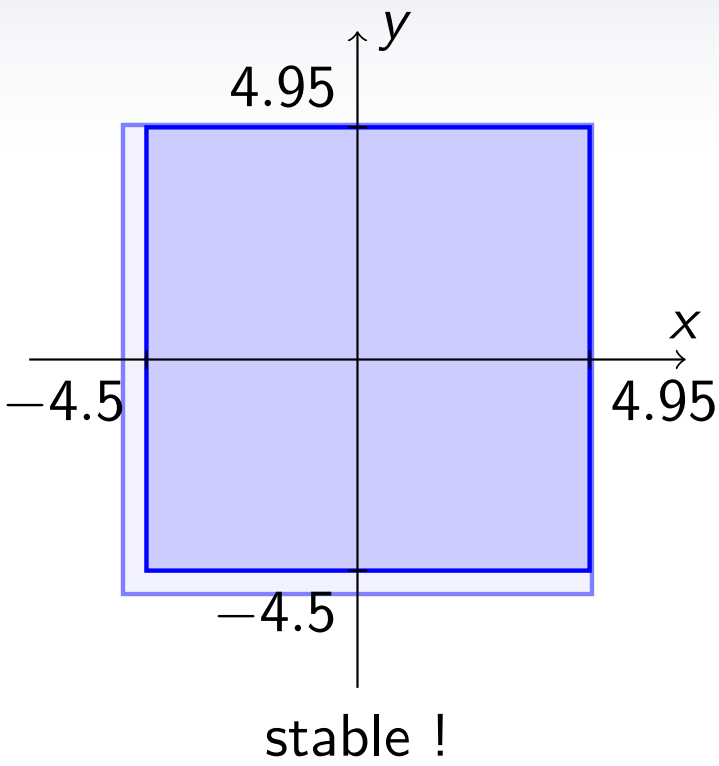

Remarks

- Worthwhile result: $x \in [-5, 5] \wedge y \in [-5, 5]$.

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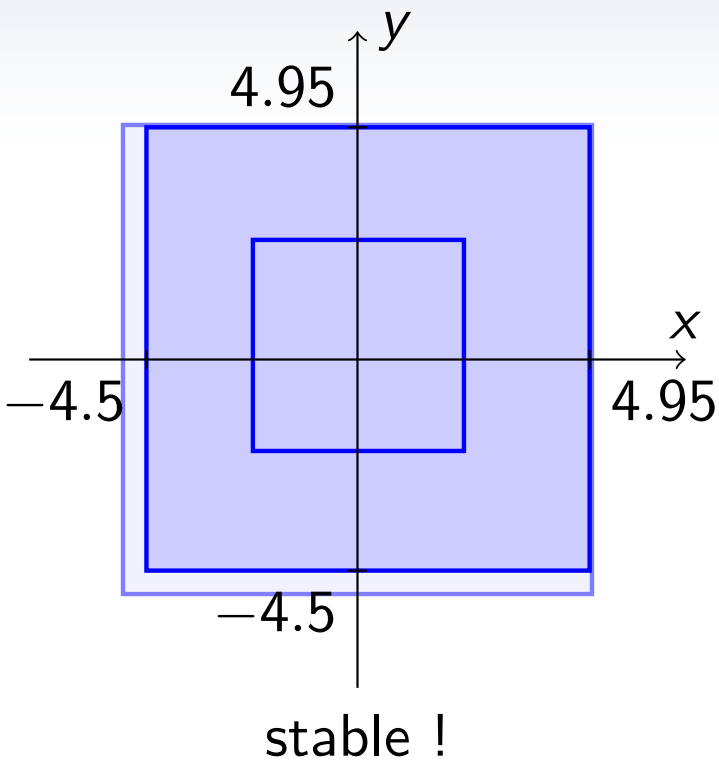

Remarks

- Worthwhile result: $x \in [-5, 5] \wedge y \in [-5, 5]$.
- Larger than least fixpoint: $x \in [-2.23, 2.27] \wedge y \in [-1.95, 2.55]$.

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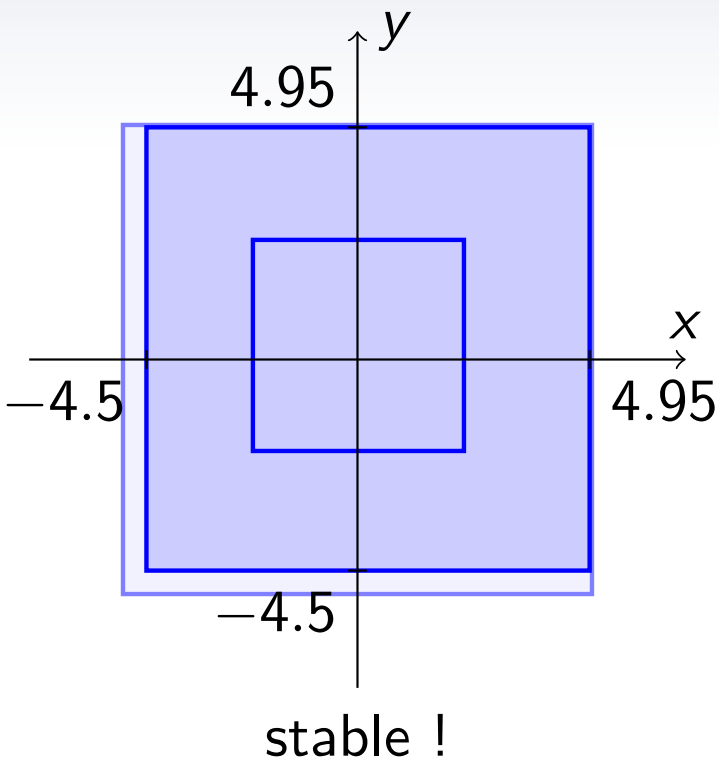

Remarks

- Worthwhile result: $x \in [-5, 5] \wedge y \in [-5, 5]$.
- Larger than least fixpoint: $x \in [-2.23, 2.27] \wedge y \in [-1.95, 2.55]$.
- Hardly refinable by narrowing.

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Basic Idea

Use **numerical optimization tools** to compute bounds that are hard to guess for widening.

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Two major approaches:

- Min-policies [GaubertGTZ07, AdjéGG10, GawlitzaSAGG12].
- Max-policies [GawlitzaS07, GawlitzaS10, GawlitzaSAGG12].

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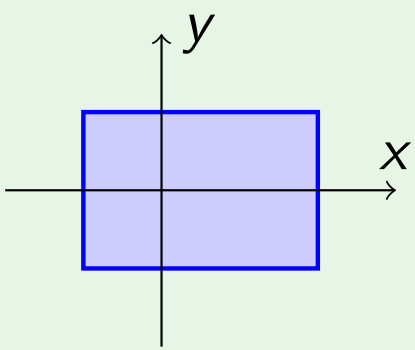

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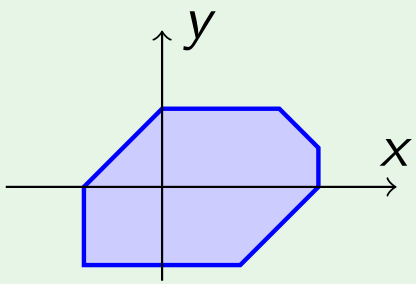

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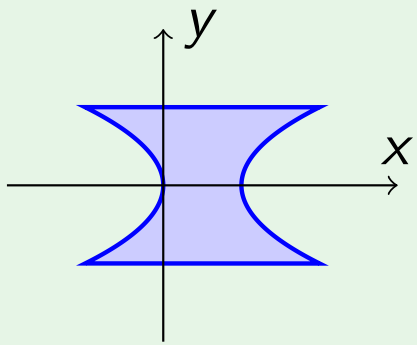

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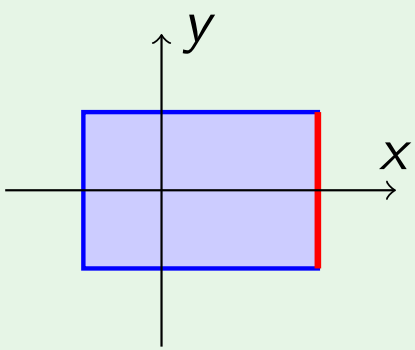

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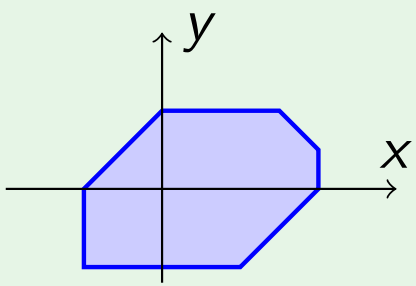

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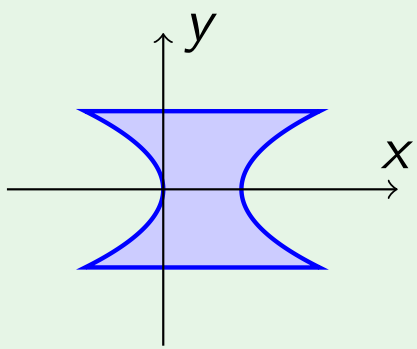

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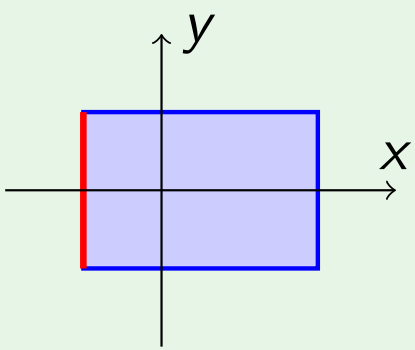

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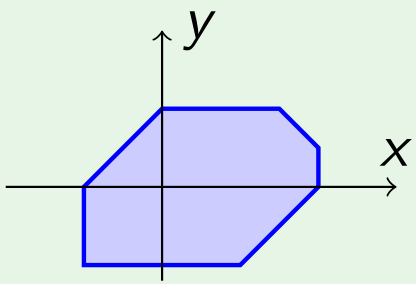

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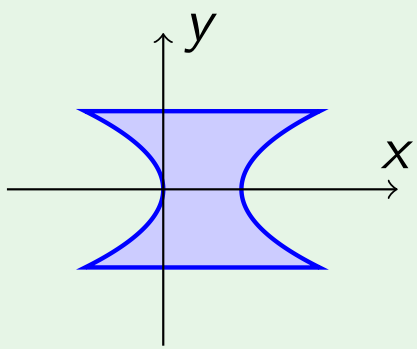

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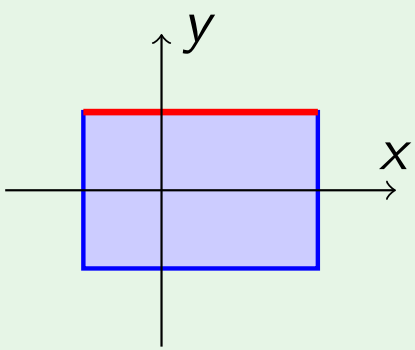

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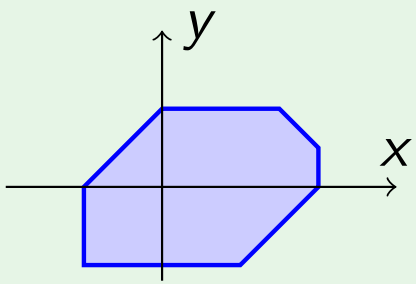

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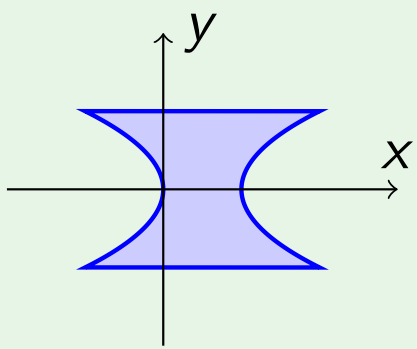

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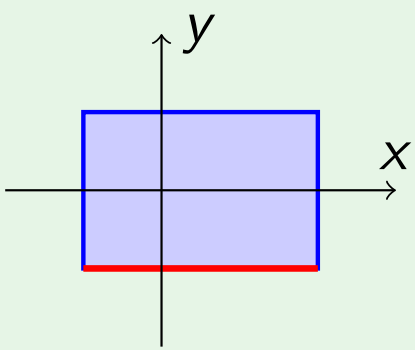

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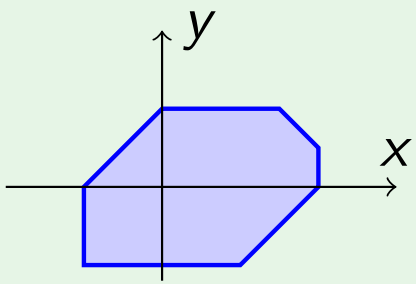

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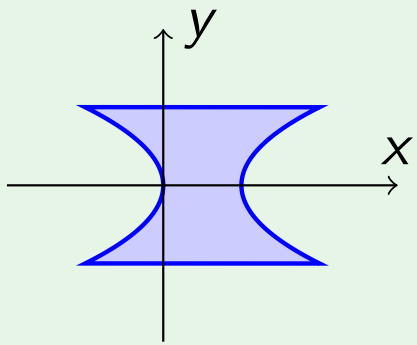

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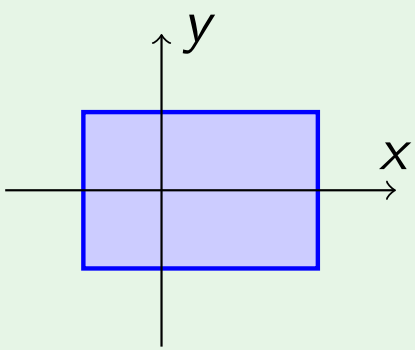

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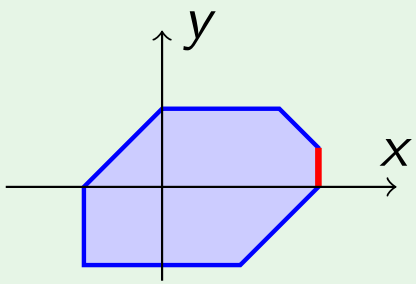

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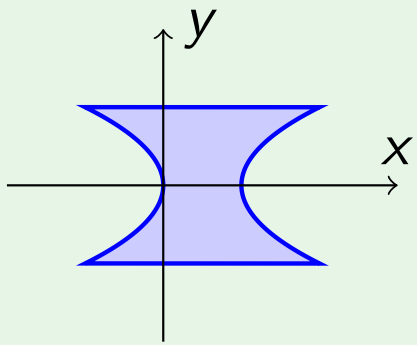

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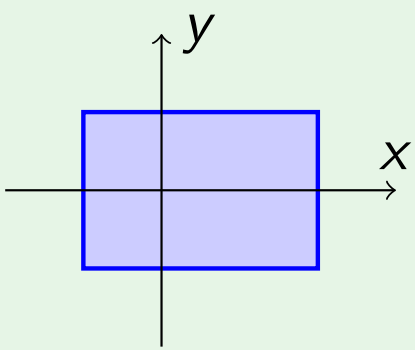

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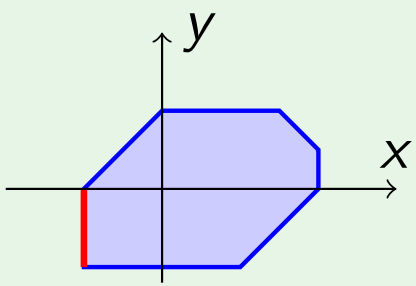

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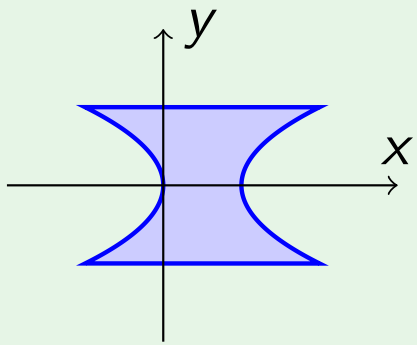

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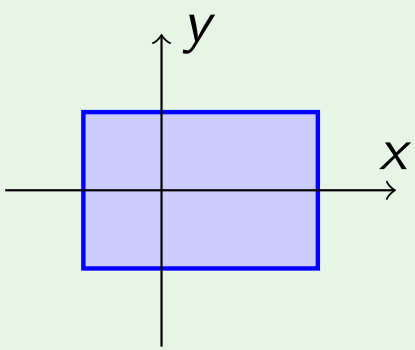

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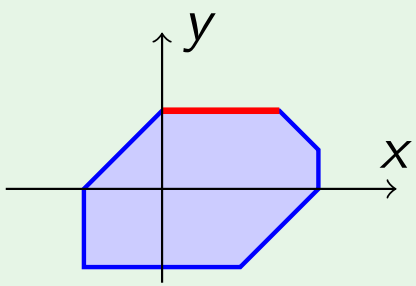

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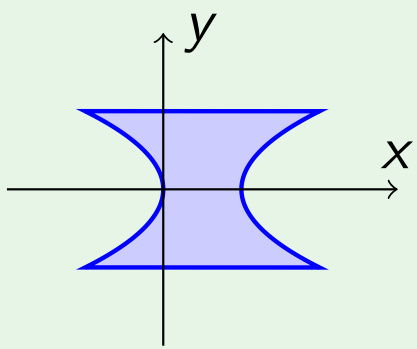

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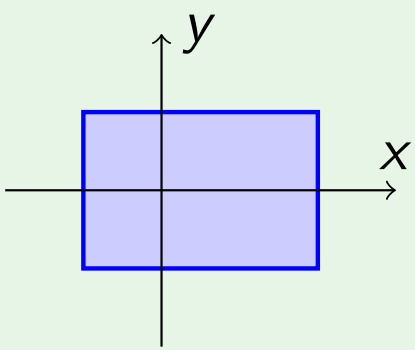

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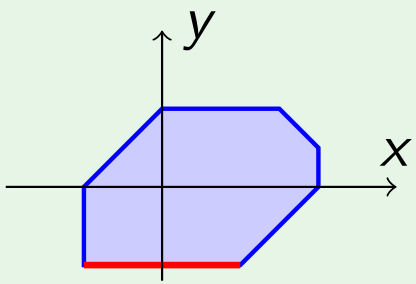

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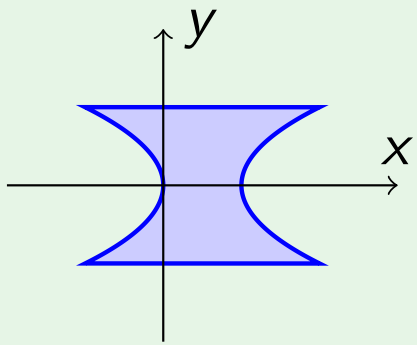

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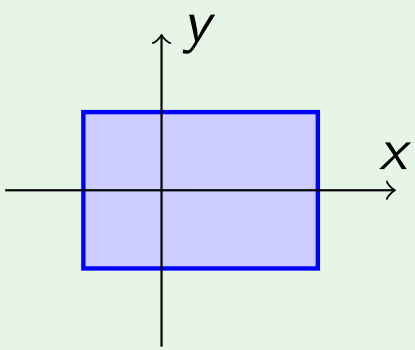

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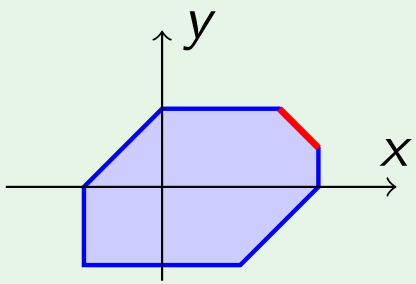

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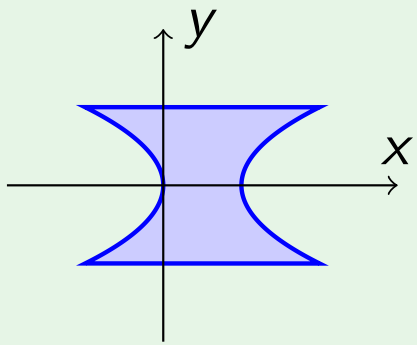

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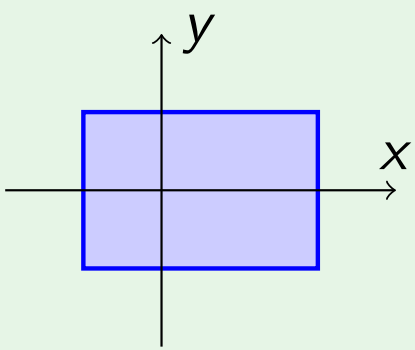

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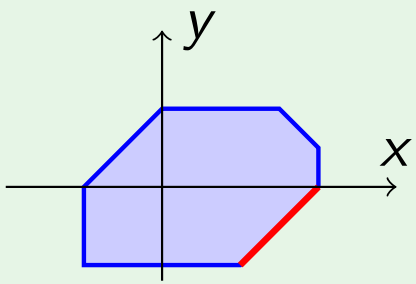

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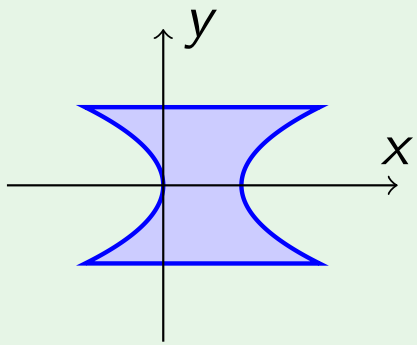

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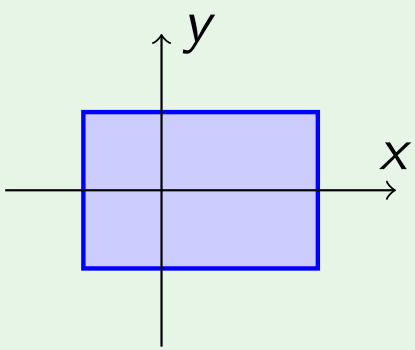

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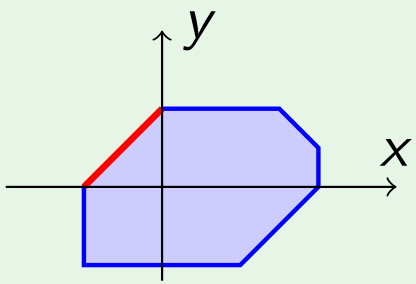

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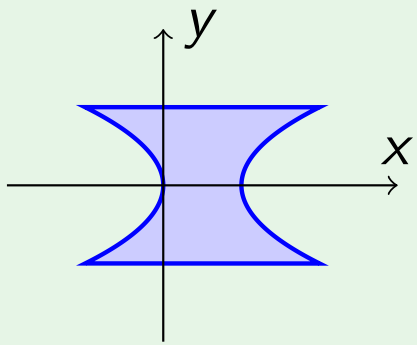

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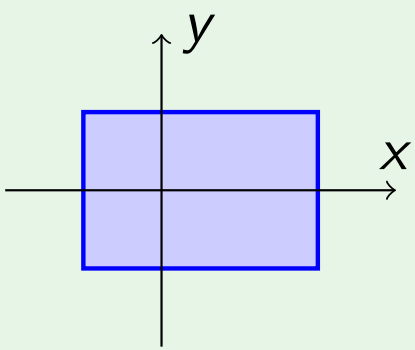

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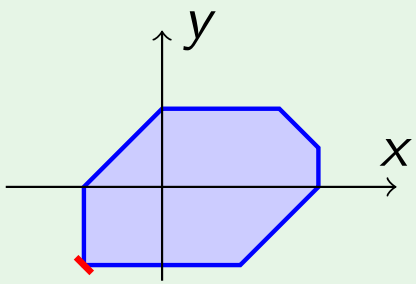

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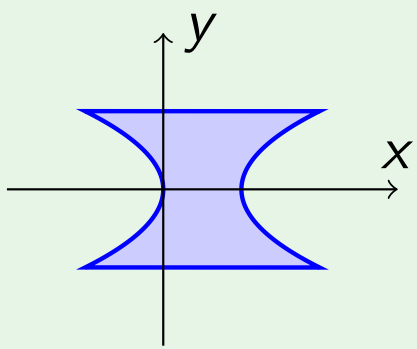

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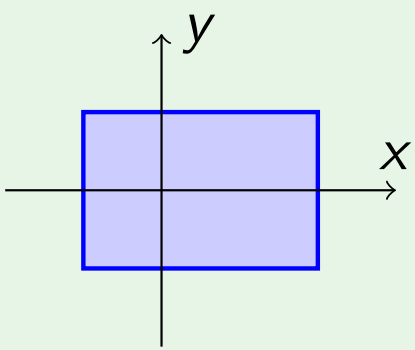

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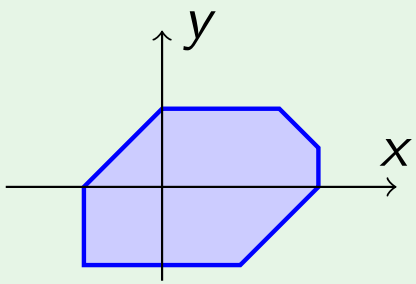

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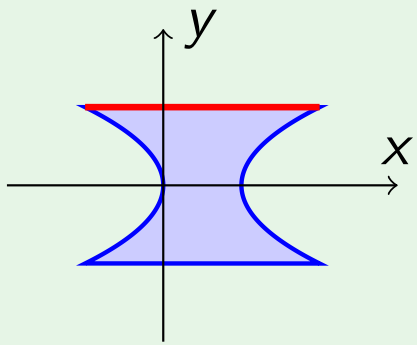

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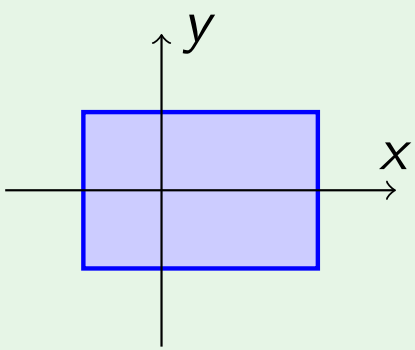

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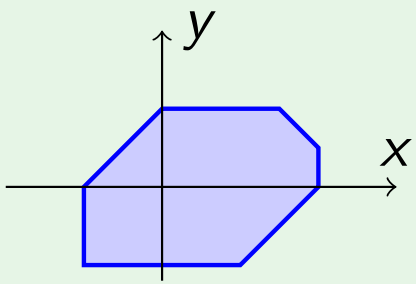

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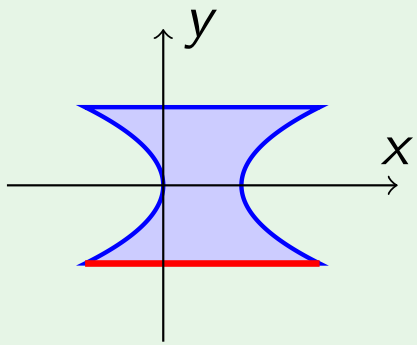

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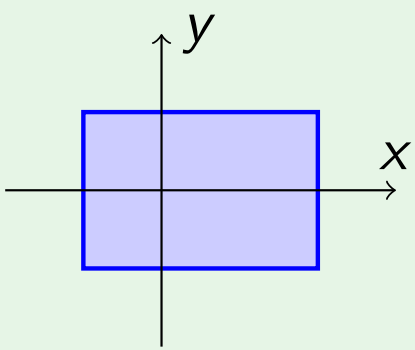

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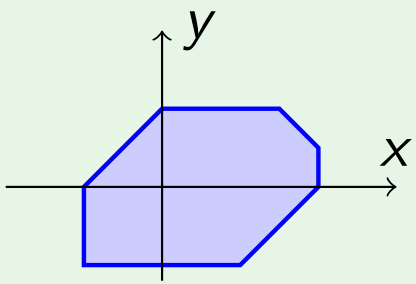

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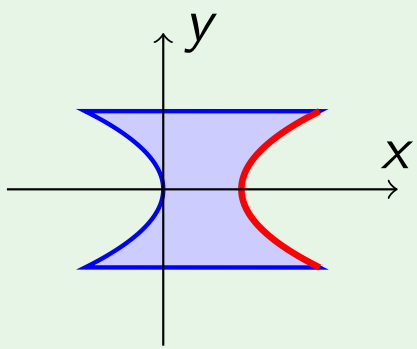

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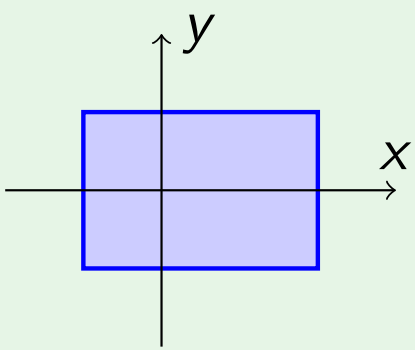

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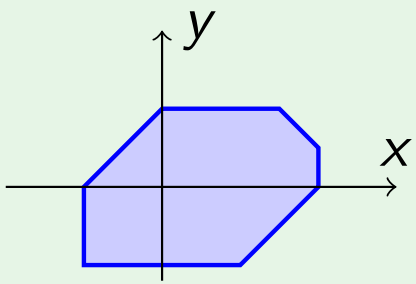

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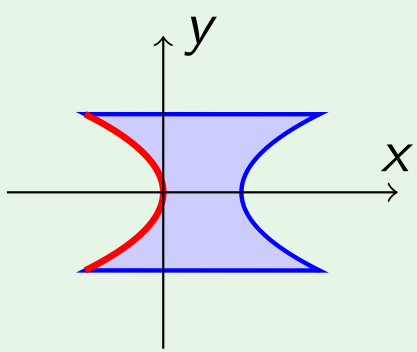

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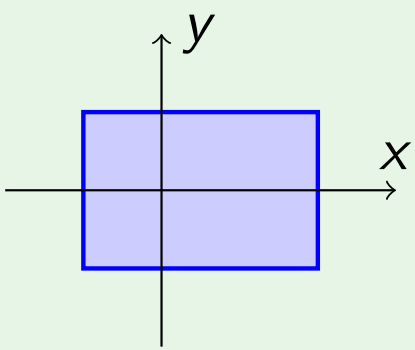

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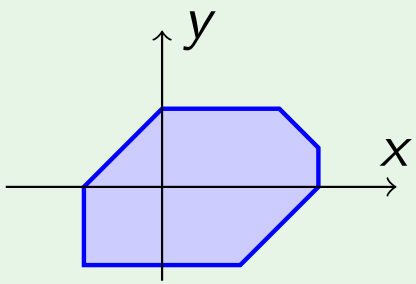

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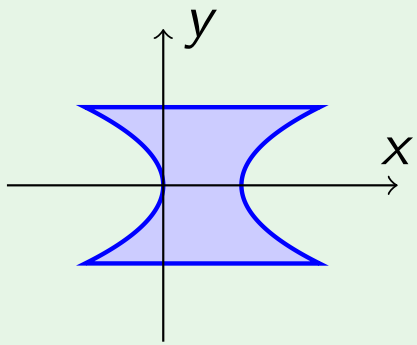

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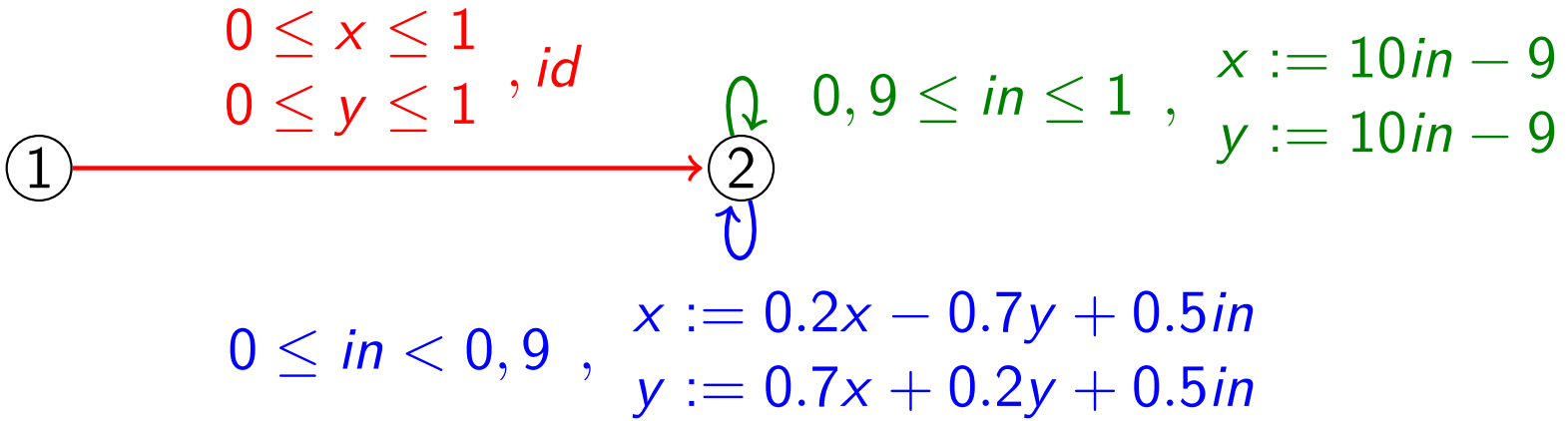
System of Equations

First step: building the **control flow graph**.



System of Equations

Second step: deriving a system of equations
from the graph and the templates ($\mathcal{T} = \{ x, -x, y, -y \}$).

$$
\begin{cases}
b_{1,1} = +\infty \qquad b_{1,2} = +\infty \qquad b_{1,3} = +\infty \qquad b_{1,4} = +\infty \\
b_{2,1} = \max\{x \mid 0 \leq x \leq 1 \wedge 0 \leq y \leq 1 \wedge \mathrm{be}(1)\} \\
\qquad \vee \max\{10\, in - 9 \mid 0.9 \leq in \leq 1 \wedge \mathrm{be}(2)\} \\
\qquad \vee \max\{0.2\, x - 0.7\, y + 0.5\, in \mid 0 \leq in \leq 0.9 \wedge \mathrm{be}(2)\} \\
b_{2,2} = \max\{-x \mid 0 \leq x \leq 1 \wedge 0 \leq y \leq 1 \wedge \mathrm{be}(1)\} \\
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$$

with $\mathrm{be}(i)$ denoting $x \leq b_{i,1} \wedge -x \leq b_{i,2} \wedge y \leq b_{i,3} \wedge -y \leq b_{i,4}$

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$\Rightarrow$ Need to integrate policy iterations into a classic abstract domain.

Policy Iterations: Integration into a Classic Abstract Domain

Basic idea:

- A (symbolic) abstract domain "recording" the control flow graph.
- Product with a template domain.
- During widening:
  reduction of template domain thanks to policy iterations on:
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Result [RouxG-ATVA13]

A **classic abstract domain** able to
fully automatically infer quadratic invariants.

Summary

State of the Art

- Policy iterations [AdjéGG10, GawlitzaS10, GawlitzaSAGG12] promising technique for quadratic invariants, but:
  - Bad handling of linear constraints.
  - Initialization not automatic.
  - Require templates to be manually provided.
  - No formal soundness check for floating point issues.
  - No implementation in a static analyzer for classic code.

Contributions [RouxG-ATVA13]

Alleviate identified shortcomings of policy iterations:

- Handling linear constraints, initialization ([Roux-PhD13]).
- Automatic computation of templates.
- Soundness check.
- Integration of policy iterations in a classic abstract domain.

Implementation

- A new domain in our Lustre analyzer;
- 4kloc of OCaml and 0.5kloc of C code:
- implemented both min and max policies, using min by default;
- uses CSDP library for semidefinite programming;
- uses OCaml-GLPK for linear programming;
- soundness check of the result by Cholesky decompositions;
- available under a GPL license at `http://cavale.enseeiht.fr/policy2013/`.

Conclusion

On the code

x0 := 0; x1 := 0; x2 := 0;
while $-1 \leq 0$ **do**
  in := **?(**-1, 1**)**;
  x0' := x0; x1' := x1; x2' := x2;
  x0 := 0.9379×x0'−0.0381×x1'−0.0414×x2'+0.0237×in;
  x1 := −0.0404×x0'+0.968×x1'−0.0179×x2'+0.0143×in;
  x2 := 0.0142×x0'−0.0197×x1'+0.9823×x2'+0.0077×in;
od

providing only the list of variables x_0, x_1, x_2, *in*
we want to apply our domain on, our tool computes

$$6.2547x_0^2 + 12.1868x_1^2 + 3.8775x_2^2 - 10.61x_0x_1 - 2.4306x_0x_2 + 2.4182x_1x_2 \leq 1.0029$$
$$\wedge x_0^2 \leq 0.1795 \wedge x_1^2 \leq 0.1136 \wedge x_2^2 \leq 0.2757$$

then

$$|x_0| \leq 0.4236 \wedge |x_1| \leq 0.3371 \wedge |x_2| \leq 0.5251.$$

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   x0' := x0; x1' := x1; x2' := x2;
   x0 := 0.9379×x0'−0.0381×x1'−0.0414×x2'+0.0237×in;
   x1 := −0.0404×x0'+0.968×x1'−0.0179×x2'+0.0143×in;
   x2 := 0.0142×x0'−0.0197×x1'+0.9823×x2'+0.0077×in;
od
```
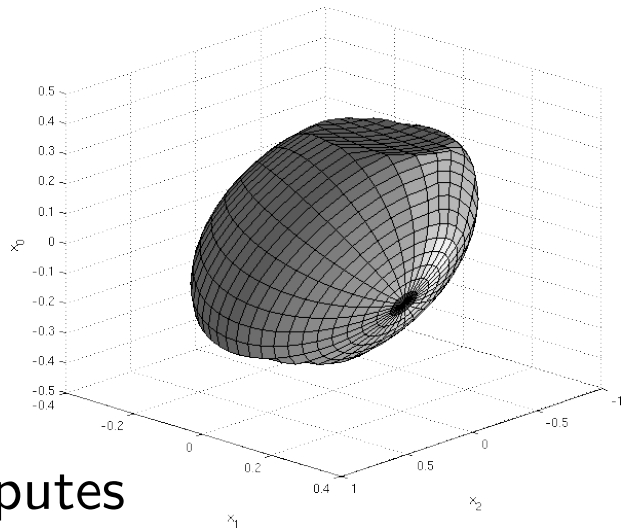



providing only the list of variables x_0, x_1, x_2, *in* we want to apply our domain on, our tool computes

$$6.2547x_0^2 + 12.1868x_1^2 + 3.8775x_2^2 - 10.61x_0x_1 - 2.4306x_0x_2 + 2.4182x_1x_2 \leq 1.0029$$
$$\wedge x_0^2 \leq 0.1795 \wedge x_1^2 \leq 0.1136 \wedge x_2^2 \leq 0.2757$$

then

$$|x_0| \leq 0.4236 \wedge |x_1| \leq 0.3371 \wedge |x_2| \leq 0.5251.$$

1 Quadratic Invariants

2 Policy Iterations

3 Open Directions

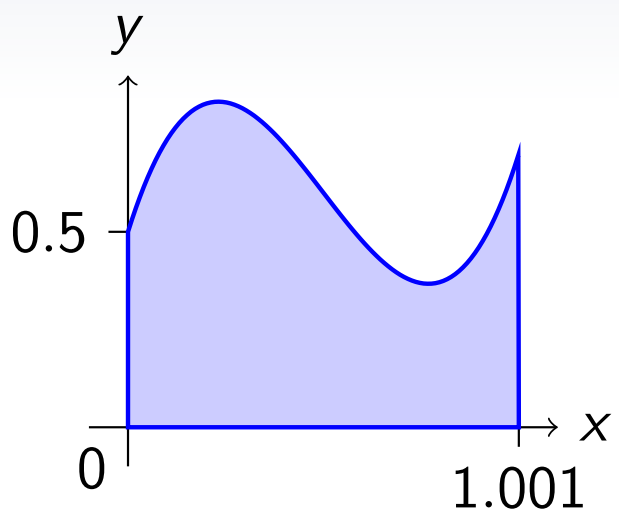
Other Properties

- Invariant generation:
  - Common Lyapunov functions.
  - Piecewise linear quadratic Lyapunov functions [RantzerJ00] (joint work with A. Adjé, ongoing work).
- Closed loop system:
  - Discretization of the plant.
  - Stability (joint work with R. Jobredeaux, submitted).
    - Generation of templates in presence of saturations.
    - Elimination of redundant variables.
  - Robustness (generalization of phase and gain margins).
  - Performance (overshoot, convergence speed).

Polynomial Invariants on Polynomial Programs

Classical template abstract domain based on **Bernstein polynomials** global optimization [RouxG-NSV13].

```
x := 0; y := ?(0, 0.5);
while x ≤ 1 do
    y := y + 0.001 × (18x² − 18x + 3);
    x := x + 0.001;
    if y ≤ 0 then y := 0 else y := y fi
od
```



$$
\begin{array}{rcl}
y - 6x^3 + 9x^2 - 3.2x & \leq & 0.5 \\
x & \leq & 1.001 \\
-x & \leq & 0 \\
y & \leq & 0.833 \\
-y & \leq & 0
\end{array}
$$

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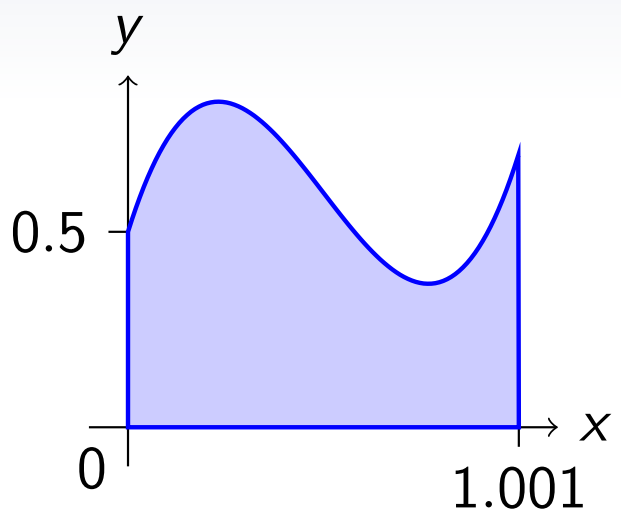

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Ideas

- Automatic generation of templates (based on SOS methods [KordaHJ13]).
- Extension to non algebraic functions (Taylor or Poisson expansion, beginning collaboration with G. Morin and X. Thirioux).